COUPLED COMMON FIXED POINT RESULTS FOR MIXED g-MONOTONE MAPPS IN PARTIALLY ORDERED G-METRIC SPACES

Sumit Chandok[1], Zead Mustafa[2], Mihai Postolache[3]

The purpose of this paper is to establish some coupled coincidence point theorems for mappings having a mixed g-monotone property in partially ordered G-metric spaces. Also, we present a result on the existence and uniqueness of coupled common fixed points. The results presented in the paper generalize and extend several well-known results in the literature.

Keywords: Coupled fixed point, mixed g-monotone property, ordered G-metric space.

MSC2010: 41A50, 47H10, 54H25.

1. Introduction

The Banach contraction mapping is one of the pivotal results of analysis. It is a famous tool for solving existence problems in various fields of mathematics. There are a lot of generalizations of the Banach contraction principle in the literature. Ran and Reurings [34] extended the Banach contraction principle in partially ordered sets with some applications to linear and nonlinear matrix equations. Nieto and Rodŕiguez-López [32] extended the result of Ran and Reurings and applied their main theorems to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Bhaskar and Lakshmikantham [6] introduced the concept of mixed monotone mappings and obtained some coupled fixed point results. Also, they applied their results on a first order differential equation with periodic boundary conditions.

In [22], Mustafa and Sims introduced G-metric spaces, which is a generalization of metric spaces, in which every triplet of elements is assigned to a non-negative real number. Recently, many researchers have obtained fixed point, common fixed point, coupled fixed point results on metric spaces, convex metric spaces, partial metric spaces, G-metric spaces, partially ordered metric spaces and partially ordered G-metric spaces (see [1]-[44]).

The purpose of this paper is to establish some new coupled coincidence point results in partially ordered G-metric spaces for mappings having mixed g-monotone

[1] Department of Mathematics, Khalsa College of Engineering & Technology (Punjab Technical University), Ranjit Avenue, Amritsar-143001, India

[2] Department of Mathematics, Hashemite University, Zarqa, Jordan

[3] University "Politehnica" of Bucharest, Splaiul Independenţei, No. 313, 060042 Bucharest, Romania, e-mail: mihai@mathem.pub.ro

property. Also, we present a result on the existence and uniqueness of coupled common fixed points.

2. **Preliminaries**

Definition 2.1. Let (X, d) be a metric space and $F\colon X \times X \to X$ and $g\colon X \to X$ two mappings. We say that F and g *commute* if $F(gx, gy) = g(F(x, y))$, for all $x, y \in X$.

Definition 2.2 ([22])**.** Let X be a non-empty set, $G\colon X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:

(G1) $G(x, y, z) = 0$ if $x = y = z$;
(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;
(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$;
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$ (symmetry in all three variables);
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a *generalized metric*, or, more specifically, a *G-metric* on X, and the pair (X, G) is called a *G-metric space.*

Definition 2.3 ([22])**.** Let (X, G) be a G-metric space, and let $\{x_n\}$ be a sequence of points of X. We say that $\{x_n\}$ is *G-convergent* to $x \in X$ if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$, that is, for any $\epsilon > 0$, there exists N in $\mathbb{N}$ such that $G(x, x_n, x_m) < \epsilon$, for all $n, m \geq N$.

We call x the limit of the sequence and write $x_n \to x$ or $\lim x_n = x$.

Proposition 2.1 ([22])**.** *Let (X, G) be a G-metric space. The following are equivalent:*

(1) *$\{x_n\}$ is G-convergent to x.*
(2) *$G(x_n, x_n, x) \to 0$ as $n \to +\infty$.*
(3) *$G(x_n, x, x) \to 0$ as $n \to +\infty$.*
(4) *$G(x_n, x_m, x) \to 0$ as $n, m \to +\infty$.*

Definition 2.4 ([22])**.** Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called a *G-Cauchy sequence* if, for any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $m, n, l \geq N$, that is, $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty$.

Proposition 2.2 ([21])**.** *Let (X, G) be a G-metric space. Then the following are equivalent*

(1) *The sequence $\{x_n\}$ is G-Cauchy;*
(2) *For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $m, n \geq N$.*

Proposition 2.3 ([22])**.** *Let (X, G) be a G-metric space. A mapping $f\colon X \to X$ is G-continuous at $x \in X$ if and only if it is G-sequentially continuous at x, that is, whenever $\{x_n\}$ is G-convergent to x, $\{f(x_n)\}$ is G-convergent to $f(x)$.*

Proposition 2.4 ([22])**.** *Let (X, G) be a G-metric space. Then, the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Definition 2.5 ([22])**.** A G-metric space (X, G) is called *G-complete* if every G-Cauchy sequence is G-convergent in (X, G).

Definition 2.6 ([15])**.** Let (X, G) be a G-metric space. A mapping $F\colon X \times X \to X$ is said to be *continuous* if for any two G-convergent sequences $\{x_n\}$ and $\{y_n\}$, to x and y respectively, $\{F(x_n, y_n)\}$ is G-convergent to $F(x, y)$.

Definition 2.7. Let $(X, \preceq)$ be a partially ordered set and $F\colon X \times X \to X$. The mapping F is said to be *non-decreasing* if for $x, y \in X$, $x \preceq y$ implies $F(x) \preceq F(y)$ and *non-increasing* if for $x, y \in X$, $x \preceq y$ implies $F(x) \succeq F(y)$.

Definition 2.8. Let $(X, \preceq)$ be a partially ordered set and $F\colon X \times X \to X$ and $g\colon X \to X$. The mapping F is said to have the *mixed g-monotone property* if $F(x, y)$ is monotone g-non-decreasing in x and monotone g-non-increasing in y, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, gx_1 \preceq gx_2 \Rightarrow F(x_1, y) \preceq F(x_2, y),$$

and

$$y_1, y_2 \in X, gy_1 \preceq gy_2 \Rightarrow F(x, y_1) \succeq F(x, y_2).$$

If g =identity mapping in Definition 2.8, then the mapping F is said to have the *mixed monotone property.*

Definition 2.9. An element $(x, y) \in X \times X$ is called a *coupled coincidence point* of the mappings $F\colon X \times X \to X$ and $g\colon X \to X$ if $F(x, y) = gx$, and $F(y, x) = gy$.

If g =identity mapping in Definition 2.9, then $(x, y) \in X \times X$ is called a *coupled fixed point.*

The following coupled fixed point theorem is the main result of Bhaskar and Lakshmikantham [6].

Theorem 2.1. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a metric space. Suppose that $F\colon X \times X \to X$ is a self mapping on X and has the mixed monotone property on X such that there exists two elements $x_0, y_0 \in X$ with $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$. Suppose that there exist $k \in [0, 1)$ such that*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}(d(x, u) + d(y, v)) \tag{2.1}$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$.

