SOLUTION OF A FRACTIONAL ORDER INTEGRAL EQUATION VIA FIXED POINT THEOREM IN PSEUDOMODULAR METRIC SPACE

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An existence theorem for a class of fractional order integral equations is established in pseudomodular metric spaces. For this purpose, we first introduce the notion of modular gauge space and prove some fixed point theorems on this setting. We also construct an example to support our result.

Keywords: Modular gauge space, fixed point, nonlinear fractional order integral equation.

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1. Introduction

As we know, the theory of both differential and integral equations is based on nonlinear analysis. If we have in mind existence and uniqueness theorems for the solutions of these classes of equations, they can be obtained by means of fixed point theory; for instance, see Ali *et al.* [1]. In the literature of fixed point theory we find several generalizations of Banach contraction principle, either made by introducing weaker contraction conditions or by using suitable structure which are more general than the metric space; Ciric [2], Suzuki [3], Choudhury *et al.* [4], Chandok and Postolache [5], Shatanawi *et al.* [6, 7, 8]. Also, an active research direction in this regard is the designing of numerical algorithms for specific problems; Thakur *et al.* [9, 10, 11], Yao *et al.* [12, 13, 14, 15]. Frigon [16] generalized the Banach contraction principle on gauge spaces. Later on Agarwal *et al.* [17], Cherichi *et al.* [18], Cherichi and Samet [19], Chis and Precup [20], Chifu and Petrusel [21], Lazar and Petrusel [22], and Jleli *et al.* [23] generalized the results of Frigon [16]. Jachymaski [24] weakened the Banach contraction condition by introducing the notion of Banach *G*-contraction and proved some fixed point theorems for such mappings on complete metric spaces endowed with graph. Afterwards, many authors extended Banach *G*-contraction in single as well as multivalued case, see for example: Kamran *et al.* [25, 26], Bojor [27, 28], Nicolae *et al.* [29], Aleomraninejad *et al.* [30], Asl *et al.* [31]. The notion of modular metric space was introduced by Chistyakov [32]. As in metric space, we know that the distance

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between two elements must be a finite positive real number but in modular metric spaces it may be infinite, further it depends upon the parameter λ (which is mentioned in the definition). Modular metric spaces is very useful generalization of metric spaces because the concept of modulars on linear spaces has been used in many nonlinear problems; for details, please see Abdou and Khamsi [33]. Many authors appreciated the work of Chistyakov [32] and proved fixed point theorems for different type of contractive mappings on modular metric space: Alfuraidan [34], Chaipunya *et al.* [35], Chistyakov [36, 37], Khamsi and Kozlowski [38].

The purpose of this paper is to introduce the notion of modular gauge space by using pseudomodular metric spaces and prove some fixed point theorems in this new space. We support our result by example. An existence theorem for a class of fractional order integral equations is also established in this new setting.

2. Preliminaries

Now, we recollect some basic notions and definitions which are required in next section.

Definition 2.1 ([32]). A function $\omega \colon (0, \infty) \times X \times X \to [0, \infty]$ is known as a modular metric on X if the following axioms hold:

(i) $\omega(\lambda, x, y) = 0 \ \forall \lambda > 0$ if and only if $x = y$;

(ii) for each $x, y \in X$, $\omega(\lambda, x, y) = \omega(\lambda, y, x) \ \forall \lambda > 0$;

(iii) for each $x, y, z \in X$, $\omega(\lambda + \mu, x, z) \leq \omega(\lambda, x, y) + \omega(\mu, y, z) \ \forall \lambda, \mu > 0$.

A modular metric on X is regular if the following weaker form of (i) is satisfied:

$$x = y \text{ if and only if } \omega(\lambda, x, y) = 0 \text{ for some } \lambda > 0.$$

If we replace (i) with

$$(i') \text{ for each } x \in X, \ \omega(\lambda, x, x) = 0 \ \forall \lambda > 0$$

then ω is known as a pseudomodular metric on X.

