WARDOWSKI TYPE CONTRACTIONS WITH APPLICATIONS ON CAPUTO TYPE NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

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In this paper, we consider mappings with contractive iterates at a point. We introduce F_S-contractions in the sense of Wardowski and Seghal and F_J-contractions in the sense of Wardowski and Jachymski. We ensure some existence and uniqueness fixed point results. We also give some applications on Ulam stability and Caputo type nonlinear fractional integrodifferential equations.

Keywords: Wardowski contractions, Ulam Stability, metric space, iterate point, fixed point, fractional differential equations of Caputo type.

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1. Introduction

Banach Fixed point theorem is one of the most important tools that can be used to solve some integral equations. Many authors extended and generalized the Banach contraction theorem to many directions. Ali et al. [3] generalized the Banach contraction theorem by generalizing Prešić type mappings in ordered metric spaces. Also, Ali et al. [4] studied a solution of Volterra integral inclusion in b-metric spaces via a new fixed point theorem. Aydi et al. [11] studied new fixed point theorems for weakly contractive mappings based on a pair of functions (ψ, ϕ). Also, Aydi et al. [15] extended the Banach contraction theorem by introducing new theorems for Boyed-Wong type contractions. Karapinar et al. [31] extended the Banach contraction theorem by introducing new type of contraction based on the notion of $\alpha-$admissible. Moreover, Karapinar et al. [1, 16, 20, 33] studied many theorems based on the notion of admissibility. Also, Karapinar et al. [29, 30] enriched the area of fixed point theory by studying many exciting fixed point theorems in partial metric spaces. For more generalizations of Banach fixed point theory,

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see [2, 7, 8, 11, 18, 25, 26, 27, 28, 32, 39, 40, 41, 42, 43, 44, 45, 46, 49]. In [17], Bryant proposed a relaxation of the Banach contractive condition involving the iteration of the function. Recall that, for a positive integer n we denote by h^n the nth iterate of h, so that $\upsilon = h^0\upsilon$ and $h^{n+1}\upsilon = h(h^n\upsilon)$ for $\upsilon \in X$ and $n \in \mathbb{N}$. Hereupon, the tripled (X, d, h) represent a metric space (X, d) with a self-mapping h on it. We shall use $(X*, d, h)$ to indicate the corresponding metric space is complete.

Theorem 1.1. *(See [17]) On $(X*, d, h)$, if for all $\upsilon, \omega \in X$, the function h satisfies the inequality*

$$d(h^n\upsilon, h^n\omega) \le \alpha d(\upsilon, \omega)$$

for some positive integer n and $\alpha \in [0, 1)$, then h has exactly one fixed point.

The next step in improving this type of contractive conditions was to consider the positive integer p depending on a point $\upsilon \in X$. The pioneer result in this way was reported by Seghal [38].

Theorem 1.2. *([38]) On $(X*, d, h)$, if, for each $\omega \in X$ there is $p(\omega) \in \mathbb{N}$ such that,*

$$d(h^{p(\omega)}\omega, h^{p(\omega)}\upsilon) \le \alpha d(\omega, \upsilon), \tag{1}$$

for each $\upsilon \in X$, where $0 < \alpha < 1$. Then h possess an unique fixed point in X.

Many other extensions were discussed in different papers. See for example, Guseman [19], Iseki [23], Matkowski [24], Singh [47]. In 1995, Jachymski unified and generalized some of these as follows:

Theorem 1.3. *([22]) On $(X*, d, h)$, we suppose also that for all $\upsilon, \omega \in X$*

$$d(h^{p(\upsilon)}\upsilon, h^{p(\upsilon)}\omega) \le \psi\left[\max\left\{d(\upsilon, h^i\omega), d(h^{p(\upsilon)}\upsilon, h^i\omega),\ i = 0, 1, 2, ...p(\upsilon)\right\}\right], \tag{2}$$

where $p : X \to \mathbb{N}$ is a mapping and ψ is a nonincreasing function such that

$$\lim_{u\to\infty}(u - \psi(u)) = \infty$$

with the property $\psi(u) < u$ for all $u > 0$. Then h possess a fixed point υ^*.

On (X, d, h), *an orbit* of $\upsilon_0 \in X$ is the set

$$O(\upsilon_0) = \{h^n\upsilon_0 : n = 0, 1, 2, ...\}$$

and $\rho(\upsilon_0)$ denoted to the diameter of the set $O(\upsilon_0)$. Note that for any subset B of X, $\rho(B) = \sup\{d(u, v) : u, v \in B\}$ is the diameter of B. We shall use the tripled (X^{o*}, d, h) if for some $\upsilon \in X$, every Cauchy sequence from $O(\upsilon)$ converges in X. In this case, the corresponding space is called orbitally complete.

We say that $F : \mathbb{R}_0^+ \to \mathbb{R}$ is a Wardowski type function [50] if it provides the following three axioms:

$(F1)$ For all $u, t \in \mathbb{R}_0^+$ if $u < t$, then $F(u) < F(t)$ (F is strictly increasing).

$(F2)$ We have $\lim_{\varsigma \to 0^+} \left(\tau^k F(\tau)\right) = 0$ for some $\tau \in (0, 1)$.

$(F3)$ We have

$$\lim_{n\to\infty} F(u_n) = -\infty \text{ if only if } \lim_{n\to\infty} u_n = 0,$$

for each sequence $\{\alpha_n\} \subset \mathbb{R}_0^+$.
Set

$$\mathcal{F} := \{F : F \text{ satisfies } (F1) - (F3)\}.$$

Definition 1.1. [50] *On $(X*, d, h)$, we say that h forms a $F-$contraction if there exist $G \in \mathcal{F}$ and $\tau > 0$ such that for all $\upsilon, \omega \in X$*

$$d(h\upsilon, h\omega) > 0 \Rightarrow \tau + G(d(h\upsilon, h\omega)) \leq G(d(\upsilon, \omega)) \tag{3}$$

The following result is belong to Wardowski.