Further suppose that either

a) F is continuous or

b) X has the following properties:

(i) if a sequence $\{x_n\} \subset X$ is a non-decreasing sequence with $x_n \to x$ in X, then $x_n \preceq x$, for every n;

(ii) if a sequence $\{y_n\} \subset X$ is a non-increasing sequence with $y_n \to y$ in X, then $y_n \succeq y$, for every n;

Then there exists $x, y \in X$ such that $F(x, y) = x$ and $y = F(y, x)$, that is, F has a coupled fixed point $(x, y) \in X \times X$.

3. **Main Results**

In this section, we prove some coupled common fixed point theorems in the context of partially ordered G-metric spaces. In this respect, let Ψ denote the set of all functions $\psi\colon [0,\infty) \to [0,\infty)$ which satisfy $\lim_{t\to r}\psi(t) > 0$ for all $r > 0$ and $\lim_{t\to 0+}\psi(t) = 0$; and Φ denote the set of all functions $\phi\colon [0,\infty) \to [0,\infty)$ such that

(i) ϕ is continuous and non-decreasing;

(ii) $\phi(t) = 0$ if and only if $t = 0$;

(iii) $\phi(t+s) \le \phi(t) + \phi(s)$, for all $t, s \in [0,\infty)$.

For example, functions $\phi_1(t) = kt$, where $k > 0$, $\phi_2(t) = \frac{t}{t+1}$, $\phi_3(t) = \ln(t+1)$, and $\phi_4(t) = \min\{t, 1\}$ are in Φ. Functions $\psi_1(t) = kt$, where $k > 0$, $\psi_2(t) = \frac{\ln(2t+1)}{2}$, and

$$\psi_3(t) = \begin{cases} 1, & t = 0 \\ \frac{t}{t+1}, & 0 < t < 1 \\ 1, & t = 1 \\ \frac{t}{2}, & t > 1 \end{cases}$$

are in Ψ.

Theorem 3.1. *Let $(X, \preceq)$ be a partially ordered set and G be a G-metric on X such that (X, G) is a complete G-metric space. Suppose that $F\colon X \times X \to X$ and $g\colon X \to X$ are continuous self mappings on X such that F has the mixed g-monotone property on X such that there exists two elements $x_0, y_0 \in X$ with $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$. Suppose that there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that*

$$\begin{aligned} \phi(G(F(x,y), F(u,v), F(w,z))) \ \le\ & \frac{1}{2}\phi(G(gx, gu, gw) + G(gy, gv, gz)) \\ & -\psi\left(\frac{G(gx, gu, gw) + G(gy, gv, gz)}{2}\right) \end{aligned} \tag{3.1}$$

for all $x, y, u, v, w, z \in X$ with $gx \succeq gu \succeq gw$ and $gy \preceq gv \preceq gz$. Further suppose that $F(X \times X) \subseteq g(X)$ and g commutes with F. Then there exists $x, y \in X$ such that $F(x, y) = gx$ and $gy = F(y, x)$, that is, F and g have a coupled coincidence point $(x, y) \in X \times X$.

Proof. Let $x_0, y_0 \in X$ be such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$. Since $F(X \times X) \subseteq g(X)$, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$gx_{n+1} = F(x_n, y_n) \text{ and } gy_{n+1} = F(y_n, x_n), \forall n \ge 0. \tag{3.2}$$

We claim that for all $n \ge 0$,

$$gx_n \preceq gx_{n+1}, \tag{3.3}$$

and

$$gy_n \succeq gy_{n+1}. \tag{3.4}$$

We shall use the mathematical induction.

Let $n = 0$. Since $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, in view of $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$, we have $gx_0 \preceq gx_1$ and $gy_0 \succeq gy_1$, that is, (3.3) and (3.4) hold for $n = 0$. Suppose that (3.3) and (3.4) hold for some $n > 0$. As F has the mixed g-monotone property and $gx_n \preceq gx_{n+1}$ and $gy_n \succeq gy_{n+1}$, from (3.2), we get

$$gx_{n+1} = F(x_n, y_n) \preceq F(x_{n+1}, y_n) \preceq F(x_{n+1}, y_{n+1}) = gx_{n+2}, \tag{3.5}$$

and

$$gy_{n+1} = F(y_n, x_n) \succeq F(y_{n+1}, x_n) \succeq F(y_{n+1}, x_{n+1}) = gy_{n+2}. \tag{3.6}$$

Now, from (3.5) and (3.6), we obtain that $gx_{n+1} \preceq gx_{n+2}$ and $gy_{n+1} \succeq gy_{n+2}$. Thus by the mathematical induction, we conclude that (3.3) and (3.4) hold for all $n \geq 0$. Therefore

$$gx_0 \preceq gx_1 \preceq gx_2 \preceq \ldots \preceq gx_n \preceq gx_{n+1} \preceq \ldots, \tag{3.7}$$

and

$$gy_0 \succeq gy_1 \succeq gy_2 \succeq \ldots \succeq gy_n \succeq gy_{n+1} \succeq \ldots. \tag{3.8}$$

Since $gx_n \preceq gx_{n+1}$ and $gy_n \succeq gy_{n+1}$, from (3.1) and (3.2), we have

$$\begin{aligned} & \phi(G(gx_{n+1}, gx_{n+1}, gx_n)) = \phi(G(F(x_n, y_n), F(x_n, y_n), F(x_{n-1}, y_{n-1}))) \\ \leq\ & \frac{1}{2}\phi(G(gx_n, gx_n, gx_{n-1}) + G(gy_n, gy_n, gy_{n-1})) \\ & -\psi\left(\frac{G(gx_n, gx_n, gx_{n-1}) + G(gy_n, gy_n, gy_{n-1})}{2}\right), \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} & \phi(G(gy_{n+1}, gy_{n+1}, gy_n)) = \phi(G(F(y_n, x_n), F(y_n, x_n), F(y_{n-1}, x_{n-1}))) \\ \leq\ & \frac{1}{2}\phi(G(gy_n, gy_n, gy_{n-1}) + G(gx_n, gx_n, gx_{n-1})) \\ & -\psi\left(\frac{G(gy_n, gy_n, gy_{n-1}) + G(gx_n, gx_n, gx_{n-1})}{2}\right). \end{aligned} \tag{3.10}$$

From (3.9) and (3.10), we have

$$\begin{aligned} & \phi(G(gx_{n+1}, gx_{n+1}, gx_n)) + \phi(G(gy_{n+1}, gy_{n+1}, gy_n)) \\ \leq\ & \phi(G(gx_n, gx_n, gx_{n-1}) + G(gy_n, gy_n, gy_{n-1})) \\ & -2\psi\left(\frac{G(gx_n, gx_n, gx_{n-1}) + G(gy_n, gy_n, gy_{n-1})}{2}\right). \end{aligned} \tag{3.11}$$

