ON OBSTINATE IDEALS IN MV-ALGEBRAS

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In this paper we introduce obstinate ideals of an MV-algebra and we state some examples and theorems. We prove that quotient algebras that constructed via obstinate ideals are Boolean algebras. Also we investigate some relationships between the obstinate ideals and the other ideals of an MV-algebra.

1. **Introduction and preliminaries**

MV-algebras are introduced by C. C. Chang in 1958 [4] to give an algebraic counterpart of the multiple-valued Łukasiewicz propositional logic. Then this class of algebras has been intensively studied by many researcher. In particular, emphasis has been put an the ideal theory of MV-algebras [7, 9, 10]. Hoo, Iseki and Tanaka [10, 11] introduced the notion of implicative and quasi-implicative ideals of MV-algebras. In 1987, S. K. Goel and A. K. Arora [8] first introduced the concept of obstinate ideals in BCK-algebras which are logical algebras introduced by Iseki in 1966 [13]. MV-algebras as well as BCK-algebras are important logical algebras. This motivates us to study the notion of obstinate ideals in MV-algebras and investigate the relations between the other ideals previously introduced. In this paper also we consider the quotient algebras induced by obstinate ideals and prove some related theorems.

Definition 1.1. [5, 14, 15] *An MV-algebra is a structure $(M, \oplus, {}^*, 0)$ where $\oplus$ is a binary operation, *, is a unary operation, and* 0 *is a constant such that the following axioms are satisfied for any $a, b \in M$:*
(MV1) $(M, \oplus, 0)$ *is an abelian monoid,*
(MV2) $(a^*)^* = a$,
(MV3) $0^* \oplus a = 0^*$,
(MV4) $(a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a$.

Note that $1 = 0^*$ and the auxiliary operation $\odot$ as follow:

$$x \odot y = (x^* \oplus y^*)^*.$$

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We say that the element $x \in M$ has order n and we write $ord(x) = n$, if n is the smallest natural number such that $nx = 1$. We say that the element x has a finite order, and write $ord(x) < \infty$. An MV-algebra M is locally finite if every non-zero element of M has finite order. We recall that the natural order determines a bounded distributive lattice structure such that

$$x \vee y = x \oplus (x^* \odot y) = y \oplus (x \odot y^*) \quad \text{and} \quad x \wedge y = x \odot (x^* \oplus y) = y \odot (y^* \oplus x).$$

Lemma 1.1. [5, 15] *In each MV-algebra, the following relations hold for all $x, y, z \in A$:*
(1) $x \le y$ *if and only if* $y^* \le x^*$,
(2) *If* $x \le y$, *then* $x \oplus z \le y \oplus z$ *and* $x \odot z \le y \odot z$,
(3) $x \le y$ *if and only if* $x^* \oplus y = 1$ *if and if* $x \odot y^* = 0$,
(4) $x, y \le x \oplus y$ *and* $x \odot y \le x, y$,
(5) $x \oplus x^* = 1$ *and* $x \odot x^* = 0$,
(6) *If* $x \in B(A)$, *then* $x \wedge y = x \odot y$, *for any* $y \in A$.
Where $B(A)$ *is the set of all complemented elements of* $L(A)$ *such that* $L(A)$ *is distributive lattice with* 0 *and* 1 *on* A.

Also for any two elements $x, y \in A$, $x \le y$ if and only if x and y satisfy the condition (3) in the above lemma.

Definition 1.2. [5, 15] *An ideal of an MV-algebra A is a nonempty subset I of A satisfying the following conditions:*
(I1) If $x \in I$, $y \in A$ *and* $y \le x$ *then* $y \in I$,
(I2) If $x, y \in I$, *then* $x \oplus y \in I$.
We denote by $Id(A)$ the set of ideals of an MV-algebra A.