Theorem 1.4. [50] *On $(X*, d, h)$, each $F-$contraction h possess a unique fixed point.*

This result, concerning the F-contractions, has aroused the interest of many authors.

Definition 1.2. [51] *On $(X*, d, h)$, we say that h forms a $F-$weak contraction if there exist $G \in \mathcal{F}$ and $\tau > 0$ such that for all $\upsilon, \omega \in X$ if it satisfies*

$$d(h\upsilon, h\omega) > 0 \Rightarrow \tau + G\left(d(h\upsilon, h\omega)\right) \leq G\left(M(\upsilon, \omega)\right) \tag{4}$$

where

$$M(\upsilon, \omega) = \max\left\{d(\upsilon, \omega), d(\upsilon, h\upsilon), d(\omega, h\omega), \frac{d(\upsilon, h\omega) + d(\omega, h\upsilon)}{2}\right\}.$$

The following theorem was expressed by Wardowski and Van Dung [51].

Theorem 1.5. [51] *On $(X, d*, h)$, each $F-$contraction h possess a unique fixed point whenever h and corresponding Wardowski function F are continuous.*

Later, it was shown, by Piri and Kumam [34], that Theorem 1.4 is still valid if we replace the axiom $(F3)$ above by
$(F3')$ F is continuous on $(0, \infty)$.
This trend has been continued by several authors, see e.g. [5, 6].
In this paper, we refine Wardowski functions and related contraction by defining two new notions, namely, F_S-contractions and F_J-contractions. and present fixed point results on these new constructions. As an application, we consider both "Caputo type nonlinear fractional differential equations" and "Ulam stability" are expressed.

2. **Main results**

We start this section by refining the family of a new family of Wardowski functions as follows:

$$\mathfrak{F} := \{F : F \text{ satisfies } (F1) - (F2)\}.$$

Based on $\mathfrak{F}$, we introduce the concept of F_S-contraction as follows.

Definition 2.1. *On $(X*, d, h)$, we say that h forms a F_S-contraction if $G \in \mathfrak{F}$ and $\tau > 0$ such that for all $\upsilon, \omega \in X$,*

$$d(h^{p(\upsilon)}\upsilon, h^{p(\upsilon)}\omega) > 0 \Rightarrow \tau + G\left(d(h^{p(\upsilon)}\upsilon, h^{p(\upsilon)}\omega)\right) \leq G\left(d(\upsilon, \omega)\right). \tag{5}$$

Our first main result is

Theorem 2.1. *On (X^{o*}, d, h), each F_S-contraction h possess a unique fixed point whenever $0 < \rho(\upsilon_0) < \infty$ for some $\upsilon_0 \in X$.*

Proof. By assumption of the theorem, there is $\upsilon_0 \in X$ such that $0 < \rho(\upsilon_0) < \infty$. Define the sequence $\{\upsilon_n\}$ as follows:

$$\upsilon_1 = h^{p_0}\upsilon_0, \quad \upsilon_2 = h^{p_1}\upsilon_1, \ \ldots \ \upsilon_{n+1} = h^{p_n}\upsilon_n,$$

where $p_n = p(\upsilon_n)$ for all $n \in \mathbb{N}$. Note that we have

$$\upsilon_1 = h^{p_0}\upsilon_0, \ \ \upsilon_2 = h^{p_1}\upsilon_1 = h^{p_1+p_0}\upsilon_0,$$

and hence inductively we find

$$\upsilon_{n+1} = h^{p_n+p_{n-1}+\ldots+p_1+p_0}\upsilon_0$$

and

$$\upsilon_{n+l} = h^{p_{n+l-1}+\ldots+p_n}\upsilon_n = h^{p_{n+l-1}+\ldots+p_n+\ldots+p_1+p_0}\upsilon_0.$$

The previous equality shows us that the sequence $\{\upsilon_n\}$ is a subsequence of the orbit $O(\upsilon_0)$. If we can find $n_0 \in \mathbb{N}\cup\{0\}$ such that $\upsilon_{n_0+1} = \upsilon_{n_0}$. Because

$$h^{p_{n_0}}\upsilon_{n_0} = \upsilon_{n_0+1} = \upsilon_{n_0},$$

we obtain that υ_{n_0} is a fixed point of $h^{p_{n_0}}$. where $p_{n_0} = p(\upsilon_{n_0})$

Consequently, without loss of generality, we suppose that $\upsilon_{n+1} \neq \upsilon_n$ for all $n \in \mathbb{N}\cup\{0\}$ that is equivalent to

$$d(\upsilon_n, \upsilon_{n+1}) > 0, \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{6}$$

Accordingly, keeping (5) in mind,

$$d(\upsilon_n, \upsilon_{n+1}) = d(\upsilon_n, h^{p_n}\upsilon_n) > 0, \text{ for all } n \in \mathbb{N} \cup \{0\}$$

implies that

$$\begin{aligned} \tau + F\left(d\left(\upsilon_n, h^{p_n}\upsilon_n\right)\right) &= \tau + F\left(d\left(h^{p_{n-1}}\upsilon_{n-1}, h^{p_{n-1}}(h^{p_n}\upsilon_{n-1})\right)\right) \\ &\leq F\left(d\left(\upsilon_{n-1}, h^{p_n}\upsilon_{n-1}\right)\right). \end{aligned}$$