By property (iii) of Φ, we have

$$\begin{aligned} & \phi(G(gx_{n+1}, gx_{n+1}, gx_n) + G(gy_{n+1}, gy_{n+1}, gy_n)) \\ \leq\ & \phi(G(gx_{n+1}, gx_{n+1}, gx_n)) + \phi(G(gy_{n+1}, gy_{n+1}, gy_n)). \end{aligned} \tag{3.12}$$

From (3.11) and (3.12), we have

$$\begin{aligned} & \phi(G(gx_{n+1}, gx_{n+1}, gx_n) + G(gy_{n+1}, gy_{n+1}, gy_n)) \\ \leq\ & \phi(G(gx_n, gx_n, gx_{n-1}) + G(gy_n, gy_n, gy_{n-1})) \\ & -2\psi\left(\frac{G(gx_n, gx_n, gx_{n-1}) + G(gy_n, gy_n, gy_{n-1})}{2}\right) \end{aligned} \tag{3.13}$$

which implies that

$$\begin{aligned} & \phi(G(gx_{n+1}, gx_{n+1}, gx_n) + G(gy_{n+1}, gy_{n+1}, gy_n)) \\ & \leq \phi(G(gx_n, gx_n, gx_{n-1}) + G(gy_n, gy_n, gy_{n-1})). \end{aligned}$$

As ϕ is non-decreasing, we get

$$\begin{aligned} & G(gx_{n+1}, gx_{n+1}, gx_n) + G(gy_{n+1}, gy_{n+1}, gy_n) \\ & \leq G(gx_n, gx_n, gx_{n-1}) + G(gy_n, gy_n, gy_{n-1}). \end{aligned}$$

Thus we proved that $\delta_n = \{G(gx_{n+1}, gx_{n+1}, gx_n) + G(gy_{n+1}, gy_{n+1}, gy_n)\}$ is a monotone decreasing sequence of non-negative real numbers. Hence there exists $r \geq 0$ such that $\delta_n \to r$ as $n \to \infty$.

Now, we shall show that $r = 0$. Suppose, to the contrary, that $r > 0$. Then taking the limit as $n \to \infty$ in (3.13), we have

$$\phi(r) = \lim \phi(\delta_n) \leq \lim \left[\phi(\delta_{n-1}) - 2\psi(\frac{\delta_{n-1}}{2})\right] < \phi(r),$$

which is a contradiction. Thus $r = 0$, that is,

$$\lim \delta_n = \lim \left[G(gx_{n+1}, gx_{n+1}, gx_n) + G(gy_{n+1}, gy_{n+1}, gy_n)\right] = 0. \tag{3.14}$$

Now, we shall prove that $\{gx_n\}$ and $\{gy_n\}$ are G-Cauchy sequences. On the contrary, assume that atleast one of $\{gx_n\}$ or $\{gy_n\}$ is not a G-Cauchy sequence. Then there exists an $\epsilon > 0$ for which we can find subsequences $\{gx_{m(k)}\}$ and $\{gx_{n(k)}\}$ of $\{gx_n\}$ and $\{gy_{m(k)}\}$ and $\{gy_{n(k)}\}$ of $\{gy_n\}$ with $n(k) > m(k) > k$ such that for every k,

$$G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) + G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}) \geq \epsilon. \tag{3.15}$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k) \geq k$ and satisfies (3.15). Then

$$G(gx_{n(k)-1}, gx_{n(k)-1}, gx_{m(k)}) + G(gy_{n(k)-1}, gy_{n(k)-1}, gy_{m(k)}) < \epsilon. \tag{3.16}$$

Using rectangle inequality, we get

$$\begin{aligned}
\epsilon \leq r_k \ &:= \ G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) + G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}) \\
&\leq \ G(gx_{n(k)}, gx_{n(k)}, gx_{n(k)-1}) + G(gx_{n(k)-1}, gx_{n(k)-1}, gx_{m(k)}) \\
&\quad +G(gy_{n(k)}, gy_{n(k)}, gy_{n(k)-1}) + G(gy_{n(k)-1}, gy_{n(k)-1}, gy_{m(k)}) \\
&= \ G(gx_{n(k)-1}, gx_{n(k)-1}, gx_{m(k)}) + G(gy_{n(k)-1}, gy_{n(k)-1}, gy_{m(k)}) + \delta_{n(k)-1} \\
&\leq \ \epsilon + \delta_{n(k)-1}.
\end{aligned}$$

Letting $k \to \infty$, in above inequality and using (3.14), we get

$$\lim r_k = \epsilon^+. \tag{3.17}$$

Also, again by using ractangle inequality, we have

$$\begin{aligned}
r_k \ &= \ G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) + G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}) \\
&\leq \ G(gx_{n(k)}, gx_{n(k)}, gx_{n(k)+1}) + G(gx_{n(k)+1}, gx_{n(k)+1}, gx_{m(k)+1}) \\
&\quad +G(gx_{m(k)+1}, gx_{m(k)+1}, gx_{m(k)}) + G(gy_{n(k)}, gy_{n(k)}, gy_{n(k)+1}) \\
&\quad +G(gy_{n(k)+1}, gy_{n(k)+1}, gy_{m(k)+1}) + G(gy_{m(k)+1}, gy_{m(k)+1}, gy_{m(k)}) \\
&= \ \delta_{m(k)} + G(gx_{n(k)}, gx_{n(k)}, gx_{n(k)+1}) + G(gx_{n(k)+1}, gx_{n(k)+1}, gx_{m(k)+1}) \\
&\quad +G(gy_{n(k)}, gy_{n(k)}, gy_{n(k)+1}) + G(gy_{n(k)+1}, gy_{n(k)+1}, gy_{m(k)+1}).
\end{aligned}$$

Using that $G(x, x, y) \leq 2G(x, yy)$, for any $x, y \in X$, in the above inequality, we have,

$$\begin{aligned}
r_k \ &\leq \ \delta_{m(k)} + 2G(gx_{n(k)}, gx_{n(k)+1}, gx_{n(k)+1}) + G(gx_{n(k)+1}, gx_{n(k)+1}, gx_{m(k)+1}) \\
&\quad +2G(gy_{n(k)}, gy_{n(k)+1}, gy_{n(k)+1}) + G(gy_{n(k)+1}, gy_{n(k)+1}, gy_{m(k)+1}) \\
&= \ \delta_{m(k)} + 2\delta_{n(k)} + G(gx_{n(k)+1}, gx_{n(k)+1}, gx_{m(k)+1}) \\
&\quad +G(gy_{n(k)+1}, gy_{n(k)+1}, gy_{m(k)+1}).
\end{aligned}$$