Note that the function $\lambda \to \omega(\lambda, x, y)$ is nonincreasing for each $\lambda > 0$ and $x, y \in X$. If $0 < \mu < \lambda$, then by using the triangle property we have

$$\omega(\lambda, x, z) \leq \omega(\lambda - \mu, x, x) + \omega(\mu, x, z) = \omega(\mu, x, z).$$

Also, when ω is a pseudomodular metric on X and $x_0 \in X$ is a fixed element, then the sets

$$X_\omega = X_\omega(x_0) = \{x \in X : \omega(\lambda, x_0, x) \to 0 \text{ as } \lambda \to \infty\}$$

and

$$X_\omega^* = X_\omega^*(x_0) = \{x \in X : \exists \lambda = \lambda(x) > 0 \text{ such that } \omega(\lambda, x_0, x) < \infty\}$$

are modular spaces (around x_0).

Definition 2.2 ([38]). Let (X, ω) be a pseudomodular metric space. Then ω is said to satisfy:

(i) Δ_2-condition, $\lim_{n \to \infty} \omega(\lambda, x_n, x) = 0$ for some $\lambda > 0$ implies $\lim_{n \to \infty} \omega(\lambda, x_n, x) = 0$ for all $\lambda > 0$.

(ii) Fatou property if for any $\{x_n\}$ ω-convergent to x and $\{y_n\}$ ω-convergent to y, then $\omega(1, x, y) \leq \liminf_{n \to \infty} \omega(1, x_n, y_n)$.

For brief study of modular metric spaces and fixed point theorems on it, we refer the reader to Abdou and Khamsi [33], Alfuraidan [34], Chaipunya *et al.* [35], Chistyakov [36, 37], Khamsi and Kozlowski [38]. It is important to note that contraction conditions on modular metric spaces make sense on bounded subsets or one needs to define a condition that the distance between any two elements must be finite.

3. Main results

In order to define the modular gauge space by using pseudomodular metrics we need to introduce our setting.

Definition 3.1. Let ω be a pseudomodular metric on X. The w-ball of radius $\mu > 0$ centered at $x \in X$ is the set

$$B[x, \omega, \mu] = \{y \in X : \forall \lambda > 0 \; \omega(\lambda, x, y) < \mu\}.$$

Example 3.1. Let $X = \mathbb{R}$ be endowed with the pseudomodular metric $\omega(\lambda, x, y) = \frac{|x-y|}{\lambda}$ for each $x, y \in X$ and $\lambda > 0$. Then

$$B[x_0, \omega, 1] = \{y \in X : \forall \lambda > 0, \; |x_0 - y| < \lambda\} = \{x_0\}.$$

Example 3.2. Let $X = \mathbb{R}$ be endowed with the pseudomodular metric $\omega(\lambda, x, y) = \frac{|x-y|}{\lceil \lambda \rceil}$ for each $x, y \in X$ and $\lambda > 0$. Then

$$B[x_0, \omega, 1] = \{y \in X : \forall \lambda > 0, \; |x_0 - y| < \lceil \lambda \rceil\} = \{y \in X : |x_0 - y| < 1\} = (-1 + x_0, 1 + x_0).$$

Definition 3.2. A family $\mathfrak{F} = \{\omega_\eta : \eta \in \mathfrak{A}\}$ of pseudomodular metrics is said to be separating if for each pair (x, y) with $x \neq y$, there exists $\omega_\eta \in \mathfrak{F}$ such that $\omega_\eta(\lambda, x, y) \neq 0 \; \forall \lambda > 0$.

Definition 3.3. Let X be a nonempty set and $\mathfrak{F} = \{\omega_\eta : \eta \in \mathfrak{A}\}$ be a family of pseudo-modular metrics on X. The topology $\mathfrak{T}(\mathfrak{F})$ having subbases the family

$$\mathfrak{B}(\mathfrak{F}) = \{B[x, \omega_\eta, \mu] : x \in X, \omega_\eta \in \mathfrak{F} \text{ and } \mu > 0\}$$

of balls is called modular topology induced by the family $\mathfrak{F}$ of pseudomodular metrics. The pair $(X, \mathfrak{T}(\mathfrak{F}))$ is called a modular gauge space.

Before going towards a next definition we define the following notion:

$$X_{\mathfrak{F}} = X_{\mathfrak{F}}(x_0) = \{x \in X : \forall \eta \in \mathfrak{A} \; \omega_\eta(\lambda, x_0, x) \to 0 \text{ as } \lambda \to \infty\}$$

where x_0 is fixed in X.

