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STOCHASTIC PERTURBATION AND CONNECTIVITY BASED ON GRUSHIN DISTRIBUTION

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The present work proves a stochastic connectivity property for a controlled stochastic perturbation of a step $k+1$ Grushin distribution. More precisely, we prove that, given the controlled stochastic perturbation and two points P, Q in the plane, it is possible, using suitable controls, to steer any admissible stochastic process, starting at the point P, into an arbitrarily small disk centered at the point Q, with the probability arbitrarily close to one.

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1. Introduction

Let M be an n-dimensional, connected smooth manifold. A distribution on M is a sub-bundle $\mathcal{D} \subset TM$. The elements of $\mathcal{D}$ are called *horizontal vector fields*. The *rank* of $\mathcal{D}_p$ at a point $p \in M$ is the dimension of $\mathcal{D}_p \subset T_pM$ as a real vector sub-space.

Suppose the distribution $\mathcal{D}$ is not integrable. A triplet $(M, \mathcal{D}, g_{\mathcal{D}})$, where $g_{\mathcal{D}}$ is a positive definite and non-degenerate metric tensor defined on $\mathcal{D} \times \mathcal{D}$, is called *sub-Riemannian manifold*. Some authors refer to sub-Riemannian geometry under the name *non-holonomic geometry* (see Vrănceanu [20]-[22]) or *Carnot-Carathéodory geometry* [14].

Locally, the distribution $\mathcal{D}$ is given by a set of linearly independent smooth vector fields $\{X_1, X_2, \cdots, X_r\}, r \leq n$ on M. When $r = n$, we have a local frame. If $r < n$, then it is possible, locally, to add the "missing" directions ([1], [20]-[22]) to create a local frame. Due a technique of Vrănceanu ([20]-[22]), to each local frame, one can attach a positive definite metric such that the frame becomes orthonormal.

If the vector fields $\{X_1, X_2, \cdots, X_r\}$, together with their iterated Lie brackets span the entire tangent space T_pM at each point $p \in M$, then we

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say that the distribution $\mathcal{D}$ is *bracket generating*. The number s of iterated Lie brackets defines the *step* of the distribution, which is $s+1$. A famous Theorem due to Chow and Rashevskii ([11], [18]) states that *any two points of a connected manifold M can be joined by a piecewise C^1 curve tangent to the distribution $\mathcal{D}$, provided that $\mathcal{D}$ is bracket generating at any point $p \in M$.*

The bracket generating condition might be seen as a link between sub-Riemannian geometry and the theory of non-elliptic differential operators (see for example [2], [3], [15]). More precisely, the theorem of Chow and Rashevskii provides a geometry which plays an important role in understanding the heat kernels of operators which satisfy the bracket generating condition. Let us remark that this condition is only a *sufficient* one.

Recently, a stochastic version of the Chow-Rashevskii Theorem was formulated and motivated for a controlled stochastic perturbation associated to step 2 Grushin operators (a particular sub-Riemannian manifold) by Călin, Udrişte and Ţevy (see [8], [9]). It states that, for *any two points P, Q, we can find suitable control functions such that the corresponding admissible stochastic process, which starts at the point P, will reach, after a certain time, any ball centered at Q, with probability closely enough to* 1.

The main contribution of the present paper is the extension of the stochastic connectivity property for a controlled stochastic perturbation associated to step $k+1$ Grushin operators, where $k \in \mathbb{N}^*$. We use a general setting in order to suggest that our theory can be extended to any distribution.

2. Stochastic admissibility associated to Grushin operators

Let us consider the Grushin vector fields (first order partial derivative operators) $X = \partial_x,\ Y = x^k \partial_y,\ k \in \mathbb{N}^*$, defined on $\mathbb{R}^2$, with coordinates (x, y). Let $\mathcal{D}_G$ be the Grushin distribution generated by $\{X, Y\}$. The sub-Riemannian geometry on $\mathcal{D}_G$ has been in the focus throughout many works, especially for the sub-Riemannian geodesics (see for example [6],[5],[4],[10]).