Using the property of ϕ, we have

$$\begin{aligned}\phi(r_k) &\leq \phi(2\delta_{n(k)} + \delta_{m(k)} + G(gx_{n(k)+1}, gx_{n(k)+1}, gx_{m(k)+1}) \\ &\quad + G(gy_{n(k)+1}, gy_{n(k)+1}, gy_{m(k)+1})) \\ &\leq \phi(2\delta_{n(k)} + \delta_{m(k)}) + \phi(G(gx_{n(k)+1}, gx_{n(k)+1}, gx_{m(k)+1}) \\ &\quad + G(gy_{n(k)+1}, gy_{n(k)+1}, gy_{m(k)+1})) \\ &\leq \phi(2\delta_{n(k)} + \delta_{m(k)}) + \phi(G(gx_{n(k)+1}, gx_{n(k)+1}, gx_{m(k)+1})) \\ &\quad + \phi(G(gy_{n(k)+1}, gy_{n(k)+1}, gy_{m(k)+1})). \end{aligned} \tag{3.18}$$

Since $n(k) > m(k)$, $gx_{n(k)} \succeq gx_{m(k)}$ and $gy_{n(k)} \preceq gy_{m(k)}$, from (3.1) and (3.2), we have

$$\begin{aligned} & \phi(G(gx_{n(k)+1}, gx_{n(k)+1}, gx_{m(k)+1})) \\ = \; & \phi(G(F(x_{n(k)}, y_{n(k)}), F(x_{n(k)}, y_{n(k)}), F(x_{m(k)}, y_{m(k)}))) \\ \leq \; & \frac{1}{2}\phi(G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) + G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)})) \\ & -\psi\left(\frac{G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) + G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)})}{2}\right) \\ = \; & \frac{1}{2}\phi(r_k) - \psi\left(\frac{r_k}{2}\right). \end{aligned} \tag{3.19}$$

Similarly, we have

$$\begin{aligned} & \phi(G(gy_{n(k)+1}, gy_{n(k)+1}, gy_{m(k)+1})) \\ = \; & \phi(G(F(y_{m(k)}, x_{m(k)}), F(y_{m(k)}, x_{m(k)}), F(y_{n(k)}, x_{n(k)}))) \\ \leq \; & \frac{1}{2}\phi(G(gy_{m(k)}, gy_{m(k)}, gy_{n(k)}) + G(gx_{m(k)}, gx_{m(k)}, gx_{n(k)})) \\ & -\psi\left(\frac{G(gy_{m(k)}, gy_{m(k)}, gy_{n(k)}) + G(gx_{m(k)}, gx_{m(k)}, gx_{n(k)})}{2}\right) \\ = \; & \frac{1}{2}\phi(r_k) - \psi\left(\frac{r_k}{2}\right). \end{aligned} \tag{3.20}$$

From (3.18)-(3.20), we have

$$\phi(r_k) \leq \phi(2\delta_{n(k)} + \delta_{m(k)}) + \phi(r_k) - 2\psi\left(\frac{r_k}{2}\right).$$

Letting $k \to \infty$ and using (3.14) and (3.17), we have

$$\phi(\epsilon) \leq \phi(0) + \phi(\epsilon) - 2\psi\left(\frac{\epsilon}{2}\right) < \phi(\epsilon),$$

which is a contradiction. This implies that $\{gx_n\}$ and $\{gy_n\}$ are G-Cauchy sequences in the G-metric space (X, G).

Now, since (X, G) is a G-complete, there is $(x, y) \in X \times X$ such that $\{gx_n\}$ and $\{gy_n\}$ are respectively G-convergent to x and y, that is from Proposition 2.1, we have

$$\lim G(gx_n, gx_n, x) = \lim G(gx_n, x, x) = 0$$

and

$$\lim G(gy_n, gy_n, y) = \lim G(gy_n, y, y) = 0.$$

Using continuity of g, we get from Proposition 2.3,

$$\lim G(g(gx_n), g(gx_n), gx) = \lim G(g(gx_n), gx, gx) = 0$$

and

$$\lim G(g(gy_n), g(gy_n), gy) = \lim G(g(gy_n), gy, gy) = 0.$$

Since $gx_{n+1} = F(x, y)$ and $gy_{n+1}F(y, x)$, the commutativity of F and g yields that $F(gx_n, gy_n) = gF(x_n, y_n) = g(gx_{n+1})$ and $F(gy_n, gx_n) = gF(y_n, x_n) = g(gy_{n+1})$.

Now we show that $F(x, y) = gx$ and $F(y, x) = gy$.

The mapping F is continuous, so since the sequences $\{gx_n\}$ and $\{gy_n\}$ are respectively G-convergent to x and y, hence using Definition 2.6, the sequence $\{F(gx_n, gy_n)\}$ is G-convergent to $F(x, y)$. Therefore, $\{g(gx_{n+1})\}$ is G-convergent to $F(x, y)$. By uniqueness of the limit, we have $F(x, y) = gx$. Similarly, we can show that $F(y, x) = gy$. Hence, (x, y) is a coupled coincidence point of F and g. □

In the next theorem, we replace the continuity of F with the topic in the following definition.

Definition 3.1. *Let $(X, \preceq)$ be a partially ordered set and G be a G-metric on X. We say that $(X, G, \preceq)$ is regular if the following conditions hold:*

(i) if a non-decreasing sequence $\{x_n\}$ is such that $x_n \to x$, then $x_n \preceq x$ for all n,

(ii) if a non-increasing sequence $\{y_n\}$ is such that $y_n \to y$, then $y \preceq y_n$ for all n.

Theorem 3.2. *Let $(X, \preceq)$ be a partially ordered set and G be a G-metric on X such that $(X, G, \preceq)$ is regular. Suppose that $F\colon X \times X \to X$ and $g\colon X \to X$ are self mappings on X such that F has the mixed g-monotone property on X such that there exists two elements $x_0, y_0 \in X$ with $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$. Suppose that there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that* (3.1) *satisfies for all $x, y, u, v, w, z \in X$ with $gx \succeq gu \succeq gw$ and $gy \preceq gv \preceq gz$. Further suppose that $F(X \times X) \subseteq g(X)$ and $(g(X), G)$ is a complete G-metric. Then there exists $x, y \in X$ such that $F(x, y) = g(x)$ and $gy = F(y, x)$, that is, F and g have a coupled coincidence point $(x, y) \in X \times X$.*