Definition 3.4. Let $(X, \mathfrak{T}(\mathfrak{F}))$ be a modular gauge space with respect to the family $\mathfrak{F} = \{\omega_\eta : \eta \in \mathfrak{A}\}$ of pseudomodular metrics on X and $\{x_n\}$ is a sequence in $X_{\mathfrak{F}}$ and $x \in X_{\mathfrak{F}}$. Then:

(i) the sequence $\{x_n\}$ ω-converges to x if for each $\eta \in \mathfrak{A}$ and $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that for some $\lambda > 0$, we have $\omega_\eta(\lambda, x_n, x) < \epsilon$ for each $n \geq N_0$. We denote it as $x_n \to^{\mathfrak{F}} x$;

(ii) the sequence $\{x_n\}$ is a ω-Cauchy sequence if for each $\eta \in \mathfrak{A}$ and $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that for some $\lambda > 0$, we have $\omega_\eta(\lambda, x_n, x_m) < \epsilon$ for each $n, m \geq N_0$;

(iii) $X_{\mathfrak{F}}$ is ω-complete if each ω-Cauchy sequence in $X_{\mathfrak{F}}$ is ω-convergent in $X_{\mathfrak{F}}$;

(iv) a subset of $X_{\mathfrak{F}}$ is said to be ω-closed if it contains the limit of each ω-convergent sequence of its elements.

(v) a subset of M of $X_{\mathfrak{F}}$ is said to be ω-bounded if we have

$$\delta_{\mathfrak{F}}(M) = \sup\{\omega_\eta(1, x, y) : x, y \in M, \eta \in \mathfrak{A}\} < \infty.$$

Note that if X is a nonempty set endowed with a separating modular gauge structure of pseudomodular metrics $\mathfrak{F} = \{\omega_\eta : \eta \in \mathfrak{A}\}$ and $\{x_n\}$ is ω-convergent in $X_{\mathfrak{F}}$, then $\{x_n\}$ ω-converges to unique limit point.

Suppose on contrary that $\{x_n\}$ ω-converges to $a, b \in X_{\mathfrak{F}}$. Then for each $\eta \in \mathfrak{A}$, there exist some $\lambda_1, \lambda_2 > 0$ such that $\lim_{n \to \infty} \omega_\eta(\lambda_1, x_n, a) = 0$ and $\lim_{n \to \infty} \omega_\eta(\lambda_2, x_n, b) = 0$. By the triangle property we have

$$\omega_\eta(\lambda_1 + \lambda_2, a, b) \leq \omega_\eta(\lambda_1, a, x_n) + \omega_\eta(\lambda_2, x_n, b) \text{ for each } n \in \mathbb{N} \text{ and } \eta \in \mathfrak{A}.$$

Letting $n \to \infty$ we get $\omega_\eta(\lambda_1 + \lambda_2, a, b) = 0$ for each $\eta \in \mathfrak{A}$. Since $\mathfrak{F} = \{\omega_\eta : \eta \in \mathfrak{A}\}$ is separating, we have $a = b$.

Subsequently, in this paper, $\mathfrak{A}$ is a directed set and X is a nonempty set endowed with a separating modular gauge structure of pseudomodular metrics $\mathfrak{F} = \{\omega_\eta : \eta \in \mathfrak{A}\}$ satisfying the Δ_2-condition and the Fatou property. Further, M is ω-complete and ω-bounded subset of $X_{\mathfrak{F}}$ under the modular gauge space $(X, \mathfrak{T}(\mathfrak{F}))$ induced by the $\mathfrak{F} = \{\omega_\eta : \eta \in \mathfrak{A}\}$. Furthermore, $G = (V, E)$ is a directed graph in $M \times M$, where the set of its vertices V is equal to M and the set of its edges E contains $\{(x, x) : x \in V\}$. Moreover, G has no parallel edges.