That is,

$$F(d\,(\upsilon_n, h^{p_n}\upsilon_n)) \leq F\,(d\,(\upsilon_{n-1}, h^{p_n}\upsilon_{n-1})) - \tau. \tag{7}$$

Continuing in the same direction, we can conclude that

$$F(d\,(\upsilon_n, \upsilon_{n+1})) = F(d\,(\upsilon_n, h^{p_n}\upsilon_n)) \leq F\,(d\,(\upsilon_0, h^{p_n}\upsilon_0)) - n\tau, \quad \text{for any } n \in \mathbb{N}. \tag{8}$$

Taking $n \to \infty$ in inequality (8) and using the fact that $0 < \rho(\upsilon_0) < \infty$, we find that $\lim_{n\to\infty} F(d\,(\upsilon_n, \upsilon_{n+1})) = -\infty$. Using $(F2)$, we have

$$\lim_{n\to\infty} d(\upsilon_n, \upsilon_{n+1}) = 0. \tag{9}$$

We will show now that $\{\upsilon_n\}$ (which is contained in the orbit $O(\upsilon_0)$) is a Cauchy sequence. Considering $q = p_{n+l-1} + p_{n+l-2} + ... + p_n$, where n, l are arbitrary natural numbers such that $l \geq 1$ and $n \geq n_0$, we can write

$$\upsilon_{n+l} = h^q \upsilon_n. \tag{10}$$

Taking into account (10), (5) and the fact that $0 < \rho(\upsilon_0) < \infty$, we have for $l \geq 1$

$$d(\upsilon_n, \upsilon_{n+l}) = d(\upsilon_n, h^q\upsilon_n) = d(h^{p_{n-1}}\upsilon_{n-1}, h^{p_{n-1}}(h^q\upsilon_{n-1})) > 0.$$

Thus,

$$\begin{aligned} F(d(\upsilon_n, h^q\upsilon_n)) &= F(d(h^{p_{n-1}}\upsilon_{n-1}, h^{p_{n-1}}(h^q\upsilon_{n-1}))) \\ &\leq F(d(\upsilon_{n-1}, h^q\upsilon_{n-1})) - \tau \\ &.... \\ &\leq F(d(\upsilon_0, h^q\upsilon_0)) - n\tau. \end{aligned}$$

Again, $F(d(\upsilon_n, \upsilon_{n+l})) = F(d(\upsilon_n, h^q\upsilon_n)) \to -\infty$, when $n \to \infty$ and from $(F2)$ we get that

$$\lim_{n\to\infty} d(\upsilon_n, \upsilon_{n+l}) = 0$$

which shows us that the sequence $\{\upsilon_n\} \subset O(\upsilon_0)$ is Cauchy in $\mathcal{X}$. Since $\mathcal{X}$ is h-orbitally complete, there is $\upsilon^* \in \mathcal{X}$ such that

$$\lim_{n\to\infty} d(\upsilon_n, \upsilon^*) = 0. \tag{11}$$

We intend to prove that υ^* is a fixed point for h. For this goal, we will first show that

$$\lim_{n\to\infty} d(h^{p(\upsilon^*)}\upsilon_n, \upsilon_n) = 0.$$

Here, we will assume that $d(h^{p(\upsilon^*)}\upsilon_n, \upsilon_n) = d(h^{p_{n-1}}(h^{p(\upsilon^*)}\upsilon_{n-1}), h^{p_{n-1}}\upsilon_{n-1}) > 0$ for all $n \geq 1$. Indeed, in the contrary, if we can find $n_0 \geq 1$ such that $h^{p(\upsilon^*)}\upsilon_{n_0} = \upsilon_{n_0}$, then υ_{n_0} is a fixed point of $h^{p(\upsilon^*)}$. From (5), we have

$$\begin{aligned} d(h^{p_{n-1}}(h^{p(\upsilon^*)}\upsilon_{n-1}), h^{p_{n-1}}\upsilon_{n-1}) &> 0 \Rightarrow \\ F(d(h^{p_{n-1}}(h^{p(\upsilon^*)}\upsilon_{n-1}), h^{p_{n-1}}\upsilon_{n-1})) &\leq F(d(h^{p(\upsilon^*)}\upsilon_{n-1}, \upsilon_{n-1})) - \tau \\ &\leq \cdots \\ &\leq F(d(h^{p(\upsilon^*)}\upsilon_0, \upsilon_0)) - n\tau. \end{aligned}$$

Hence, $\lim_{n\to\infty} F(d(h^{p_{n-1}}(h^{p(\upsilon^*)}\upsilon_{n-1}), h^{p_{n-1}}\upsilon_{n-1})) = -\infty$. By $(F2)$, we get that

$$\lim_{n\to\infty} d(h^{p_{n-1}}(h^{p(\upsilon^*)}\upsilon_{n-1}), h^{p_{n-1}}\upsilon_{n-1}) = d(h^{p(\upsilon^*)}\upsilon_n, \upsilon_n) = 0. \tag{12}$$

Now, we assert that $h^{p(\upsilon^*)}\upsilon^* = \upsilon^*$, that is, υ^* is a fixed point of $h^{p(\upsilon^*)}$. We reason by contradiction and assume that $h^{p(\upsilon^*)}\upsilon^* \neq \upsilon^*$. By using the triangle inequality,

$$0 < d(h^{p(\upsilon^*)}\upsilon^*, \upsilon^*) \leq d(h^{p(\upsilon^*)}\upsilon^*, h^{p(\upsilon^*)}\upsilon_n) + d(h^{p(\upsilon^*)}\upsilon_n, \upsilon_n) + d(\upsilon_n, \upsilon^*). \tag{13}$$