Proof. Following the proof of Theorem 3.1, we will get two G-Cauchy sequences $\{gx_n\}$ and $\{gy_n\}$ in the complete G-metric space $(g(X), G)$. Then, there exist x, y in X such that $gx_n \to gx$ and $gy_n \to gy$. Since $\{gx_n\}$ is non-decreasing and $\{gy_n\}$ is non-increasing, using the regularity of $(X, G, \preceq)$, we have $gx_n \preceq gx$ and $gy \preceq gy_n$ for all $n \geq 0$. If $gx_n = gx$ and $gy_n = gy$ for some $n \geq 0$, then $gx = gx_n \preceq gx_{n+1} \preceq gx = gx_n$ and $gy \preceq gy_{n+1} \preceq gy_n = gy$, which implies that $gx_n = gx_{n+1} = F(x_n, y_n)$ and $gy_n = gy_{n+1} = F(y_n, x_n)$, that is, (x_n, y_n) is a coupled coincidence points of F and g. Then, we suppose that $(gx_n, gy_n) \neq (gx, gy)$ for all $n \geq 0$. Using rectangle inequality (3.1) and properties of ϕ and ψ, we have

$$\phi(G(F(x, y), gx_{n+1}, gx_{n+1})) = \phi(G(F(x, y), F(x_n, y_n), F(x_n, y_n)))$$
$$\leq \frac{1}{2}\phi(G(gx, gx_n, gx_n) + G(gy, gy_n, gy_n)) - \psi\Big(\frac{G(gx, gx_n, gx_n) + G(gy, gy_n, gy_n)}{2}\Big).$$

Taking $n \to \infty$, we get $G(F(x, y), gx, gx) = 0$ and hence $gx = F(x, y)$. Similarly, we can show that $gy = F(y, x)$. Thus F and g have a coupled coincidence point. □

If $\phi(t) = t$, in the above result we have the following result.

Corollary 3.1. *Let $(X, \preceq)$ be a partially ordered set and G be a G-metric on X such that (X, G) is a complete G-metric space. Suppose that $F\colon X \times X \to X$ and $g\colon X \to X$ are continuous self mappings on X such that F has the mixed g-monotone property on X such that there exists two elements $x_0, y_0 \in X$ with $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$. Suppose that there exists $\psi \in \Psi$ such that*

$$G(F(x,y), F(u,v), F(w,z)) \leq \frac{1}{2}[G(gx, gu, gw) + G(gy, gv, gz)] - \psi\left(\frac{G(gx, gu, gw) + G(gy, gv, gz)}{2}\right) \tag{3.21}$$

for all $x, y, u, v, w, z \in X$ with $gx \succeq gu \succeq gw$ and $gy \preceq gv \preceq gz$. Further suppose that $F(X \times X) \subseteq g(X)$ and g commutes with F. Then there exists $x, y \in X$ such that $F(x, y) = gx$ and $gy = F(y, x)$, that is, F and g have a coupled coincidence point $(x, y) \in X \times X$.

Corollary 3.2. *Let $(X, \preceq)$ be a partially ordered set and G be a G-metric on X such that $(X, G, \preceq)$ is regular. Suppose that $F\colon X \times X \to X$ and $g\colon X \to X$ are self mappings on X such that F has the mixed g-monotone property on X such that there exists two elements $x_0, y_0 \in X$ with $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$. Suppose that there exists $\psi \in \Psi$ such that (3.21) satisfies for all $x, y, u, v, w, z \in X$ with $gx \succeq gu \succeq gw$ and $gy \preceq gv \preceq gz$. Further suppose that $F(X \times X) \subseteq g(X)$ and $(g(X), G)$ is a complete G-metric. Then there exists $x, y \in X$ such that $F(x, y) = g(x)$ and $gy = F(y, x)$.*

If $\psi(t) = \frac{(1-k)t}{2}$, in the the above corollaries we have the following results.

Corollary 3.3. *Let $(X, \preceq)$ be a partially ordered set and G be a G-metric on X such that (X, G) is a complete G-metric space. Suppose that $F\colon X \times X \to X$ and $g\colon X \to X$ are continuous self mappings on X such that F has the mixed g-monotone property on X such that there exists two elements $x_0, y_0 \in X$ with $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$. Suppose that there exists $k \in [0, 1)$ such that*

$$G(F(x,y), F(u,v), F(w,z)) \leq \frac{k}{2}[G(gx, gu, gw) + G(gy, gv, gz)], \tag{3.22}$$

for all $x, y, u, v, w, z \in X$ with $gx \succeq gu \succeq gw$ and $gy \preceq gv \preceq gz$. Further suppose that $F(X \times X) \subseteq g(X)$ and g commutes with F. Then there exists $x, y \in X$ such that $F(x, y) = gx$ and $gy = F(y, x)$, that is, F and g have a coupled coincidence point $(x, y) \in X \times X$.

Corollary 3.4. *Let $(X, \preceq)$ be a partially ordered set and G be a G-metric on X such that $(X, G, \preceq)$ is regular. Suppose that $F\colon X \times X \to X$ and $g\colon X \to X$ are self mappings on X such that F has the mixed g-monotone property on X such that there exists two elements $x_0, y_0 \in X$ with $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$. Suppose that there exists $k \in [0, 1)$ such that (3.22) satisfies for all $x, y, u, v, w, z \in X$ with $gx \succeq gu \succeq gw$ and $gy \preceq gv \preceq gz$. Further suppose that $F(X \times X) \subseteq g(X)$ and $(g(X), G)$ is a complete G-metric. Then there exists $x, y \in X$ such that $F(x, y) = g(x)$ and $gy = F(y, x)$.*

Now, we shall prove the existence and uniqueness of a coupled common fixed point.

For this purpose, if $(X, \preceq)$ is a partially ordered set, then we endow the product space $X \times X$ with the following partial order relation:

$$\text{for } (x, y), (u, v) \in X \times X, \quad (u, v) \preceq (x, y) \Leftrightarrow x \preceq u, y \succeq v.$$

Theorem 3.3. *In addition to hypotheses of Theorem 3.1, suppose that for every (x, y) and (z, t) in $X \times X$, there exists $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable to $(F(x, y), F(y, x))$ and $(F(z, t), F(t, z))$. Then F and g have a unique coupled common fixed point, that is, there exists a unique $(x, y) \in X \times X$ such that $x = gx = F(x, y)$ and $y = gy = F(y, x)$.*

Proof. From Theorem 3.1, the set of coupled coincidence points of F and g is non-empty. Suppose that (x, y) and (z, t) are coupled coincidence points of F and g, that is, $gx = F(x, y)$, $gy = F(y, x)$, $gz = F(z, t)$ and $gt = F(t, z)$. We shall show that $gx = gz$ and $gy = gt$. By the assumption, there exists $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable with $(F(x, y), F(y, x))$ and $(F(z, t), F(t, z))$. Put $u_0 = u$, $v_0 = v$ and choose $u_1, v_1 \in X$ so that $gu_1 = F(u_0, v_0)$ and $gv_1 = F(v_0, u_0)$. Then, similarly as in the proof of Theorem 3.1, we can inductively define sequences $\{gu_n\}$, $\{gv_n\}$ as $gu_{n+1} = F(u_n, v_n)$ and $gv_{n+1} = F(v_n, u_n)$ for all n. Further, set $x_0 = x$, $y_0 = y$, $z_0 = z$, $t_0 = t$ and on the same way define the sequences $\{gx_n\}$, $\{gy_n\}$, and $\{gz_n\}$, $\{gt_n\}$.