Definition 1.3. [5, 15] *Let I be an ideal of an MV-algebra A. Then I is a proper if $I \neq A$. Proper ideal P is a prime if and only if for all $x, y \in A$, $x \odot y^* \in P$ or $y \odot x^* \in P$.*

• [3] *An ideal I of an MV-algebra A is called a Boolean ideal if $x \wedge x^* \in I$, for all $x \in A$.*

• [3] *P is a primary ideal of an MV-algebra A if it is a proper ideal such that for every $a, b \in A$ such that $a \odot b \in P$, there exists an integer $n > 0$ such that $a^n \in P$ or $b^n \in P$.*

• [3, 6] *An ideal I is called a prefect ideal if for each $x \in A$, there exists an integer $n \ge 1$ such that $x^n \in I$ if and only if $(x^*)^m \notin I$ for all integers m.*

• [10, 11, 12] *An ideal I of an MV-algebra A is called an implicative if for any $x, y, z \in A$ such that $z \odot (y^* \odot x^*) \in I$ and $y \odot x^* \in I$, then $z \odot x^* \in I$.*

• [10, 11, 12] *An ideal I is a quasi-implicative if for any $x \in A$ such that $x^n \in I$ for some $n \ge 1$, then $x \in I$.*

Lemma 1.2. [3] *Every prime ideal of an MV-algebra is a primary ideal of MV-algebra.*

Lemma 1.3. [10, 11, 12] *Let I be an implicative ideal. Then I is a quasi-implicative ideal.*

Definition 1.4. [3, 6] *An MV-algebra A is called a prefect if every nonzero element $x \in A$, $ord(x) = \infty$ if and only if $ord(x^*) < \infty$.*

Also A is a prefect MV-algebra if and only if any proper ideal of A is a prefect.

Definition 1.5. [2] *Let X be a nonempty subset of an MV-algebra A and $X^{\perp}$ be the annihilator of X defined by $X^{\perp} = \{a \in A : a \wedge x = 0 \quad \text{for any} \quad x \in X\}$.*

Lemma 1.4. [5, 15] *M is a maximal ideal of an MV-algebra A if and only if for any $x \notin M$, $(nx)^* \in M$, for some integer $n \geq 1$.*

Definition 1.6. [1, 9] *A nonzero element m of a poset P with 0 is a molecule if whenever $0 < x, y \leq m$, then $\{x, y\}$ has a nonzero lower bound. Thus $m \in A$ is a molecule if and only if whenever $x, y \in A$ satisfy $0 < x, y \leq m$, then $x \wedge y > 0$. $Mol(A)$ denote the set of all molecules of A.*

In an MV-algebra M, the distance function is
$d : \quad M \times M \longrightarrow M, \qquad d(x, y) := (x \odot y^*) \oplus (y \odot x^*)$.
Suppose that I is an ideal of an MV-algebra A. Define $x \sim_I y$ if and only if $d(x, y) \in I$ if and only if $x \odot y^* \in I$ and $y \odot x^* \in I$. Then $\sim_I$ is a congruence relation on A. The set of all congruence classes is denoted by A/I then $A/I = \{[x] : x \in A\}$, where $[x] = \{y \in A : x \sim_I y\}$. We can easily to see that $x \in I$ if and only if $x/I = 0/I$. The MV-algebra operations on A/I given by $x/I \oplus y/I = (x \oplus y)/I$ and $(x/I)^* = x^*/I$, are well defined. Hence $(A/I, \oplus, *, [0])$ becomes an MV-algebra [5, 15].

Definition 1.7. [5, 15] *An MV-algebra A is simple, if A is nontrivial and $\{0\}$ is its only proper ideal.*

2. Obstinate ideals in MV-algebras

Form now on $(A, \oplus, *, 0, 1)$ or simply A is an MV-algebra.

Definition 2.1. *A proper ideal of A is called an obstinate ideal of A if $x, y \notin I$ imply $x \odot y^* \in I$ and $y \odot x^* \in I$, for all $x, y \in A$.*

Example 2.1. *Let $\Omega = \{1, 2\}$ and $\mathcal{A} = P(\Omega) = \{\{1\}, \{2\}, \{1, 2\}, \emptyset\}$, which is an MV-algebra with operations $\oplus = \cup$, $\odot = \cdot = \cap$ and $A^* = \Omega - A$, for any $A \in \mathcal{A}$. It is clear that $I_1 = \{\emptyset, \{1\}\}$ and $I_2 = \{\emptyset\}$ are ideals of $\mathcal{A}$. I_1 is an obstinate ideal of $\mathcal{A}$. $\{1, 2\} \odot \{2\}^* = \{1, 2\} \cap \{1\} = \{1\} \in I_1$ and $\{2\} \odot \{1, 2\}^* = \{2\} \cap \emptyset = \emptyset \in I_1$. It follows that I_1 is an obstinate ideal of $\mathcal{A}$. But $I_2 = \{\emptyset\}$ is not an obstinate ideal of $\mathcal{A}$. In fact $\{1\} \odot \{2\}^* = \{1\} \cap \{1\} = \{1\} \notin I_2$ and $\{2\} \odot \{1\}^* = \{2\} \cap \{2\} = \{2\} \notin I_2$, where $\{1\}$, $\{2\} \notin I_2$.*