The rank of the distribution $\mathcal{D}_G$ drops to 1 on the vertical axis. Consequently, the Grushin distribution $\mathcal{D}_G := \{X,\ Y\}$ is a non-holonomic frame only on $\mathbb{R}^2 \setminus Oy$. By taking the first k iterated Lie brackets

$$\begin{aligned}
&[X, Y] = kx^{k-1}\partial_y, \\
&[X, [X, Y]] = k(k-1)x^{k-2}\partial_y, \\
&\vdots \\
&[X, [X, \cdots, [X, Y]] \cdots] = k!\partial_y,
\end{aligned}$$

we see that the distribution $\mathcal{D}_G$ is bracket generating and it is of step $k+1$. Thus the condition of the Chow-Rashevskii Theorem is satisfied.

A controlled vector field in the Grushin distribution $\{X, Y\}$ is of the form

$$\varphi(x, y)X(x, y) + \phi(x, y)Y(x, y).$$

Its controlled field lines $c : [0,t] \longrightarrow \mathbb{R}^2,\ c(s) = (x(s), y(s))$, are described by the Pfaff system

$$\begin{pmatrix} dx(s) \\ dy(s) \end{pmatrix} = (u_1(s)X(x(s), y(s)) + u_2(s)Y(x(s), y(s)))ds, \tag{1}$$

or explicitly

$$\begin{cases} dx(s) = u_1(s)ds \\ dy(s) = u_2(s)x^k(s)ds, \end{cases} \tag{2}$$

where the controls $u_1(s), u_2(s)$ are piecewise smooth and take values in a bounded and closed set $U \in \mathbb{R}$. The set of such controls, denoted by $\mathcal{U}$, is called the set of *admissible controls*. The solutions of the Pfaff system (1) are called *admissible horizontal curves*.

To the Pfaff system (1), we attach a controlled stochastic differential equations system (SDE), called *stochastic perturbation* (see [8], [9]), in the sense,

$$\begin{pmatrix} dx(s) \\ dy(s) \end{pmatrix} = (u_1(s)X(x(s), y(s)) + u_2(s)Y(x(s), y(s)))ds + \begin{pmatrix} \sigma_1 dW_s^1 \\ \sigma_2 dW_s^2 \end{pmatrix}. \tag{3}$$

Generally, a stochastic controlled dynamics is described by an Itô process

$$c_s = \left(x^1(s), x^2(s), \cdots, x^n(s)\right),$$

satisfying (see for example [17], [12])

$$dc_s = b(s, c_s, u_s)ds + \sigma(s, c_s, u_s)dW_s, \tag{4}$$

where $b : \mathbb{R} \times \mathbb{R}^n \times U \to \mathbb{R}^n$, $\sigma : \mathbb{R} \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times m}$ are given functions, and W_s is an m-dimensional Wiener process (Brownian motion). The control $u_s \in U \subset \mathbb{R}^k$, which takes values in a given Borel set U, at any instant s, is a stochastic process $u_s = u(s,\omega)$ measurable w. r. t. the σ-algebra generated by $\{W_{s\wedge\tau}, \tau \geq 0\}$. Different problems use different types of controls (see for instance [17], Chapter 11). The controls $u(s,\omega) = u(s)$, not depending on ω, are called *deterministic* or *open loop controls*. Denote the set of deterministic controls by $\mathcal{U}_1$. Functions of the form $u(s,\omega) = u_0(s, c_s(\omega))$, for some function $u_0 : \mathbb{R}^{n+1} \to U \subset \mathbb{R}^k$, are called *Markov controls*. It is assumed that the function u does not depend on the starting point but only on the state of the system. Denote the set of Markov controls by $\mathcal{U}_2$.

3. The stochastic connectivity property

In the present section we shall state and prove our main result concerning the stochastic connectivity associated to a stochastic perturbation of step $k+1$ Grushin distribution.

Consider a 2-dimensional Brownian motion (W_s^1, W_s^2) together with a pair of non-negative constants (σ_1, σ_2), which control the amplitude of the error.

Definition 3.1. *A stochastic process $c_s = (x(s), y(s))$, which satisfies the SDE system*

$$\begin{cases} dx(s) = u_1(s)ds + \sigma_1 dW_s^1 \\ dy(s) = u_2(s)x^k(s)ds + \sigma_2 dW_s^2, \end{cases} \tag{5}$$

with $u_1, u_2 \in \mathcal{U}_1 \cup \mathcal{U}_2$, is called admissible stochastic process.