Since $(F(x, y), F(y, x)) = (gx_1, gy_1) = (gx, gy)$, $(F(u, v), F(v, u)) = (gu_1, gv_1)$ are comparable, then $gx \succeq gu_1$ and $gy \preceq gv_1$. Now, we shall show that (gx, gy) and (gu_n, gv_n) are comparable, that is, $gx \succeq gu_n$ and $gy \preceq gv_n$ for all n. Suppose that it holds for some $n \geq 0$, then by the mixed g-monotone property of F, we have $gu_{n+1} = F(u_n, v_n) \preceq F(x, y) = gx$ and $gv_{n+1} = F(v_n, u_n) \succeq F(y, x) = gy$. Hence $gx \succeq gu_n$ and $gy \preceq gv_n$ hold for all n. Thus from (3.1), we have

$$\begin{aligned} \phi(G(gx, gx, gu_{n+1})) &= \phi(G(F(x, y), F(x, y), F(u_n, v_n))) \\ &\leq \frac{1}{2}\phi(G(gx, gx, gu_n) + G(gy, gy, gv_n)) \\ &\quad -\psi\left(\frac{G(gx, gx, gu_n) + G(gy, gy, gv_n)}{2}\right) \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} \phi(G(gy, gy, gv_{n+1})) &= \phi(G(F(y, x), F(y, x), F(v_n, u_n))) \\ &\leq \frac{1}{2}\phi(G(gy, gy, gv_n) + G(gx, gx, gu_n)) \\ &\quad -\psi\left(\frac{G(gy, gy, gv_n) + G(gx, gx, gu_n)}{2}\right) \end{aligned} \tag{3.24}$$

From (3.23), (3.24) and properties of ϕ and ψ, we have

$$\begin{aligned} &\phi(G(gx, gx, gu_{n+1}) + G(gy, gy, gv_{n+1})) \leq \phi(G(gx, gx, gu_{n+1})) + \phi(G(gy, gy, gv_{n+1})) \\ &\quad \leq \phi(G(gx, gx, gu_n) + G(gy, gy, gv_n)) - 2\psi\left(\tfrac{G(gx, gx, gu_n) + G(gy, gy, gv_n)}{2}\right) \end{aligned} \tag{3.25}$$

which implies that $\phi(G(gx, gx, gu_{n+1}) + G(gy, gy, gv_{n+1})) \leq \phi(G(gx, gx, gu_n) + G(gy, gy, gv_n))$. Hence the sequence $\{\delta_n := G(gx, gx, gu_n) + G(gy, gy, gv_n)\}$ is non-negative and decreasing and so $\lim \delta_n = \delta$, for some $\delta \geq 0$.

We shall show that $\delta = 0$. On the contrary, assume that $\delta > 0$. From (3.25), taking $n \to \infty$, we obtain

$$\phi(\delta) \leq \phi(\delta) - 2 \lim_{n\to\infty} \psi\left(\frac{G(gx, gx, gu_n) + G(gy, gy, gv_n)}{2}\right) < \phi(\delta),$$

which is contradiction. Thus, $\lim G(gx, gx, gu_n) = 0 = \lim G(gy, gy, gv_n)$. Similarly, we can prove that $\lim G(gz, gz, gu_n) = 0 = \lim G(gt, gt, gv_n)$. Hence $gx = gz$ and $gy = gt$. Since $gx = F(x, y)$ and $gy = F(y, x)$, by the commutativity of F and g, we have

$$g(g(x)) = g(F(x, y)) = F(gx, gy), \text{ and } g(gy) = g(F(y, x)) = F(gy, gx). \tag{3.26}$$

Denote $gx = p$ and $gy = q$. Then $gp = F(p, q)$ and $gq = F(q, p)$. Thus (p, q) is a coupled coincidence point. Then from (3.26), with $z = p$ and $t = q$, it follows $gp = gx$ and $gq = gy$, that is, $gp = p$ and $gq = q$. Hence $p = gp = F(p, q)$ and $q = gq = F(q, p)$. Therefore, (p, q) is a coupled common fixed point of F and g.

To prove the uniqueness, assume that (r, s) is another coupled common fixed point. Then by (3.26), we have $r = gr = gp = p$ and $s = gs = gq = q$. Hence we get the result. □

Example 3.1. Let $X = [0, +\infty)$ be endowed with usual metric, and with usual order in $\mathbb{R}$. Consider the function

$$G\colon [0, +\infty)^3 \to [0, +\infty), \quad G(x, y, z) = \max\{|x - y|, |x - z|, |y - z|\}.$$

It is known that (X, G) is a G-metric space (see [22]). Define

$$g\colon X \to X, \quad g(x) = x^2; \quad F\colon X \times X \to X, \quad F(x, y) = \begin{cases} \frac{1}{4}(x^2 - y^2), & x \geq y \\ 0, & x < y. \end{cases}$$

Then it is clear that $(g(X), G)$ is complete, $F\colon X \times X \to X \subseteq g(X) = X$, and F has the g-monotone property. Moreover, taking $x_0 = y_0 = 0$, then $x_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq y_0$.

Consider $\phi(t) = t$ and $\psi(t) = \frac{t}{2}$ for all $t \geq 0$. Now, we verify inequality 3.1,

$$\begin{aligned}
&\frac{1}{2}\phi(G(gx, gu, gw) + G(gy, gv, gz)) - \psi\left(\frac{G(gx, gu, gw) + G(gy, gv, gz)}{2}\right) \\
= \;& \frac{1}{2}\phi(\max\{|x^2 - u^2|, |x^2 - w^2|, |u^2 - w^2|\} + \max\{|y^2 - v^2|, |y^2 - z^2|, |v^2 - z^2|\}) \\
& -\psi\Big(\frac{\max\{|x^2 - u^2|, |x^2 - w^2|, |u^2 - w^2|\} + \max\{|y^2 - v^2|, |y^2 - z^2|, |v^2 - z^2|\}}{2}\Big) \\
= \;& \frac{1}{2}[\max\{|x^2 - u^2|, |x^2 - w^2|, |u^2 - w^2|\} + \max\{|y^2 - v^2|, |y^2 - z^2|, |v^2 - z^2|\}] \\
& -\frac{1}{4}[\max\{|x^2 - u^2|, |x^2 - w^2|, |u^2 - w^2|\} + \max\{|y^2 - v^2|, |y^2 - z^2|, |v^2 - z^2|\}] \\
= \;& \frac{1}{4}[\max\{|x^2 - u^2|, |x^2 - w^2|, |u^2 - w^2|\} + \max\{|y^2 - v^2|, |y^2 - z^2|, |v^2 - z^2|\}].
\end{aligned}$$