If for $n \in \mathbb{N}$, there is $i_{n+1} > i_n$ where $i_0 = 1$ such that $h^{p(\upsilon^*)}\upsilon_{i_n} = h^{p(\upsilon^*)}\upsilon^*$, then

$$d(h^{p(\upsilon^*)}\upsilon^*, \upsilon^*) = d(h^{p(\upsilon^*)}\upsilon_{i_n}, \upsilon^*) = \lim_{n\to\infty} d(h^{p(\upsilon^*)}\upsilon_{i_n}, \upsilon_{i_n}) = 0,$$

so, υ^* is a fixed point of $h^{p(\upsilon^*)}$. Suppose there is $m \in \mathbb{N}$ so that for all $n \geq m$,

$$d(h^{p(\upsilon^*)}\upsilon^*, h^{p(\upsilon^*)}\upsilon_n) > 0.$$

Thereby,

$$F(d(h^{p(\upsilon^*)}\upsilon^*, h^{p(\upsilon^*)}\upsilon_n) \leq F(d(\upsilon^*, \upsilon_n)) - \tau < F(d(\upsilon^*, \upsilon_n)).$$

As F is increasing, we have

$$d(h^{p(\upsilon^*)}\upsilon^*, h^{p(\upsilon^*)}\upsilon_n) < d(\upsilon^*, \upsilon_n),$$

and returning in (13), we obtain that

$$\begin{aligned} 0 < d(h^{p(\upsilon^*)}\upsilon^*, \upsilon^*) &\leq d(h^{p(\upsilon^*)}\upsilon^*, h^{p(\upsilon^*)}\upsilon_n) + d(h^{p(\upsilon^*)}\upsilon_n, \upsilon_n) + d(\upsilon_n, \upsilon^*) \\ &\leq d(\upsilon^*, \upsilon_n) + d(h^{p(\upsilon^*)}\upsilon_n, \upsilon_n) + d(\upsilon_n, \upsilon^*). \end{aligned} \tag{14}$$

Letting $n \to \infty$ and taking into account (11) and (12), we can conclude that

$$d(h^{p(\upsilon^*)}\upsilon^*, \upsilon^*) = 0. \tag{15}$$

Thus $h^{p(\upsilon^*)}\upsilon^* = \upsilon^*$, and so the existence of a fixed point for $h^{p(\upsilon^*)}$. Let us show now that $h^{p(\upsilon^*)}$ has at most one fixed point. Suppose on contrary that υ^* and ω^* are two distinct fixed points of $h^{p(\upsilon^*)}$, so $d(\upsilon^*, \omega^*) = d(h^{p(\upsilon^*)}\upsilon^*, h^{p(\upsilon^*)}\omega^*) > 0$. Thus,

$$\tau + F(d(h^{p(\upsilon^*)}\upsilon^*, h^{p(\upsilon^*)}\omega^*)) \leq F(d(\upsilon^*, \omega^*)),$$

that is,

$$\tau + F(d(\upsilon^*, \omega^*)) \leq F(d(\upsilon^*, \omega^*)),$$

which is a contradiction. Note that

$$h(\upsilon^*) = h(h^{p(\upsilon^*)}\upsilon^*) = h^{p(\upsilon^*)}(h\upsilon^*).$$

Due to the uniqueness of the fixed point of $h^{p(\upsilon^*)}$, we have $h\upsilon^* = \upsilon^*$. □

As a corollary of Theorem 2.2, we have

Corollary 2.1. *On $(X, d*, h)$, for a given mapping $p : X \to \mathbb{N}$, we suppose there exists $\tau > 0$ such that for all $\upsilon, \omega \in X$,*

$$d(h^{p(\upsilon)}\upsilon, h^{p(\upsilon)}\omega) \leq e^{-\tau} d(\upsilon, \omega). \tag{16}$$

Assume there exists $\upsilon_0 \in X$ such that $0 < \rho(\upsilon_0) < \infty$. Moreover, suppose that $(\mathcal{X}, d)$ is h-orbitally complete. Then h has a unique fixed point.

Proof. It suffices to take $F(t) = \ln(t)$ for $t > 0$ in Theorem 2.2. Then

$$d(h^{p(\upsilon)}\upsilon, h^{p(\upsilon)}\omega) > 0 \Rightarrow d(h^{p(\upsilon)}\upsilon, h^{p(\upsilon)}\omega) \leq e^{-\tau} d(\upsilon, \omega). \tag{17}$$

Note that (16) also holds if $d(h^{p(\upsilon)}\upsilon, h^{p(\upsilon)}\omega) = 0$. Thus, (16) is verified for all $\upsilon, \omega \in \mathcal{X}$. □

The concept of F_J-contractions is introduced in the following.

Definition 2.2. *On $(X*, d, h)$, we say that h forms a F_J-contraction if $G \in \mathfrak{F}$ and $\tau > 0$ such that for all $\upsilon, \omega \in X$,*

$$d(h^{p(\upsilon)}\upsilon, h^{p(\upsilon)}\omega) > 0 \Rightarrow \tau + G\left(d(h^{p(\upsilon)}\upsilon, h^{p(\upsilon)}\omega)\right) \leq G\left(M_J d(\upsilon, \omega)\right) \tag{18}$$

where

$$M_J(\upsilon, \omega) = \max\left\{d(\upsilon, h^s\omega), d(h^{p(\upsilon)}\upsilon, h^s\omega) : s = 0, 1, ..., p(\upsilon)\right\}. \tag{19}$$

Our second main result is the following:

Theorem 2.2. *On (X^{0*}, d, h), each F_J-contraction h possess a unique fixed point whenever there exists $\upsilon_0 \in X$ such that $0 < \rho(\upsilon_0) < \infty$.*