Theorem 3.1. *Let* $T \colon M \to M$ *be a mapping such that for each* $\eta \in \mathfrak{A}$, *we have*

$$\omega_\eta(1, Tx, Ty) \leq a_\eta \omega_\eta(1, x, y) + b_\eta \omega_\eta(1, x, Tx) + c_\eta \omega_\eta(1, y, Ty) + e_\eta \omega_\eta(2, x, Ty) + L_\eta \omega_\eta(1, y, Tx)$$
$$(1)$$

for all $(x, y) \in E$, *where* $a_\eta, b_\eta, c_\eta, e_\eta, L_\eta \geq 0$, *and* $a_\eta + b_\eta + c_\eta + 2e_\eta < 1 \; \forall \eta \in \mathfrak{A}$. *Further, assume that the following conditions hold:*

(i) there exists $x_0 \in M$ *such that* $(x_0, Tx_0) \in E$;

(ii) T is edge preserving, that is, if $(x, y) \in E$ *then* $(Tx, Ty) \in E$;

(iii) if $\{x_n\}$ *is a sequence in* M *such that* $(x_n, x_{n+1}) \in E$ *for each* $n \in \mathbb{N}$ *and* $x_n \to^{\mathfrak{F}} x$ *as* $n \to \infty$, *then* $(x_n, x) \in E$ *for each* $n \in \mathbb{N}$.

Then T has a fixed point.

Proof. By hypothesis (i), there exists $x_0 \in M$ such that $(x_0, Tx_0) \in E$. Since T is edge preserving we get $(Tx_0, T^2 x_0) \in E$. Continuing we get $(T^n x_0, T^{n+1} x_0) \in E$ for each $n \in \mathbb{N}$.

Define $x_n = Tx_{n-1} = T^n x_0$ for each $n \in \mathbb{N}$. From (1) we have

$$
\begin{aligned}
\omega_\eta(1, x_n, x_{n+1}) &= \omega_\eta(1, Tx_{n-1}, Tx_n) \\
&\leq a_\eta \omega_\eta(1, x_{n-1}, x_n) + b_\eta \omega_\eta(1, x_{n-1}, Tx_{n-1}) + c_\eta \omega_\eta(1, x_n, Tx_n) \\
&\quad + e_\eta \omega_\eta(2, x_{n-1}, Tx_n) + L_\eta \omega_\eta(1, x_n, Tx_{n-1}) \\
&= a_\eta \omega_\eta(1, x_{n-1}, x_n) + b_\eta \omega_\eta(1, x_{n-1}, x_n) + c_\eta \omega_\eta(1, x_n, x_{n+1}) \\
&\quad + e_\eta \omega_\eta(2, x_{n-1}, x_{n+1}) + L_\eta \omega_\eta(1, x_n, x_n) \\
&\leq (a_\eta + b_\eta + e_\eta)\omega_\eta(1, x_{n-1}, x_n) + (c_\eta + e_\eta)\omega_\eta(1, x_n, x_{n+1}) + L_\eta 0 \ \forall \ \eta \in \mathfrak{A}.
\end{aligned}
$$

After some simplification we get

$$\omega_\eta(1, x_n, x_{n+1}) \leq \xi_\eta \omega_\eta(1, x_{n-1}, x_n) \ \forall \ \eta \in \mathfrak{A}$$

where, $\xi_\eta = \frac{a_\eta + b_\eta + e_\eta}{1 - c_\eta - e_\eta} < 1$. Iteratively we get

$$\omega_\eta(1, x_n, x_{n+1}) \leq (\xi_\eta)^n \omega_\eta(1, x_0, x_1) \ \forall \ \eta \in \mathfrak{A} \text{ and } n \in \mathbb{N}.$$

Now we show that $\{x_n\}$ is ω-Cauchy sequence. For each $m, p \in \mathbb{N}$ and $\eta \in \mathfrak{A}$, we have

$$
\begin{aligned}
\omega_\eta(p, x_m, x_{m+p}) &\leq \sum_{i=m}^{m+p-1} \omega_\eta(1, x_i, x_{i+1}) \\
&\leq \sum_{i=m}^{m+p-1} (\xi_\eta)^i \omega_\eta(1, x_0, x_1) \\
&\leq \sum_{i=m}^{\infty} (\xi_\eta)^i \delta_{\mathfrak{F}}(M) \to 0 \text{ as } m \to \infty.
\end{aligned}
$$

This shows that $\{x_n\}$ is ω-Cauchy sequence in M. Since M is ω-complete, there exists $x^* \in M$ such that $\{x_n\}$ is ω-convergent to x^*, that is, for each $\eta \in \mathfrak{A}$ we have $\lim_{n \to \infty} \omega_\eta(\lambda, x_n, x^*) = 0$ for some $\lambda > 0$. Since $\mathfrak{F}$ satisfies Δ_2-condition, thus we have

$$\lim_{n \to \infty} \omega_\eta(\lambda, x_n, x^*) = 0 \text{ for all } \lambda > 0 \text{ and } \eta \in \mathfrak{A}.$$