By definition of g, we shall prove that (3.1) holds for all $x, y, u, v, w, z \in X$ with $x \leq u \leq w$ and $z \leq v \leq y$. For this, we distinguish the following cases:

Case 1. If $x < y$, $u < v$, and the case when $w \leq z$, (3.1) is obvious, so let $w \geq z$. It follows

$$G(F(x,y),F(u,v),F(w,z)) = G\Big(0,0,\frac{|w^2-z^2|}{4}\Big) = \frac{|w^2-z^2|}{4}$$
$$\leq \frac{1}{4}[\max\{|x^2-u^2|,|x^2-w^2|,|u^2-w^2|\} + \max\{|y^2-v^2|,|y^2-z^2|,|v^2-z^2|\}].$$

Case 2. If $x < y$ and $u \geq v$, then $z \leq v \leq u \leq w$, so $w \geq z$. We have

$$G(F(x,y),F(u,v),F(w,z)) = G\Big(0,\frac{u^2-v^2}{4},\frac{w^2-z^2}{4}\Big) = \frac{w^2-z^2}{4}$$
$$\leq \frac{1}{4}[\max\{|x^2-u^2|,|x^2-w^2|,|u^2-w^2|\} + \max\{|y^2-v^2|,|y^2-z^2|,|v^2-z^2|\}].$$

Case 3. If $x \geq y$, in this case, we have $z \leq v \leq y \leq x \leq u \leq w$. So, we obtain

$$G(F(x,y),F(u,v),F(w,z)) = G\Big(\frac{x^2-y^2}{4},\frac{u^2-v^2}{4},\frac{w^2-z^2}{4}\Big)$$
$$= \max\Big\{\Big|\frac{x^2-y^2}{4}-\frac{u^2-v^2}{4}\Big|,\Big|\frac{x^2-y^2}{4}-\frac{w^2-z^2}{4}\Big|,\Big|\frac{u^2-v^2}{4}-\frac{w^2-z^2}{4}\Big|\Big\}$$
$$= \frac{x^2-y^2}{4}-\frac{w^2-z^2}{4}$$
$$\leq \frac{1}{4}[\max\{|x^2-u^2|,|x^2-w^2|,|u^2-w^2|\} + \max\{|y^2-v^2|,|y^2-z^2|,|v^2-z^2|\}].$$

In all the cases, inequality (3.1) of Theorem 3.1 is satisfied. Hence by Theorem 3.1, $(0,0)$ is coupled coincidence point. Indeed, for $x > y$, $F(y,x) = 0$ and since $F(y,x) = g(y)$, we have $g(y) = 0$. Then $F(x,0) = g(x)$ implies $x = 0$.

Other consequences of our results are the given in the following, for mappings involving contractions of integral type.

Denote by Λ the set of functions $\mu\colon [0,\infty) \to [0,\infty)$ satisfying the following hypotheses:

(h1) μ is a Lebesgue-integrable mapping on each compact of $[0,\infty)$;
(h2) for any $\epsilon > 0$, we have $\int_0^\epsilon \mu(t) > 0$.

Corollary 3.5. *Let $(X,\preceq)$ be a partially ordered set and G be a G-metric on X such that (X,G) is a complete G-metric space. Suppose that $F\colon X\times X \to X$ and $g\colon X \to X$ are continuous self mappings on X such that F has the mixed g-monotone property on X such that there exists two elements $x_0, y_0 \in X$ with $g(x_0) \preceq F(x_0,y_0)$ and $g(y_0) \succeq F(y_0,x_0)$. Suppose that*

$$\int_0^{G(F(x,y),F(u,v),F(w,z))} \alpha(s)ds \leq \int_0^{\frac{1}{2}[G(gx,gu,gw)+G(gy,gv,gz)]} \alpha(s)ds - \int_0^{[\frac{G(gx,gu,gw)+G(gy,gv,gz)}{2}]} \beta(s)ds,$$

hold for all $x,y,u,v,w,z \in X$ with $gx \succeq gu \succeq gw$ and $gy \preceq gv \preceq gz$, where $\alpha,\beta \in \Lambda$. Further suppose that $F(X\times X) \subseteq g(X)$ and g commutes with F. Then there exists $x,y \in X$ such that $F(x,y) = gx$ and $gy = F(y,x)$, that is, F and g have a coupled coincidence point $(x,y) \in X\times X$.

Corollary 3.6. *Let $(X, \preceq)$ be a partially ordered set and G be a G-metric on X such that (X, G) is a complete G-metric space. Suppose that $F\colon X \times X \to X$ and $g\colon X \to X$ are continuous self mappings on X such that F has the mixed g-monotone property on X such that there exists two elements $x_0, y_0 \in X$ with $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$. Suppose that there exists $k \in [0, 1)$ such that*

$$\int_0^{G(F(x,y),F(u,v),F(w,z))} \alpha(s)ds \leq k \int_0^{[G(gx,gu,gw)+G(gy,gv,gz)]} \alpha(s)ds$$

for all $x, y, u, v, w, z \in X$ with $gx \succeq gu \succeq gw$ and $gy \preceq gv \preceq gz$, where $\alpha \in \Lambda$. Further suppose that $F(X \times X) \subseteq g(X)$ and g commutes with F. Then there exists $x, y \in X$ such that $F(x, y) = gx$ and $gy = F(y, x)$, that is, F and g have a coupled coincidence point $(x, y) \in X \times X$.

REFERENCES

[1] M. Abbas, T. Nazir, W. Shatanawi and Z. Mustafa, Fixed and related fixed point theorems for three maps in G-metric spaces, accepted in Hacet. J. Math. Stat.

[2] H. Aydi, M. Postolache and W. Shatanawi, Coupled fixed point results for (ψ, ϕ)-weakly contractive mappings in ordered G-metric spaces, Comput. Math. Appl. 63(2012), No. 1, 298-309.

[3] H. Aydi, E. Karapınar and M. Postolache: Tripled coincidence point theorems for weak φ-contractions in partially ordered metric spaces, Fixed Point Theory Appl., ID: 2012:44.

[4] H. Aydi, W. Shatanawi, M. Postolache, Z. Mustafa and N. Tahat, Theorems for Boyd-Wong type contractions in ordered metric spaces, Abstr. Appl. Anal. 2012, ID: 359054, 13 pp.

[5] H. Aydi, E. Karapinar and Z. Mustafa, Mixed g-monotone property and tripled fixed point theorems in partially ordered metric spaces, Tamkang J. Math., accepted, 2012.

[6] T.G. Bhaskar and V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65(2006), 1379-1393.

[7] S. Chandok, Some common fixed point theorems for generalized f-weakly contractive mappings, J. Appl. Math. Informatics 29(2011), 257-265.

[8] S. Chandok, Some common fixed point theorems for generalized nonlinear contractive mappings, Comput. Math. Appl. 62(2011), 3692-3699.