Proof. Due to the assumption of the theorem, there is $\upsilon_0 \in \mathcal{X}$ such that $0 < \rho(\upsilon_0) < \infty$. By following the initial corresponding parts in the proof of Theorem 2.2, we define $\{\upsilon_{n+1}\}_{n\in\mathbb{N}} = \{h^{p_n}\upsilon_n\}_{n\in\mathbb{N}}$ starting the mentioned υ_0 in $\mathcal{X}$ with

$$d(\upsilon_n, \upsilon_{n+1}) > 0, \text{ for all } n \in \mathbb{N}\cup\{0\}\,. \tag{20}$$

If for some n, $d(h^{p_n}\upsilon_n, h^{p_n}(h^l\upsilon_0)) = 0$, then υ_{n+1} is a fixed point of h^l. Indeed, we have

$$\upsilon_{n+1} = h^{p_n}\upsilon_n = h^{p_n}(h^l\upsilon_0) = h^{p_n+l}(\upsilon_0) = h^l(h^{p_n}\upsilon_n) = h^l(\upsilon_{n+1}).$$

Now, assume that for all n, $d(h^{p_n}\upsilon_n, h^{p_n}(h^l\upsilon_0)) > 0$. By replacing υ by υ_n and ω by $h^l\upsilon_0$ in (18), we find

$$\tau+F(d(h^{p_n}\upsilon_n, h^{p_n}(h^l\upsilon_0)) \leq F(\max\left\{d(\upsilon_n, h^{l+s}\upsilon_0), d(\upsilon_{n+1}, h^{l+s}\upsilon_0) : s \in \{0,1,2,\cdots,p(n)\}\right\}), \tag{21}$$

for some $\tau > 0$. Due to the hypothesis that $\rho(\upsilon_0) < \infty$, denoting by $\xi_n = \lim_{l\to\infty}\sup d(\upsilon_n, h^l\upsilon_0)$, we have that $\{\xi_n\}$ is a bounded real sequence. For n fixed, we can find a sequence $\{l(i)\}_i$ of positive integers such that

$$\lim_{i\to\infty} d(\upsilon_{n+1}, h^{p_n+l(i)}\upsilon_0) = \lim_{i\to\infty} d(h^{p_n}\upsilon_n, h^{p_n+l(i)}\upsilon_0) = \xi_n.$$

Furthermore, we may assume that for each $s = 0,1,2,\cdots,p_n$, the subsequences $\left\{d(\upsilon_n, h^{l(i)+s}\upsilon_0)\right\}_i$ and $\left\{d(\upsilon_{n+1}, h^{l(i)+s}\upsilon_0)\right\}_i$ are convergent. Hence there exist α_s and β_s such that

$$\lim_{i\to\infty} d(\upsilon_n, h^{l(i)+s}\upsilon_0) = \alpha_s,$$

and

$$\lim_{i\to\infty} d(\upsilon_{n+1}, h^{l(i)+s}\upsilon_0) = \beta_s,$$

for $s = 0,1,2,\cdots,p_n$. For such s, $\alpha_s \leq \xi_n$ and $\beta_s \leq \xi_{n+1}$, so the monotonicity of the function F yields that $F(\max\{\alpha_s,\beta_s\}) \leq F(\max\{\xi_n,\xi_{n+1}\})$. Returning to the inequality (21) and putting $l = l(i)$, one writes

$$\tau+F(d(\upsilon_{n+1}, h^{p_n+l(i)}\upsilon_0)) \leq F(\max\left\{d(\upsilon_n, h^{l(i)+s}\upsilon_0), d(\upsilon_{n+1}, h^{l(i)+s}\upsilon_0) : s \in \{0,1,2,\cdots,p(n)\}\right\}).$$

Letting $i \to \infty$ and using $(F3')$, we get

$$F(\xi_{n+1}) \leq F(\max\{\alpha_s,\beta_s : s = 0,1,2,\cdots,p_n\}) - \tau \leq F(\max\{\xi_n,\xi_{n+1}\}) - \tau.$$

If for some n, $\max\{\xi_n,\xi_{n+1}\} = \xi_{n+1}$, then

$$F(\xi_{n+1}) \leq F(\xi_{n+1}) - \tau$$

so $\tau \leq 0$, which is a contradiction. Attendantly, we assume that $\max\{\xi_n,\xi_{n+1}\} = \xi_n$ for each $n \geq 0$. In this case,

$$F(\xi_{n+1}) \leq F(\xi_n) - \tau < F(\xi_n), \text{ for } n \in \mathbb{N}.$$

That implies that

$$F(\xi_{n+1}) \leq F(\xi_n) - \tau \leq ... \leq F(\xi_0) - n\tau.$$

Since $\lim_{n\to\infty}[F(\xi_0) - n\tau] = -\infty$, it follows from $(F2)$ that $\lim_{n\to\infty}\xi_n = 0$.
In the next step, we will show that the sequence $\{h^n\upsilon_0\}$ is Cauchy. Since $\lim_{n\to\infty}\xi_n = 0$, for $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that $d(\upsilon_m, h^l\upsilon_0) = \xi_m < \varepsilon$. A ttendantly, $\{h^n\upsilon_0\} \subset O(\upsilon_0)$ is a Cauchy sequence. But, the space $\mathcal{X}$ is h-orbitally complete, thereby $\lim_{n\to\infty} h^n\upsilon_0 = \upsilon^*$ for some $\upsilon^* \in \mathcal{X}$. We assert now that $\lim_{n\to\infty} d(h^{p(\upsilon^*)}\upsilon^*, h^{p(\upsilon^*)}(h^n\upsilon_0) = 0$.