Example 2.2. *Let $A = \{0, a, b, 1\}$, where $0 < a, b < 1$. Define $\odot$, $\oplus$ and $*$ as follows:*

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\oplus$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

$*$	0	a	b	1
	1	b	a	0

Then $(A, \oplus, \odot, *, 0, 1)$ *is an* MV*-algebra, it is clear that* $I_1 = \{0, a\}$ *and* $I_2 = \{0, b\}$ *are obstinate ideals of* A.

Example 2.3. *Let* $A = \{0, a, b, c, d, 1\}$*, where* $0 < a, b < c < 1$ *and* $0 < b < d < 1$. *Define* $\oplus$*,* $\odot$ *and* $*$ *as follows:*

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	c	c	1	1
b	b	c	d	1	d	1
c	c	c	1	1	1	1
d	d	1	d	1	d	1
1	1	1	1	1	1	1

$*$	0	a	b	c	d	1
	1	d	c	b	a	0

Then $(A, \oplus, \odot, *, 0, 1)$ *is an* MV*-algebra and it is clear* $I = \{0, a\}$ *is an ideal of* A *but it is not an obstinate ideal of* A. $d, c \notin I$ *and* $d \odot c^* = d \odot b = b \notin I$.

In the following proposition, we give a necessary and sufficient condition on a proper ideal to be an obstinate ideal.

Proposition 2.1. *A proper ideal* I *of* A *is an obstinate ideal if and only if for any* $x \in A$ *if* $x \notin I$ *then there exists* $n \geq 1$ *such that* $nx^* \in I$.

Proof. Suppose that I is an obstinate proper ideal and $x \notin I$. Since $1 \notin I$, then $0 = x \odot 1^* \in I$ and $x^* = 1 \odot x^* \in I$. So $nx^* \in I$, for $n = 1$.
Conversely, let $x, y \notin I$. We show that $x \odot y^* \in I$ and $y \odot x^* \in I$.
By hypothesis $nx^* \in I$ and $my^* \in I$, for some $n, m \geq 1$. We know that $x^* \leq nx^*$ and $y^* \leq ny^*$. By ideal property $x^* \in I$ and $y^* \in I$. Since $y \odot x^* \leq x^*$ and $x \odot y^* \leq y^*$, then $x \odot y^* \in I$ and $y \odot x^* \in I$. □

Theorem 2.1. *Let* I *be an obstinate ideal of* A. *Then* I *is a maximal ideal of* A.

Proof. Let I be an obstinate ideal which is not a maximal. So there exists a proper ideal J such that $I \subset J$. Suppose that $a \in J - I$. Then by Proposition 2.5, $na^* \in I$ for some $n \geq 1$. We know that $a^* \leq na^*$. By the ideal property $a^* \in I$ and also $a^* \in J$. Since $a \in J$, $a \oplus a^* = 1 \in J$, which is a contradiction. □

In the following example we show that the converse of the above theorem may not hold.

Example 2.4. *In Example* 2.4*, we can check that* $I = \{0, a\}$ *is a maximal ideal. But* I *is not an obstinate ideal.*

Lemma 2.1. *Let* I *be a proper ideal of* A*. Then* I *is an obstinate ideal if and only if* $x \in I$ *or* $x^* \in I$*, for all* $x \in A$*.*

Proof. Assume that I is an obstinate ideal and $x \notin I$. By Proposition 2.5, we get that $nx^* \in I$, for some $n \geq 1$. We know that $x^* \leq nx^*$, then $x^* \in I$ and obtain the result.
Conversely, let $x \notin I$. We need to show that $nx^* \in I$, for some $n \geq 1$. By hypothesis, we get that $1x^* \in I$. Hence I is an obstinate ideal of A. □