Theorem 3.1. *Let $P = (x_P, y_P), Q = (x_Q, y_Q)$ be two points in $\{\mathbb{R}^2, \mathcal{D}_G\}$ and denote by $D(Q, r)$ the Euclidean disk of radius r, centered at Q. Then, for any $\varepsilon, r > 0$, there exists a striking time $t < \infty$ and an admissible stochastic process c_s, satisfying the boundary conditions*

$$(x(0), y(0)) = (x_P, y_P), \ (\mathbb{E}[x(t)], \mathbb{E}[y(t)]) = (x_Q, y_Q),$$

such that

$$\mathbb{P}(c_t \in D(Q, r)) \geq 1 - \varepsilon. \tag{6}$$

Proof. Since the Grushin distribution is translation invariant along the Oy axis, the general starting point (x_P, y_P) can be reduced to the point $(x_P, 0)$.

As $\|c_t - Q\|^2$ is a nonnegative random variable, from Markov's inequality, we have

$$\mathbb{P}\left(\|c_t - Q\|^2 \geq r^2\right) \leq \frac{1}{r^2}\mathbb{E}\left[\|c_t - Q\|^2\right],$$

which is equivalent to

$$\mathbb{P}\left(\|c_t - Q\|^2 \leq r^2\right) \geq 1 - \frac{1}{r^2}\mathbb{E}\left[\|c_t - Q\|^2\right]. \tag{7}$$

Since the distance between two arbitrary points on the Oy axis is infinite, due to the rank variation of the Grushin distribution, we shall consider two cases.

Case 1. Suppose $x_P \neq x_Q$, $y_P = 0$. In this case, we can select the controls

$$\begin{aligned} &u_1(s) \in \mathcal{U}_1, \ u_1(s) = a, \ a \in \mathbb{R}, \\ &u_2(s) \in \mathcal{U}_1, \ u_2(s) = \frac{\operatorname{sgn}(y_Q)}{\left(|a|s + \sigma_1 s^{1/2} + \delta + x_P\right)^k}, \end{aligned} \tag{8}$$

where $\delta > 0$. The integral variant of the stochastic system (5), with initial conditions $c_0 = P$, has the form

$$\begin{aligned} x(t) &= at + \sigma_1 W_t^1 + x_P, \\ y(t) &= \int_0^t u_2(s)x^k(s)ds + \sigma_2 W_t^2. \end{aligned} \tag{9}$$

The properties of the Wiener processes

$$\mathbb{E}[W_t^i] = 0, \ \mathbb{E}\left[(W_t^i)^2\right] = t, \ i = 1, 2,$$

and the boundary conditions

$$\begin{aligned} &\mathbb{E}[x(t)] = x_Q \implies at + x_P = x_Q, \\ &\mathbb{E}[y(t)] = y_Q \implies \int_0^t u_2(s)\mathbb{E}\left[x^k(s)\right] ds = y_Q, \end{aligned}$$

together with the independence of W_t^1 and W_t^2, respectively, give

$$\mathbb{E}\left[\|c_t - Q\|^2\right] = \mathbb{E}\left[(x(t) - x_Q)^2\right] + \mathbb{E}\left[(y(t) - y_Q)^2\right]$$
$$= \mathbb{E}\left[(at + x_P - x_Q + \sigma_1 W_t^1)^2\right] + \mathbb{E}\left[\left(\int_0^t u_2(s)x^k(s)ds - y_Q + \sigma_2 W_t^2\right)^2\right]$$
$$= \sigma_1^2 t + \mathbb{E}\left[\left(\int_0^t u_2(s)x^k(s)ds\right)^2\right] + y_Q^2 + \sigma_2^2 t - 2y_Q \int_0^t u_2(s)\mathbb{E}\left[x^k(s)\right] ds$$
$$= \left(\sigma_1^2 + \sigma_2^2\right) t - y_Q^2 + \mathbb{E}\left[\left(\int_0^t u_2(s)x^k(s)ds\right)^2\right].$$

Let us find now some estimates for the last term of the above equality. By the Cauchy-Schwartz inequality, we get

$$\left(\int_0^t u_2(s)x^k(s)ds\right)^2 \leq \int_0^t \left(u_2(s)x^k(s)\right)^2 ds \int_0^t ds,$$

which implies

$$\mathbb{E}\left[\left(\int_0^t u_2(s)x^k(s)ds\right)^2\right] \leq t \int_0^t u_2^2(s)\mathbb{E}\left[x^{2k}(s)\right] ds. \tag{10}$$

Recall that (see for example [17])

$$\mathbb{E}\left[\left(W_t^1\right)^i\right] = \begin{cases} \dfrac{i!}{2^{i/2}(i/2)!}\, t^{i/2}, & i \in 2\mathbb{N}, \\ 0, & i \in 2\mathbb{N}+1. \end{cases}$$

Consequently, $\mathbb{E}\left[\left(W_t^1\right)^i\right] \leq \left(t^{1/2}\right)^i, \ \forall i \in \mathbb{N}$.