By using hypothesis (iii), the triangular inequality and (1), we have

$$
\begin{aligned}
\omega_\eta(1, x_{m+1}, Tx^*) &= \omega_\eta(1, Tx_m, Tx^*) \\
&\leq a_\eta \omega_\eta(1, x_m, x^*) + b_\eta \omega_\eta(1, x_m, Tx_m) + c_\eta \omega_\eta(1, x^*, Tx^*) \\
&\quad + e_\eta \omega_\eta(2, x_m, Tx^*) + L_\eta \omega_\eta(1, x^*, Tx_m) \\
&\leq a_\eta \omega_\eta(1, x_m, x^*) + b_\eta \omega_\eta(1, x_m, x_{m+1}) + c_\eta \omega_\eta(1, x^*, Tx^*) \\
&\quad + e_\eta [\omega_\eta(1, x_m, x^*) + \omega_\eta(1, x^*, Tx^*)] + L_\eta \omega_\eta(1, x^*, x_{m+1}) \ \forall \ \eta \in \mathfrak{A}.
\end{aligned}
$$

Letting $m \to \infty$ and by using the Fatou property on the left side of the above inequality, we get

$$\omega_\eta(1, x^*, Tx^*) \leq (c_\eta + e_\eta)\omega_\eta(1, x^*, Tx^*) < \omega_\eta(1, x^*, Tx^*) \ \forall \ \eta \in \mathfrak{A},$$

which is not possible if $\omega_\eta(1, x^*, Tx^*) \neq 0$. Thus, $\omega_\eta(1, x^*, Tx^*) = 0 \ \forall \ \eta \in \mathfrak{A}$. As we known that the structure $\{\omega_\eta : \eta \in \mathfrak{A}\}$ on $X_{\mathfrak{F}}$ is separating, thus we conclude that $x^* = Tx^*$. $\square$

We denote by Ψ the family of nondecreasing functions, $\psi \colon [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$, where ψ^n is the nth iterate of ψ.

Note that in the following theorem it is not necessary that $\mathfrak{F}$ satisfies the Fatou property. Following theorem even holds without this condition.

Theorem 3.2. *Let $T \colon M \to M$ be a mapping such that for each $\eta \in \mathfrak{A}$, we have*

$$\omega_\eta(1, Tx, Ty) + \varphi(Tx) + \varphi(Ty) \leqslant \psi(\omega_\eta(1, x, y) + \varphi(x) + \varphi(y)) \text{ for each } (x, y) \in E \quad (2)$$

where $\psi \in \Psi$ and $\varphi : M \to [0, \infty)$ be a lower semi-continuous function. Further, assume that the following conditions hold:

(i) there exists $x_0 \in M$ such that $(x_0, Tx_0) \in E$;

(ii) T is edge preserving, that is, if $(x, y) \in E$, then $(Tx, Ty) \in E$;

(iii) if $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}) \in E$ for each $n \in \mathbb{N}$ and $x_n \to^{\mathfrak{F}} x$ as $n \to \infty$, then $(x_n, x) \in E$ for each $n \in \mathbb{N}$.

Then T has a fixed point.

Proof. By hypothesis (i), there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E$. Since T is edge preserving we get $(Tx_0, T^2x_0) \in E$. Continuing we get $(T^nx_0, T^{n+1}x_0) \in E$ for each $n \in \mathbb{N}$. Define $x_n = Tx_{n-1} = T^nx_0$, for each $n \in \mathbb{N}$. From (2), for each $n \in \mathbb{N}$ we have

$$\begin{aligned} \omega_\eta(1, x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) &= \omega_\eta(1, Tx_{n-1}, Tx_n) + \varphi(Tx_{n-1}) + \varphi(Tx_n) \\ &\leq \psi(\omega_\eta(1, x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)) \ \forall \ \eta \in \mathfrak{A}. \end{aligned}$$