Theorem 2.2. *Let* I *be an ideal of* A*. Then the following conditions are equivalent:*
(i) *I is a maximal and Boolean ideal,*
(ii) *I is a prime and Boolean ideal,*
(iii) *I is an obstinate ideal.*

Proof. $(i) \to (ii)$ It is clear.
$(ii) \to (iii)$ Let I be a prime and Boolean ideal. Then we have $x \wedge x^* \in I$, for any $x \in A$. Since P is prime, it follows that $x \in I$ or $x^* \in I$. By Lemma 2.8, I is an obstinate ideal.
$(iii) \to (i)$ Let I be an obstinate ideal. By Theorem 2.6, I is a maximal ideal. On the other hand, by Lemma 2.8, we deduce that $x \in I$ or $x^* \in I$. Also we know that $x \wedge x^* \leq x$ and $x \wedge x^* \leq x^*$, then $x \wedge x^* \in I$. Hence I is a Boolean ideal. □

Theorem 2.3. *Any simple* MV*-algebra* $A \neq \{0, 1\}$ *has no obstinate ideal.*

Proof. Let A be simple MV-algebra. Then $I = \{0\}$ is its only proper ideal, if I is an obstinate ideal and $x, y \notin I$ such that $x \neq y$. Hence $x \odot y^* \in I$ and $y \odot x^* \in I$. On the other hand $x \odot y^* = 0$ and $y \odot x^* = 0$, by Lemma 1.2, we have $x \leq y$ and $y \leq x$. This results $x = y$, which is a contradiction. Hence A has not obstinate ideal. □

Example 2.5. *In Example* 2.4*, we have* $I_1 = \{0, a\}$ *and* $I_2 = \{0, b, d\}$ *are ideals of* A *but* I_1 *is not an obstinate ideal. Since* $d \odot c^* = d \odot b = b \notin I_1$ *and* I_2 *is an obstinate ideal of* A*.*

Corollary 2.1. (*Extension property for obstinate ideal*) *Suppose that* I *and* J *are two proper ideals such that* $I \subseteq J$*. If* I *is an obstinate ideal, then* J *is also an obstinate ideal.*

Proof. Let I be an obstinate ideal and $I \subseteq J$. Then by Theorem 2.6, I is a maximal ideal. Since J is a proper ideal, we get that $I = J$. Hence J is an obstinate ideal. □

Remark 2.1. *Let* I *and* J *be ideals of* A*. We have*

$$I \vee J = (I \cup J] = \{a \in A : a \leq b \oplus c, \quad \text{for some } b \in I \text{ and } c \in J\}.$$

It is an ideal of A*,* [5, 15]*. If* I *or* J *is an obstinate ideal, then by Corollary* 2.12*, we get that* $I \vee J$ *is an obstinate ideal.*

Lemma 2.2. $\{0\}$ *is an obstinate ideal of* A *if and only if every ideal* I *of* A *is an obstinate ideal.*

Proof. Suppose that I is an arbitrary ideal of A. Since $\{0\} \subseteq I$ and $\{0\}$ is an obstinate, then by Corollary 2.12, I is an obstinate. Conversely, it is clear. □

Corollary 2.2. *Let* I *be an obstinate ideal of* A. *Then* A/I *is a Boolean algebra.*

Proof. Using Theorem 2.6, we obtain I is a maximal ideal of A. Also, [7, Theorem 4.9], we deduce that A/I is a Boolean algebra. □

Corollary 2.3. *The following are equivalent:*
(*i*) *A is a Boolean algebra,*
(*ii*) *Every ideal is obstinate ideal,*
(*iii*) $\{0\}$ *is an obstinate ideal.*

We recall that if I is a maximal ideal of A, then A/I is a locally finite MV-algebra [15].

Lemma 2.3. *If* $\{0\}$ *is an obstinate ideal of* A, *then* A *is a locally finite* MV*-algebra.*

Proof. Suppose that $\{0\}$ is an obstinate ideal of A. It follows that from Theorem 2.6, $\{0\}$ is a maximal ideal of A. Hence $A/\{0\} \simeq A$ is a locally finite. □

In the following example, we show that the converse of the above lemma is not true.