Taking into account the expression for $x(t)$, we find

$$\begin{aligned} &\mathbb{E}\left[\left(at + \sigma_1 W_t^1 + x_P\right)^{2k}\right] = \sum_{i=0}^{2k} C_{2k}^i \left(at + x_P\right)^{2k-i} \sigma_1^i \mathbb{E}\left[\left(W_t^1\right)^i\right] \\ &< \sum_{i=0}^{2k} C_{2k}^i \left(at + x_P\right)^{2k-i} \sigma_1^i \left(t^{1/2}\right)^i = \left(at + x_P + \sigma_1 t^{1/2}\right)^{2k}, \end{aligned} \tag{11}$$

where we have used the fact that the odd terms $\mathbb{E}\left[\left(W_t^1\right)^i\right]$ are zero.

Substituting the inequality (11) in (10), together with the control function $u_2(t)$, yields

$$\mathbb{E}\left[\left(\int_0^t u_2(s)x^k(s)ds\right)^2\right] \leq t \int_0^t \frac{\left(as + \sigma_1 s^{1/2} + x_P\right)^{2k}}{\left(|a|s + \sigma_1 s^{1/2} + \delta + x_P\right)^{2k}}\, ds \leq t^2. \tag{12}$$

Thus,

$$\mathbb{E}\left[\|c_t - Q\|^2\right] \leq \left(\sigma_1^2 + \sigma_2^2\right) t - y_Q^2 + t^2. \tag{13}$$

Finally, for any $\varepsilon, r > 0$, the equation,

$$t^2 + \left(\sigma_1^2 + \sigma_2^2\right) t - \left(y_Q^2 + \varepsilon r^2\right) = 0 \tag{14}$$

has a strictly positive solution

$$t = \frac{-\left(\sigma_1^2 + \sigma_2^2\right) + \sqrt{\left(\sigma_1^2 + \sigma_2^2\right)^2 + 4\left(y_Q^2 + \varepsilon r^2\right)}}{2}, \tag{15}$$

for which

$$\mathbb{E}\left[\|c_t - Q\|^2\right] \leq \varepsilon r^2 \Longleftrightarrow 1 - \frac{1}{r^2}\mathbb{E}\left[\|c_t - Q\|^2\right] \geq 1 - \varepsilon, \text{ i.e.,} \tag{16}$$

$$\mathbb{P}\left(c_t \in D(Q, r)\right) \geq 1 - \varepsilon.$$

From the boundary conditions we also have $a = \dfrac{x_Q - x_P}{t}$.

Case 2. Suppose $x_P = x_Q, y_P = 0$. Let us consider the following control functions $u_1(s), u_2(s) \in \mathcal{U}_1$

$$\begin{aligned} &u_1(s) \in \mathcal{U}_1,\ \ u_1(s) = \sin(as),\ \ a \neq 0, \\ &u_2(s) \in \mathcal{U}_1,\ \ u_2(s) = \frac{\operatorname{sgn}(y_Q)}{\left(\delta + |\sin(as)| + \sigma_1 s^{\frac{1}{2}}\right)^k}, \end{aligned} \tag{17}$$

for some $\delta > 0$. It is easy to see that the above computations hold also in this case. Notice also that $a = \dfrac{\pi}{t}$, where t is given in (15).

For Markov controls, take $u_1(s) = \dfrac{x_Q - x_P}{t}$, $u_2(s) = \dfrac{b}{x^k(s)}\dfrac{y_Q - y_P}{t}$. □

4. Extended stochastic connectivity

As it can be seen in Theorem 3.1, one boundary condition is deterministic whereas the other is expressed in probabilistic terms. It is possible to express both conditions probabilistically such that the roles of the endpoints become interchangeable.