Iteratively, for each $n \in \mathbb{N}$ we get

$$\begin{aligned} \omega_\eta(1, x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) &\leq \psi^n(\omega_\eta(1, x_0, x_1) + \varphi(x_0) + \varphi(x_1)) \\ &\leq \psi^n(\delta_{\mathfrak{F}}(M) + \varphi(x_0) + \varphi(x_1)) = \psi^n(\xi) \quad (3) \end{aligned}$$

for all $\eta \in \mathfrak{A}$, where $\xi = \delta_{\mathfrak{F}}(M) + \varphi(x_0) + \varphi(x_1)$. Letting $n \to \infty$ in the above inequality, we get

$$\lim_{n \to \infty} \omega_\eta(1, x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) = 0 \ \forall \ \eta \in \mathfrak{A}.$$

Consequently,

$$\lim_{n \to \infty} \omega_\eta(1, x_n, x_{n+1}) = 0 \ \forall \ \eta \in \mathfrak{A} \text{ and } \lim_{n \to \infty} \varphi(x_n) = 0.$$

To prove that $\{x_n\}$ is ω-Cauchy sequence, take arbitrary $m, p \in \mathbb{N}$, and by using the triangle inequality and (3), for each $\eta \in \mathfrak{A}$ we have

$$\begin{aligned} \omega_\eta(p, x_m, x_{m+p}) &\leq \sum_{i=m}^{m+p-1} \omega_\eta(1, x_i, x_{i+1}) \\ &\leq \sum_{i=m}^{m+p-1} \omega_\eta(1, x_i, x_{i+1}) + \varphi(x_i) + \varphi(x_{i+1}) \\ &\leq \sum_{i=m}^{\infty} \psi^i(\xi) \to 0 \text{ as } m \to \infty. \end{aligned}$$

This shows that $\{x_n\}$ is a ω-Cauchy sequence in M. Since M is ω-complete, there exists $x^* \in M$ such that $\{x_n\}$ is ω-convergent to x^*, that is, for each $\eta \in \mathfrak{A}$ we have

$\lim_{n\to\infty} \omega_\eta(\lambda, x_n, x^*) = 0$ for some $\lambda > 0$. Since $\mathfrak{F}$ satisfies the Δ_2-condition, we have

$$\lim_{n\to\infty} \omega_\eta(\lambda, x_n, x^*) = 0 \text{ for all } \lambda > 0 \text{ and } \eta \in \mathfrak{A}.$$

As φ is lower semi-continuous, we have $\varphi(x^*) \leq \liminf_{n\to\infty} \varphi(x_n) = 0$. This implies that $\varphi(x^*) = 0$. By using hypothesis (iii), the triangle inequality and (2), we have

$$
\begin{aligned}
\omega_\eta(2, x^*, Tx^*) &\leq \omega_\eta(1, x^*, x_n) + \omega_\eta(1, x_n, Tx^*) \\
&= \omega_\eta(1, x^*, x_n) + \omega_\eta(1, Tx_{n-1}, Tx^*) \\
&\leq \omega_\eta(1, x^*, x_n) + \psi(\omega_\eta(1, x_{n-1}, x^*) + \varphi(x_{n-1}) + \varphi(x^*)) \\
&< \omega_\eta(1, x^*, x_n) + \omega_\eta(1, x_{n-1}, x^*) + \varphi(x_{n-1}) + \varphi(x^*) \ \forall \ \eta \in \mathfrak{A}.
\end{aligned}
$$

Letting $n \to \infty$ in the above inequality, we get $\omega_\eta(2, x^*, Tx^*) = 0 \ \forall \ \eta \in \mathfrak{A}$. As the structure $\{\omega_\eta : \eta \in \mathfrak{A}\}$ is separating, thus we conclude $x^* = Tx^*$. $\qquad\square$

The following remarks are necessary:

1. If M is a subset of X and $\mathfrak{F} = \{\omega_\eta : \eta \in \mathfrak{A}\}$ is a family of pseudomodular metrics on X such that for each $\eta \in \mathfrak{A}$, we have $\omega_\eta(\lambda, x, y) < \infty$ for all $\lambda > 0$ and $x, y \in M$, then we may ignore the ω-boundedness of M from above theorems. Because in this case distance between any two points of M must be finite.

2. Hypothesis (iii) of the above theorems can be replaced by continuity of the operator.

As novel application, we prove the existence theorem for fractional-order integral equation of the form:

$$x(t) = f(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s, x(s)) ds, \quad \alpha \in (0, 1), \quad t \in I \tag{4}$$

where Γ is the Euler gamma function given by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$, $f: I \to \mathbb{R}$ is a continuous function and $g: I \times \mathbb{R} \to \mathbb{R}$ is continuous and increasing function, that is, $g(t, \cdot)$ is increasing for all $t \in I$.