Example 2.6. *Let* $A = \{0, 1, 2\}$, *where* $0 < 1 < 2$. *Define* $\odot$, $\oplus$ *and* $*$ *as follows:*

$\odot$	0	1	2
0	0	0	0
1	0	0	1
2	0	1	2

$\oplus$	0	1	2
0	0	1	2
1	1	2	2
2	2	2	2

$*$	0	1	2
	2	1	0

Then $(A, \oplus, *, 0, 2)$ *is a locally finite* MV*-algebra but* $I = \{0\}$ *is not an obstinate ideal of* A. *Since* $2 \odot 1^* = 2 \odot 1 = 1 \notin I$.

We recall that An MV-algebra A is said to be semisimple if and only if non-trivial and $Rad(A) = \{0\}$ [5].

Corollary 2.4. *If* $\{0\}$ *is an obstinate ideal of* A, *then* A *is a semisimple .*

Proof. Since $\{0\}$ is an obstinate ideal of A, it follows that from Theorem 2.6, $\{0\}$ is a maximal ideal of A. Hence $Rad(A) = \{0\}$. Thus A is a semisimple MV-algebra. □

In the following example, we show that the converse of the above corollary may not hold.

Example 2.7. *In Example* 2.3, *we have* $I_1 = \{0, a\}$ *and* $I_2 = \{0, b\}$ *are ideals of* A *such that* $Rad(A) = I_1 \cap I_2 = \{0\}$. *Hence* A *is a semisimple* MV*-algebra but* $I = \{0\}$ *is not an obstinate ideal of* A. *Since* $a \odot b^* = a \odot a = a \notin I$ *and* $b \odot a^* = b \odot b = b \notin I$.

Theorem 2.4. *Let* I *be an ideal of* A. *Then* I *is an obstinate ideal if and only if every ideal of* A/I *is an obstinate ideal.*

Proof. Assume that I is an obstinate ideal of A. Let $x/I \notin \{[0]\}$, from Lemma 2.8, it is suffices to show $(x/I)^* \in \{[0]\}$. Since $x/I \notin \{[0]\}$, $x/I \neq 0/I$, hence $d(x,0) \notin I$, then $x \notin I$. We apply the hypothesis and obtain $x^* \in I$, then $x^* = d(x^*,0) \in I$. On the other hand $x^*/I = 0/I$ or $(x/I)^* = x^*/I \in \{[0]\}$. Hence $\{[0]\}$ is an obstinate ideal of A/I. By Theorem 2.6, we deduce that $\{[0]\}$ is a maximal ideal of A/I. Hence every ideal of A/I is an obstinate ideal.
Conversely, assume that every ideal of the quotient algebra A/I is an obstinate ideal and $x \in A$ such that $x \notin I$.
We must show that $x^* \in I$. By hypothesis, we get that $x/I \neq 0/I$, hence $x/I \notin \{[0]\}$. Since $\{[0]\}$ is an ideal of A/I. By hypothesis, $\{[0]\}$ is an obstinate ideal. Therefore $(x/I)^* = x^*/I \in \{[0]\}$, then $x^*/I = 0/I$. So $x^* \in I$. Hence I is an obstinate ideal. □

Remark 2.2. *We describe the notion of $A/\{0\}$. Let $[x]$ be an arbitrary element of $A/\{0\} = \{[x] : x \in A\}$. We have*

$$\begin{aligned}
[x] &= \{y \in A : x \sim_{\{0\}} y\}, \\
&= \{y \in A : d(x,y) \in \{0\}\} \\
&= \{y \in A : x \odot y^* \in \{0\} \textit{ and } y \odot x^* \in \{0\}\}, \\
&= \{y \in A : x \leq y \textit{ and } y \leq x\}, \\
&= \{y \in A : x = y\}, \\
&= \{x\}.
\end{aligned}$$

Define $f \in Hom(A, A/\{0\})$ such that $f(x) = [x]$, for any $x \in A$. Clearly $A/\{0\} \cong A$.

The following example shows that the MV-homomorphic image of an obstinate ideal is not even an ideal.