We should consider two cases:

(1) For any $n \in \mathbb{N}$, there exists $i_n > i_{n-1}$ with $i_0 = 1$ such that $h^{p(\upsilon^*)+i_n}\upsilon_0 = h^{p(\upsilon^*)}\upsilon^*$. Then $\upsilon^* = \lim_{n\to\infty} h^{p(\upsilon^*)+i_n}\upsilon_0 = h^{p(\upsilon^*)}\upsilon^*$, so υ^* is a fixed point of h.

(2) There is $m \in \mathbb{N}$ so that for each $n \geq m$, $d(h^{p(\upsilon^*)}\upsilon^*, h^{p(\upsilon^*)}(h^n\upsilon_0)) > 0$. For $\upsilon = \upsilon^*$ and $\omega = h^n\upsilon_0$, we have

$$\tau + F(d(h^{p(\upsilon^*)}\upsilon^*, h^{p(\upsilon^*)}(h^n\upsilon_0)) \leq$$

$$\leq F(\max\left\{d(\upsilon^*, h^{n+s}\upsilon_0), d(h^{p(\upsilon^*)}\upsilon^*, h^{n+s}\upsilon_0) : s = 0, 1, ..., p(\upsilon^*)\right\}).$$

Letting $n \to \infty$ and keeping in mind $(F3')$, we obtain that

$$F(d(h^{p(\upsilon^*)}\upsilon^*, \upsilon^*)) \leq F(d(h^{p(\upsilon^*)}\upsilon^*)) - \tau.$$

Since $\tau > 0$, it is a contradiction. Now, let us indicate that $h^{p(\upsilon^*)}$ have only one fixed point. Assume, on the contrary, there is $\omega^* \in \mathcal{X}$ with $\upsilon^* \neq \omega^* = h^{p(\upsilon^*)}\omega^*$. For $\upsilon = \upsilon^*$ and $\omega = h^n\upsilon_0$, we have

$$\tau + F(d(h^{p(\upsilon^*)}\omega^*, h^{p(\upsilon^*)}(h^n\upsilon_0))) \leq$$

$$F(\max\left\{d(\omega^*, h^{n+s}\upsilon_0), d(h^{p(\upsilon^*)}\omega^*, h^{n+s}\upsilon_0) : s = 0, 1, ..., p(\upsilon^*)\right\}).$$

Letting $n \to \infty$, we get that

$$F(d(\upsilon^*, \omega^*)) \leq F(d(\upsilon^*, \omega^*)) - \tau.$$

It is a contradiction, attendantly, the obtained fixed point of $h^{p(\upsilon^*)}$ is unique. Finally, since $h(\upsilon^*) = h(h^{p(\upsilon^*)}\upsilon^*) = h^{p(\upsilon^*)}(h(\upsilon^*))$, it follows that $h(\upsilon^*)$ is a fixed point of $h^{p(\upsilon^*)}$. Due to the uniqueness, we can conclude that $h\upsilon^* = \upsilon^*$, which ends the proof. □

Corollary 2.2. *On $(X, d*, h)$, let $p : \mathcal{X} \to \mathbb{N}$. If there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that for all $\upsilon, \omega \in X$,*

$$d(h^{p(\upsilon)}\upsilon, h^{p(\upsilon)}\omega) > 0$$

implies that

$$\tau + F\left(d(h^{p(\upsilon)}\upsilon, h^{p(\upsilon)}\omega)\right) \leq F\left(\max\left\{d(\upsilon, \omega), d(\upsilon, h^{p(\upsilon)}\omega), d(h^{p(\upsilon)}\upsilon, \omega)\right\}\right). \quad (22)$$

Then h possess a unique fixed point.

3. Applications

In this section, we present a Ulam stability and we resolve a nonlinear fractional differential type equation using Corollary 2.1.

3.1. Ulam stability

Ulam [21, 36] stability has attracted attention of several authors in fixed point theory, see [48]. On (X^{0*}, d, h), we investigate the fixed point equation

$$\upsilon = h\upsilon \quad (23)$$

and the inequality (for $\varepsilon > 0$)

$$d(h^{p(\varkappa)}\varkappa, \varkappa) \leq \varepsilon. \quad (24)$$

Equation (23) is called Ulam stable if satisfies the condition:

(US) there is a constant $\Delta > 0$, for each $\varepsilon > 0$ and for every solution $\varkappa^*$ of the inequality (24), there is a solution $\upsilon^* \in X$ of the equation (23) such that

$$d(\upsilon^*, \varkappa^*) \leq \Delta\varepsilon. \quad (25)$$

Theorem 3.1. *Under the hypotheses of Corollary 2.1, the fixed point equation (23) is Ulam stable.*

Proof. On account of Corollary 2.1, we guarantee a unique $\upsilon^* \in X$ such that $\upsilon^* = h\upsilon^*$, in other words, $\upsilon^* \in X$ forms a solution of (23). Let $\varepsilon > 0$ and $\varkappa^* \in X$ be an ε-solution, that is,

$$d(h^{p(\varkappa^*)}\varkappa^*, \varkappa^*) \leq \varepsilon.$$

We have

$$\begin{aligned} d(\upsilon^*, \varkappa^*) = d(h\upsilon^*, \varkappa^*) = d(h^{p(\varkappa^*)}\upsilon^*, \varkappa^*) \leq & d(h^{p(\varkappa^*)}\upsilon^*, h^{p(\varkappa^*)}\varkappa^*) + d(h^{p(\varkappa^*)}\varkappa^*, \varkappa^*) \\ \leq & e^{-\tau} d(\upsilon^*, \varkappa^*) + \varepsilon. \end{aligned}$$

We deduce

$$d(\upsilon^*, \varkappa^*) \leq \frac{1}{1 - e^{-\tau}}\varepsilon = c\varepsilon,$$

where $c = \frac{1}{1-e^{-\tau}} > 0$. Accordingly, the equation (23) is Ulam stable. □

3.2. Application to Nonlinear fractional integrodifferential equations

Here, we give a solution for a Caputo type nonlinear fractional integrodifferential equation. For more details on fractional calculus, see [9, 10, 12, 13, 14, 37]. The Caputo derivative of a continuous mapping $g : [0, \infty) \to \mathbb{R}$, (order $\delta > 0$) is given by

$${}^{C}D^{\delta}(g(t)) := \frac{1}{\Gamma(n - \delta)} \int_0^t (t - s)^{n-\delta-1} g^{(n)}(s) ds \qquad (n - 1 < \delta < n, \quad n = [\delta] + 1),$$

where Γ represent the gamma function $[\delta]$ refer to the integer part of the positive real number δ .