Corollary 4.1. *Let P, Q be two arbitrary points in $\{\mathbb{R}^2, \mathcal{D}\}$. Then, for any $r_1, r_2 > 0$, $0 < \varepsilon_1, \varepsilon_2 < 1$, there exist t_1 and t_2, respectively, and an admissible stochastic process c_s, satisfying the boundary conditions*

$$\left(\mathbb{E}\left[x(t_1)\right], \mathbb{E}\left[y(t_1)\right]\right) = \left(x_P, y_P\right),\ \left(\mathbb{E}\left[x(t_2)\right], \mathbb{E}\left[y(t_2)\right]\right) = \left(x_Q, y_Q\right),$$

such that

$$\mathbb{P}\left(c_{t_1} \in D(P, r_1)\right) \geq 1 - \varepsilon_1,\ \mathbb{P}\left(c_{t_2} \in D(Q, r_2)\right) \geq 1 - \varepsilon_2. \tag{18}$$

Proof. Since at least one endpoint is not the origin, suppose P is that point. We implicitly consider that the controls have precisely the same form as above, and that the constants are determined similarly.

Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ and $r = \min\{r_1, r_2\}$. By Theorem 3.1, there exist a striking time t_1 and an admissible stochastic process c_s, starting at the origin $O \in \mathbb{R}^2$, such that $\mathbb{E}\left[\|c_{t_1} - P\|^2\right] \le \varepsilon \left(\frac{r}{2}\right)^2$, hence

$$\mathbb{P}\left(c_{t_1} \in D\left(P, \frac{r}{2}\right) \subset D\left(P, r_1\right)\right) \ge 1 - \varepsilon \ge 1 - \varepsilon_1. \tag{19}$$

Integrating (5), one obtains

$$\begin{aligned} x(t_2) &= \int_{t_1}^{t_2} u_1(s)ds + \sigma_1 W^1_{(t_2 - t_1)} + x(t_1) \\ y(t_2) &= \int_{t_1}^{t_2} u_2(s)x^k(s)ds + \sigma_2 W^2_{(t_2 - t_1)} + y(t_1), \end{aligned} \tag{20}$$

whereas the boundary conditions imply

$$\int_{t_1}^{t_2} u_1(s)ds + x_P = x_Q, \ \mathbb{E}\left[\int_{t_1}^{t_2} u_2(s)x^k(s)ds\right] + y_P = y_Q.$$

Noticing that $\mathbb{E}\left[W^i_{(t_2-t_1)}\right] = 0$, $\mathbb{E}\left[\left(W^i_{(t_2-t_1)}\right)^2\right] = t_2 - t_1$, $i = 1, 2$, we evaluate

$$\begin{aligned} \mathbb{E}\left[(x(t_2) - x_Q)^2\right] &= \mathbb{E}\left[(x(t_1) - x_P)^2\right] + \sigma_1^2\left(t_2 - t_1\right), \\ \mathbb{E}\left[(y(t_2) - x_Q)^2\right] &= \mathbb{E}\left[(y(t_1) - x_P)^2\right] + \sigma_2^2\left(t_2 - t_1\right) \\ &\quad - \left(x_Q - x_P\right)^2 + \mathbb{E}\left[\left(\int_{t_1}^{t_2} u_2(s)x^k(s)ds\right)^2\right]. \end{aligned}$$

The inequality (12) becomes, in this case,

$$\mathbb{E}\left[\left(\int_{t_1}^{t_2} u_2(s)x^k(s)ds\right)^2\right] \le \left(t_2 - t_1\right)^2.$$

Similarly with (14), the equation

$$\left(\sigma_1^2 + \sigma_2^2\right)\left(t_2 - t_1\right) - \left(x_Q - x_P\right)^2 + \left(t_2 - t_1\right)^2 = \frac{3}{4}\varepsilon r^2,$$

with $(t_2 - t_1)$ as the unknown, has a strictly positive solution, for which

$$\mathbb{E}\left[\|c_{t_2} - Q\|^2\right] \le \mathbb{E}\left[\|c_{t_1} - P\|^2\right] + \frac{3}{4}\varepsilon r^2 \le \varepsilon r^2,$$

and hence $\mathbb{P}\left(c_{t_2} \in D(Q, r_2)\right) \ge 1 - \varepsilon_2$. This completes the proof. □

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