Example 2.8. *In Example 2.3, consider MV-homomorphism $f : A \to A$ such that $f(0) = 0$, $f(a) = 1$, $f(b) = 0$ and $f(1) = 1$. It is clear $I = \{0, a\}$ is an obstinate ideal of A, while $f(I) = \{0, 1\}$ is not an ideal of A.*

In the following theorem, we study inverse image of an obstinate ideal under a MV-homomorphism.

Theorem 2.5. *Let $f : A \to B$ be a MV-homomorphism and I be an obstinate ideal of B. Then inverse image of I is an obstinate ideal of A.*

Proof. Let I be an obstinate ideal of B and $x \in A$ such that $x \notin f^{-1}(I)$. Then $f(x) \notin I$, since I is an obstinate ideal of B, by Lemma 2.8, we deduce that $[f(x)]^* \in I$. We deduce that $f(x^*) \in I$. Then we get that $x^* \in f^{-1}(I)$. Hence $f^{-1}(I)$ is an obstinate ideal of A. □

3. The relations obstinate ideal and the other ideals

Theorem 3.1. *If P is an obstinate ideal of A, then P is a primary ideal of A.*

Proof. Let $a \odot b \in P$ such that for every $n \geq 1$, $a^n \notin P$. Since P is an obstinate ideal and we have $a \notin P$ and $1 \notin P$, so $1 \odot a^* \in P$ and $a \odot 1^* \in P$. Hence $a^* \in P$, for every $n \geq 1$. From ideal property, we have $a^* \oplus (a \odot b) \in P$. On the other hand $b \leq a^* \vee b \in P$. Hence $b^1 \in P$, for $n = 1$. Therefore P is a primary ideal of A. □

The following example shows that a primary ideal may not be an obstinate ideal.

Example 3.1. *Let $A = \{0, a, b, c, d, 1\}$. where $0 < a, c < d < 1$ and $0 < a < b < 1$. Define $\odot$, $\oplus$ and $*$ as follows:*

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	0	0	a
b	0	a	b	0	a	b
c	0	0	0	c	c	c
d	0	0	a	c	c	d
1	0	a	b	c	d	1

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	b	b	d	1	1
b	b	b	b	1	1	1
c	c	d	1	c	d	1
d	d	1	1	d	1	1
1	1	1	1	1	1	1

$*$	0	a	b	c	d	1
	1	d	c	b	a	0

Then $(A, \oplus, \odot, *, 0, 1)$ is an MV-algebra. It is clear that $I = \{0, c\}$ is a maximal ideal, then I is a prime ideal, by Lemma 1.5, we deduce that I is a primary ideal but I is not an obstinate ideal, if $a, b \notin I$, we have $b \odot a^* = b \odot d = a \notin I$.

Theorem 3.2. *Let I be a primary ideal and quasi-implicative ideal of A. Then I is an obstinate ideal.*

Proof. By Lemma 2.8, it is sufficient to show that $x \in I$ or $x^* \in I$, for any $x \in A$. We have $0 = x \odot x^* \in I$, since I is a primary ideal, so $x^n \in I$ or $(x^*)^n \in I$, for some integer $n \geq 1$. Since I is a quasi-implicative ideal, then $x \in I$ or $x^* \in I$. Hence I is an obstinate ideal. □

Corollary 3.1. *Let M be a maximal and quasi-implicative ideal of A. Then M is an obstinate ideal of A.*

Proof. Since M is a maximal ideal, then M is a prime ideal of A. By Lemma 1.5, we deduce that M is a primary ideal. Also by Theorem 3.3, we implies that M is an obstinate ideal of A. □

Proposition 3.1. *Let A be a prefect MV-algebra and I be a quasi-implicative ideal of A. Then I is an obstinate ideal of A.*

Proof. Let I be a quasi-implicative ideal of A. Since A is a prefect MV-algebra, I is a prefect ideal of A. By Lemma 2.8, it is sufficient to show that $x \in I$ or $x^* \in I$, for any $x \in A$. Suppose that $x \notin I$, since I is a quasi-implicative ideal of A, then $(x^n) \notin I$, for every $n > 1$. Since I is a prefect, $(x^*)^m \in I$, for some integer $m \geq 1$. Hence since I is a quasi-implicative ideal, $x^* \in I$. □

Theorem 3.3. *Let A be a Boolean algebra and $m \in Mol(A)$. Then $(m]^{\perp}$ is an obstinate ideal.*