In this subsection, we examine the nonlinear fractional integrodifferential equation of Caputo type:

$${}^{C}D^{\delta}(\xi(t)) = G(t, \xi(t)), \tag{26}$$

under the conditions

$$\xi(0) = 0, \qquad \xi(1) = \int_0^{\theta} \xi(s) ds, (\text{ integral boundary condition, })$$

where $G : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\xi \in C([0, 1], [0, \infty)$, $0 < \theta < 1$, and $1 < \delta \leq 2$, (for more details, see [12]). Clearly, a solution of the equation (26) is a fixed point of the integral equation

$$\begin{aligned} F\xi(t) \quad = \quad & \frac{1}{\Gamma(\delta)} \int_0^t (t - s)^{\delta-1} G(s, \xi(s)) ds \\ & - \frac{2t}{(2 - \theta^2)\Gamma(\delta)} \int_0^1 (1 - s)^{\delta-1} G(s, \xi(s)) ds \\ & + \frac{2t}{(2 - \theta^2)\Gamma(\delta)} \int_0^{\theta} \Big(\int_0^s (s - m)^{\delta-1} G(m, \xi(m)) dm \Big) ds. \end{aligned} \tag{27}$$

Theorem 3.2. *On account of the nonlinear fractional differential equation (26), we assume that*

$$|G(s, \xi(s)) - G(s, \eta(s))| \leq \frac{\Gamma(\delta + 1)}{5} e^{-\tau} |\sqrt{|\xi(s)|} - \sqrt{|\eta(s)|}|$$

and

$$|G(s, \xi(s))| + |G(s, \eta(s))| \leq \frac{\Gamma(\delta + 1)}{5} e^{-\tau} |\sqrt{|\xi(s)|} + \sqrt{|\eta(s)|}|$$

for each $s \in [0, 1]$, for some $\tau > 0$ and for all $\xi, \eta \in C([0, 1], \mathbb{R})$, we have Then, the equation (26) possess a unique solution.

Proof. The space $X =: C([0,1],\mathbb{R})$ endowed with the metric $d : X \times X \to [0,\infty)$ defined as

$$d(\xi,\eta) = \sup_{t\in[0,1]} |\xi(t) - \eta(t)|$$

for all $\xi,\eta \in X$, is complete . We claim that F verifies the condition (16). For this, consider $\xi,\eta \in X$ and $t \in [0,1]$. We have

$$\begin{aligned}
|F\xi(t) - F\eta(t)| \quad &= \quad \Big|\frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}G(s,\xi(s))ds \\
&\quad -\frac{2t}{(2-\theta^2)\Gamma(\delta)}\int_0^1 (1-s)^{\delta-1}G(s,\xi(s))ds \\
&\quad +\frac{2t}{(2-\theta^2)\Gamma(\delta)}\int_0^\theta \Big(\int_0^s (s-m)^{\delta-1}G(m,\xi(m))dm\Big)ds \\
&\quad +\frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}G(s,\eta(s))ds \\
&\quad -\frac{2t}{(2-\theta^2)\Gamma(\delta)}\int_0^1 (1-s)^{\delta-1}G(s,\eta(s))ds \\
&\quad +\frac{2t}{(2-\theta^2)\Gamma(\delta)}\int_0^\theta \Big(\int_0^s (s-m)^{\delta-1}G(m,\eta(m))dm\Big)ds\Big| \\
&\leq \quad \frac{1}{\Gamma(\delta)}\int_0^t |t-s|^{\delta-1}(|G(s,\xi(s)) - G(s,\eta(s))|)ds,
\end{aligned}$$