Let $X = (C[0, 10], \mathbb{R})$ be the space of all continuous and bounded functions defined on $I = [0, 10]$. Define the family of pseudonorms by

$$\|x\|_n = \max_{t\in[0,n]} |x(t)|, \quad n \in J = \{1, 2, 3, \dots, 9, 10\}.$$

By using this family of pseudonorms we get a family of pseudomodular metrics as

$$\omega_n(\lambda, x, y) = \frac{1}{|\lambda|} \|x - y\|_n.$$

Clearly, $\mathfrak{F} = \{\omega_n : n \in J\}$ defines a modular gauge structure on X, which is ω-complete, separating and satisfying both Δ_2-condition and Fatou property. Define the graph $G = (V, E)$ such that $V = X$ and $E = \{(x, y) : x(t) \leq y(t), \forall t \in I\}$.

Theorem 3.3. *Let $X = (C[0, 10], \mathbb{R})$ and let the operator*

$$T: X \to X, \quad Tx(t) = f(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s, x(s)) ds, \quad \alpha \in (0, 1), \quad t \in I = [0, 10]$$

where Γ is the Euler gamma function given by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$, $f \colon I \to \mathbb{R}$ is a continuous function and $g \colon I \times \mathbb{R} \to \mathbb{R}$ is continuous and increasing function, that is, $g(t, \cdot)$ is increasing for all $t \in I$. Further, assume that the following conditions hold:

(i) for each $t, s \in [0, n]$ and $x, y \in X$ with $(x, y) \in E$, we have

$$|g(s, x(s)) - g(s, y(s))| \leq \frac{\Gamma(\alpha + 1)}{10} \|x - y\|_n \ \text{for each } n \in J;$$

(ii) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E$.

Then the integral equation (4) *has at least one solution.*

Proof. First we show that for each $(x, y) \in E$, inequality (1) holds. For any $(x, y) \in E$ and $t \in [0, n]$, for each $n \in J$, we have

$$
\begin{aligned}
|Tx(t) - Ty(t)| &\leq \int_0^t \left| \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [g(s, x(s)) - g(s, y(s))] \right| ds \\
&\leq \int_0^t \left| \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{10} \right| \|x - y\|_n ds \\
&= \frac{t^\alpha}{10} \|x - y\|_n.
\end{aligned}
$$

Thus, we get $\omega_n(\lambda, Tx, Ty) \leq a\omega_n(\lambda, x, y)$ for each $(x, y) \in E$ and $n \in J$ with $a = \frac{t^\alpha}{10} < 1$. This implies that (1) holds with $a_n = a$, and $b_n = c_n = e_n = L_n = 0$ for each $n \in J$. As $g(t, \cdot)$ is increasing for all $t \in I$, thus for each $(x, y) \in E$, we have $(Tx, Ty) \in E$. Therefore, by Theorem 3.1, there exists a fixed point of the operator T, that is the integral equation (4) has at least one solution. $\qquad\square$

Example 3.3. Let $M = \left\{ \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} : x_1, x_2, x_3, x_4 \in \mathbb{R} \right\}$ be the set of all 2×2 matrices. Consider the family $\mathfrak{F} = \{\omega_n : n \in \{1, 2, 3, 4\}\}$ of pseudomodular metrics defined as

$$\omega_n(\lambda, X, Y) = \frac{1}{\lceil \lambda \rceil} \max_{1 \leq i \leq n} \{|x_i - y_i|\} \ \forall X, Y \in M.$$

Define the operator $T \colon M \to M$ by $TX = AX$, where $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ such that a, b are nonnegative real numbers with $\max\{a, b\} < 1$ and $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$. It is easy to see that all the conditions of Theorem 3.1 and inequality (1) with $a_n = \max\{a, b\} < 1$, and $b_n = c_n = e_n = L_n = 0$ for each $n \in \{1, 2, 3, 4\}$ hold. Thus the operator T has a fixed point.

4. Conclusion

In this paper, we introduce the notion of modular gauge space and prove some fixed point theorems on this setting. An existence theorem for a class of fractional order integral equations is established in pseudomodular metric spaces. We also construct an example to support our result.

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