[9] S. Chandok, Common fixed points, invariant approximation and generalized weak contractions, Internat. J. Math. Math. Sci. 2012, ID: 102980.

[10] S. Chandok and J.K. Kim, Fixed point theorem in ordered metric spaces for generalized contractions mappings satisfying rational type expressions, to appear in J. Nonlinear Funct. Anal. Appl. (2012)

[11] S. Chandok, Some common fixed point results for generalized weak contractive mappings in partially ordered metric spaces, to appear in J. Nonlinear Anal. Optimization.

[12] S. Chandok, M.S. Khan and K.P.R. Rao, Some coupled common fixed point theorems for a pair of mappings satisfying a contractive condition of rational type, to appear in J. Nonlinear Anal. Appl.

[13] S. Chandok and E. Karapinar, Some common fixed point results for generalized rational type weak contraction mappings in partially ordered metric spaces, to appear in Thai J. Math.

[14] S. Chandok and M. Postolache, Fixed point theorem for weekly Chatterjea-type contractions, Fixed Point Theory Appl., 2013, ID: 2013:28, 9 p

[15] B.S. Choudhury and P. Maity, Coupled fixed point results in generalized metric spaces, Math. Comput. Modelling 54(2011), 73-79.

[16] B.S. Choudhury, N. Metiya and M. Postolache, A generalized weak contraction principle with applications to coupled coincidence point problems, Fixed Point Theory Appl. 2013, ID: 2013:152, 21 p

[17] R.H. Haghi, M. Postolache and Sh. Rezapour, On T-stability of the Picard iteration for generalized φ-contraction mappings, Abstr. Appl. Anal. 2012, ID: 658971, 7 pp.

[18] J. Harjani, B. López and K. Sadarangani, Fixed point theorems for mixed monotone operators and applications to intgeral equations, Nonlinear Anal. 74(2011), 1749-1760.

[19] E. Karapinar, W. Shatanawi and Z. Mustafa, Quadruple fixed point theorems under nonlinear contractive conditions in partially ordered metric spaces, J. Appl. Math. 2012, ID: 951912.
[20] V. Lakshmikantham and L. Ćirić, Coupled fixed point theorems for nonlinear contractions in partially orderd metric spaces, Nonlinear Anl. 70(2009), 4341-4349.
[21] Z. Mustafa and B. Sims, Some remarks concerning D-metric spaces, in: Proceedings of the Int. Conf. on Fixed Point Theory and Applications, Yokohama, Japan, 2004, pp. 189-198.
[22] Z. Mustafa and B. Sims, A new approach to generalized metric spaces, J. Nonlinear Convex Anal. 7(2006), No. 2, 289-297.
[23] Z. Mustafa, H. Obiedat, F. Awawdeh, Some of fixed point theorem for mapping on complete G-metric spaces, Fixed Point Theory Appl. 2008, ID: 189870, 12 pp.
[24] Z. Mustafa and B. Sims, Fixed point theorems for contractive mappings in complete G-metric spaces, Fixed Point Theory Appl. 2009, ID: 917175, 10 pp.
[25] Z. Mustafa, W. Shatanawi and M. Bataineh, Existence of fixed point results in G-metric spaces, Int. J. Math. Math. Sci. 2009, ID: 283028, 10 pp.
[26] Z. Mustafa, M. Khandagjy and W. Shatanawi, Fixed point results on complete G-metric spaces, Stud. Sci. Math. Hung. 48(2011), No. 3, 304-319.
[27] Z. Mustafa, F. Awawdeh, W. Shatanawi, Fixed point theorem for expansive mappings in G-metric spaces, Int. J. Contemp. Math. Sci. 5(2010), No. 49-52.
[28] Z. Mustafa, H. Aydi and E. Karapinar, On common fixed points in G-metric spaces using (E.A) property, Comput. Math. Appl. 64(2012) 1944-1956.
[29] Z. Mustafa, H. Aydi and E. Karapinar, Mixed g-monotone property and quadruple fixed point theorems in partial ordered G-metric space, Fixed Point Theory Appl. 2012, 2012:71.
[30] Z. Mustafa, Common fixed points of weakly compatible mappings in G-metric spaces, Appl. Math. Sci. 6(2012), No. 92, 4589-4600.
[31] Z. Mustafa, Some new common fixed point theorems under strict contractive conditions in G-metric spaces, J. Appl. Math. 2012, ID: 248937, 21 pp.
[32] J.J. Nieto and R.R. López, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, Order 22(2005), 223-239.
[33] M.O. Olatinwo and M. Postolache, Stability results for Jungck-type iterative processes in convex metric spaces, Appl. Math. Comput. 218(2012), No. 12, 6727-6732.
[34] A.C.M. Ran and M.C.B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc. 132(5)(2004), 1435-1443.
[35] K.P.R. Rao, K. Bhanu Lakshmi and Z. Mustafa, Fixed and related fixed point theorems for three maps in G-metric space, J. Adv. Stud. Topology, Accepted, 2012.
[36] W. Shatanawi and A. Pitea, Omega-distance and coupled fixed point in G-metric spaces, Fixed Point Theory Appl. 2013, Article ID: 2013:208, 15 p.
[37] W. Shatanawi and A. Pitea, Fixed and coupled fixed point theorems of Omega-distance for nonlinear contraction, Fixed Point Theory Appl. 2013, ID: 2013:xxx (editorially accepted).
[38] W. Shatanawi and A. Pitea, Some coupled fixed point theorems in quasi-partial metric spaces, Fixed Point Theory Appl. 2013, ID: 2013:153.
[39] W. Shatanawi and M. Postolache, Some fixed point results for a G-weak contraction in G-metric spaces, Abstr. Appl. Anal. 2012, ID: 815870, 19 pp.
[40] W. Shatanawi and M. Postolache, Coincidence and fixed point results for generalized weak contractions in the sense of Berinde on partial metric spaces, Fixed Point Theory Appl. Volume 2013, ID: 2013:54, 17 pp.
[41] W. Shatanawi and M. Postolache, Common fixed point results of mappings for nonlinear contractions of cyclic form in ordered metric spaces, Fixed Point Theory Appl. Volume 2013, ID: 2013:60, 13 pp.
[42] W. Shatanawi, M. Postolache, Z. Mustafa and N. Tahat, Theorems for Boyd-Wong type contractions in ordered metric spaces, Abstr. Appl. Anal. 2012, ID: 359054, 14 pp.
[43] W. Shatanawi, S. Chauhan, M. Postolache, M. Abbas and S. Radenović, Common fixed points for contractive mappings of integral type in G-metric spaces, J. Adv. Math. Stud. 6(2013), No. 1, 53-72.
[44] W. Shatanawi, M. Postolache and Z. Mustafa, Tripled and coincidence fixed point theorems for contractive mappings satisfying Φ-maps in partially ordered metric spaces, submitted.