and so

$$\begin{aligned}
|F\xi(t) - F\eta(t)| \quad &\leq \quad \frac{1}{\Gamma(\delta)}\int_0^t |t-s|^{\delta-1}\frac{\Gamma(\delta+1)}{5}\left[e^{-\tau}\sup_{s\in[0,1]}|\sqrt{|\xi(s)|} - \sqrt{|\eta(s)|}|\right]ds \\
&\quad +\frac{2t}{(2-\theta^2)\Gamma(\delta)}\int_0^1 (1-s)^{\delta-1}\frac{\Gamma(\delta+1)}{5}\left[e^{-\tau}\sup_{s\in[0,1]}|\sqrt{|\xi(s)|} - \sqrt{|\eta(s)|}|\right]ds \\
&\quad +\frac{2t}{(2-\theta^2)\Gamma(\delta)}\int_0^\theta \left|\int_0^s (s-m)^{\delta-1}\frac{\Gamma(\delta+1)}{5}\left[e^{-\tau}\sup_{s\in[0,1]}|\sqrt{|\xi(s)|} - \sqrt{|\eta(s)|}|\right]dm\right|ds \\
&\leq \quad \frac{\Gamma(\delta+1)}{5}[e^{-\tau}\sup_{s\in[0,1]}|\sqrt{\xi(s)} - \sqrt{\eta(s)}|] \times \sup_{t\in[0,1]}\Big(\frac{1}{\Gamma(\delta)}\int_0^1 |t-s|^{\delta-1}ds \\
&\qquad +\frac{2t}{(2-\theta^2)\Gamma(\delta)}\int_0^1 (1-s)^{\delta-1}ds + \frac{2t}{(2-\theta^2)\Gamma(\delta)}\int_0^\theta\int_0^s |s-m|^{\delta-1}dmds\Big) \\
&\leq \quad e^{-\tau}\sup_{s\in[0,1]}|\sqrt{|\xi(s)|} - \sqrt{|\eta(s)|}|.
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
|F\xi(t)|+|F\eta(t)| &= \Big|\frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}G(s,\xi(s))ds \\
&\quad -\frac{2t}{(2-\theta^2)\Gamma(\delta)}\int_0^1 (1-s)^{\delta-1}G(s,\xi(s))ds \\
&\quad +\frac{2t}{(2-\theta^2)\Gamma(\delta)}\int_0^\theta \Big(\int_0^s (s-m)^{\delta-1}G(m,\xi(m))dm\Big)ds\Big| \\
&\quad +\Big|\frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}G(s,\eta(s))ds \\
&\quad -\frac{2t}{(2-\theta^2)\Gamma(\delta)}\int_0^1 (1-s)^{\delta-1}G(s,\eta(s))ds \\
&\quad +\frac{2t}{(2-\theta^2)\Gamma(\delta)}\int_0^\theta \Big(\int_0^s (s-m)^{\delta-1}G(m,\eta(m))dm\Big)ds\Big|,
\end{aligned}
$$

and so

$$
\begin{aligned}
|F\xi(t)|+|F\eta(t)| &\leq \frac{1}{\Gamma(\delta)}\int_0^t |t-s|^{\delta-1}(\frac{\Gamma(\delta+1)}{5}e^{-\tau}|\sqrt{|\xi(s)|}+\sqrt{|\eta(s)|}|)ds \\
&\quad +\frac{2t}{(2-\theta^2)\Gamma(\delta)}\int_0^1 (1-s)^{\delta-1}(|\frac{\Gamma(\delta+1)}{5}e^{-\tau}|\sqrt{\xi(s)}+\sqrt{\eta(s)}|)ds \\
&\quad +\frac{2t}{(2-\theta^2)\Gamma(\delta)}\int_0^\theta \Big|\int_0^s (s-m)^{\delta-1}\frac{\Gamma(\delta+1)}{5}e^{-\tau}|\sqrt{\xi(m)}+\sqrt{\eta(m)}|dm\Big|ds \\
&\leq \frac{\Gamma(\delta+1)}{5}\sup_{s\in[0,1]}|\sqrt{|\xi(s)|}+\sqrt{|\eta(s)|} \\
&\quad \sup_{t\in[0,1]}\Bigg(\frac{1}{\Gamma(\delta)}\int_0^t |t-s|^{\delta-1}ds+\frac{2t}{(2-\theta^2)\Gamma(\delta)}\int_0^1 (1-s)^{\delta-1}ds \\
&\quad +\frac{2t}{(2-\theta^2)\Gamma(\delta)}\int_0^\theta \Big|\int_0^s (s-m)^{\delta-1}dm\Big|ds\Bigg) \\
&\leq \sup_{s\in[0,1]}|\sqrt{|\xi(s)|}+\sqrt{|\eta(s)|}|.
\end{aligned}
$$

This leads to

$$
\sup_{t\in[0,1]}\Bigg(|F\xi(t)|+|F\eta(t)|\Bigg) \leq \sup_{s\in[0,1]}|\sqrt{|\xi(s)|}+\sqrt{|\eta(s)|}|.
$$

We deduce that

$$\begin{aligned}
d(F^2\xi, F^2\eta) &= \sup_{t\in[0,1]}\Bigg(|F\xi(t)-F\eta(t)|\times|F\xi(t)+F\eta(t)|\Bigg)\\
&= \sup_{t\in[0,1]}\Bigg(|F\xi(t)-F\eta(t)|\Bigg)\times\sup_{t\in[0,1]}\Bigg(|F\xi(t)+F\eta(t)|\Bigg)\\
&\leq \sup_{t\in[0,1]}\Bigg(|F\xi(t)-F\eta(t)|\Bigg)\times\sup_{t\in[0,1]}\Bigg(|F\xi(t)|+|F\eta(t)|\Bigg)\\
&\leq e^{-\tau}\sup_{s\in[0,1]}|\sqrt{|\xi(s)|}-\sqrt{|\eta(s)|}|\times\sup_{s\in[0,1]}|\sqrt{|\xi(s)|}+\sqrt{|\eta(s)|}|\\
&= e^{-\tau}\sup_{s\in[0,1]}||\xi(s)|-|\eta(s)||\\
&\leq e^{-\tau}\sup_{s\in[0,1]}|\xi(s)-\eta(s)|\\
&= e^{-\tau}d(\xi,\eta),
\end{aligned}$$

for all $\xi, \eta \in X$. Therefore, the condition (16) holds with $p : X \to \mathbb{N}$ such that $p(\xi) = 2$ for each $\xi \in X$ (it is a constant). Accordingly all axioms of Corollary 2.1 are verified and attendantly F possess a unique fixed point. It yields that the Caputo type nonlinear fractional differential equation (26) possess a unique solution. □

4. Conclusions

The notions of F_S-contractions in the sense of Wardowski and Seghal and F_J-contractions in the sense of Wardowski and Jachymski were introduced. We studied some existence and uniqueness fixed point results in such new notions. Some applications on Ulam stability and Caputo type nonlinear fractional integrodifferential equations were investigated.

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