Proof. Suppose that $m \in Mol(B(A))$ and $x \notin (m]^{\perp}$ and $x^* \notin (m]^{\perp}$ for any $x \in B(A)$. Hence $0 < x \wedge m$ and $0 < x^* \wedge m \leq m$, hence $0 < (x \wedge x^*) \wedge m$ which is a contradiction because by Lemma 1.2, we have $(x \wedge x^*) \wedge m = (x \odot x^*) \wedge m = 0$. □

Theorem 3.4. *If $(m]^{\perp}$ is an obstinate ideal, then $m \in Mol(A)$.*

Proof. Suppose that $(m]^{\perp}$ is an obstinate ideal. By Theorem 2.6, we deduce that $(m]^{\perp}$ is a maximal ideal, hence $(m]^{\perp}$ is a prime ideal. Let $x, y \in A$ satisfy $0 < x, y \leq m$. Then $x = x \wedge m \neq 0$ and $y = y \wedge m \neq 0$, that is $x \notin (m]^{\perp}$ and $y \notin (m]^{\perp}$. This means that $x \wedge y \notin (m]^{\perp}$, hence $x \wedge y \wedge m \neq 0$. Thus $x \wedge y \neq 0$. Therefore $m \in Mol(A)$. □

Theorem 3.5. *Any obstinate ideal of A is an implicative ideal of A.*

Proof. Let I be an obstinate ideal of A. Suppose that $z \odot (y^* \odot x^*) \in I$ and $y \odot x^* \in I$, for any $x, y, z \in A$ but $z \odot x^* \notin I$, by contrary. Since I is an obstinate, by Lemma 2.8, we deduce that $z^* \oplus x = (z \odot x^*)^* \in I$. It follows from Lemma 1.2 (4), $x \leq z^* \oplus x \in I$, so $x \in I$. By ideal properties and by hypothesis, we have $y \leq x \vee y = x \oplus (y \odot x^*) \in I$, so $y \in I$. Also by hypothesis, we deduce that $z \odot x^* \leq (z \odot x^*) \vee y = [(z \odot x^*) \odot y^*] \oplus y \in I$. Thus $z \odot x^* \in I$, which is a contradiction. Therefore I is an implicative ideal of A. □

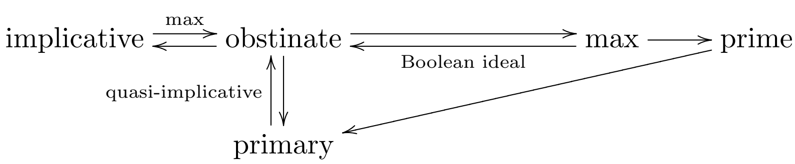
Corollary 3.2. *If I is an implicative and maximal ideal, then I is an obstinate ideal.*

Proof. By [7, Theorem 5.1], we implies that M is a maximal and quasi-implicative ideal. It follows that from Corollary 3.4, I is an obstinate ideal. □

The following example shows that an implicative ideal may not be an obstinate ideal.

Example 3.2. *In Example* 2.2*, consider $I = \{\emptyset\}$. It is clear that I is an implicative ideal but is not obstinate ideal. Since $\{1\} \odot \{2\}^* = \{1\} \cap \{1\} = \{1\} \notin I$ and $\{2\} \odot \{1\}^* = \{2\} \cap \{2\} = \{2\} \notin I$, where $\{1\}, \{2\} \notin I$.*

In the following diagram we show that the relationships between an obstinate ideal and the other ideals in MV-algebra are described.



4. **Conclusion and future research**

MV-algebras were originally introduced by Chang in [4] in order to give an algebraic counterpart of the Łukasiewics many valued logic.
In this paper, we introduced the notion of an obstinate ideal in MV-algebras.
We have also presented several characterizations and many important properties of obstinate ideals in MV-algebras. We proved that obstinate ideals are Boolean ideals and I is an obstinate ideal if and only if I is a maximal and Boolean ideal. Hence

if I is an obstinate ideal of A, then A/I is a Boolean algebra.
Also we studied relations between an obstinate ideal and the other ideals of an MV-algebra.
In our future work, we will try to define other types of ideals in MV-algebras and other logical algebraic structures.

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