FAMILIES OF ITERATED FUNCTION SYSTEMS. CONVERGENCES PROPERTIES OF THE ASSOCIATED ATTRACTORS AND FRACTAL MEASURES

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In this paper, we consider a sequence of iterated function system (I.F.S.), which is built using a finite family of contractions and a sequence of linear and contin-uous operators. We study the problem of the convergence for the sequence of attractors and fractal measures associated to the sequence I.F.S. We also study the case of vector fractal measures. Some examples are, also, provided.

Keywords: Iterated function system, attractor, fractal measure, vector measure, weak convergences of operators.

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1. Introduction

Fractal sets play, nowadays, a very important role, both in mathematics and in other domains of science. That's why they should be studied from more and more directions.

In this paper, we discuss about fractal sets which are obtained as attractors of a, (so called), iterated function system (I.F.S.). These attractors are, in fact, fixed points of some contractions (obtained using the I.F.S.), defined and taking values on the class $\mathcal{K}(T)$ of the compact and nonempty subsets of a complete metric space (T, d), where, on $\mathcal{K}(T)$, we consider the Hausdorff-Pompeiu metric.

The problem that we study in this paper is: being given a sequence of I.F.S., having, each of them, the associated attractor $K_n \in \mathcal{K}(T)$, such that the sequence of I.F.S. converges (in some sense) to an I.F.S. with the associated attractor K, it is true that K_n converges to K, in the Hausdorff-Pompeiu metric?

More than that, being given an I.F.S. (considering this time, (T, d) compact), one can consider an operator (called the Markov operator) on the set of normalized Borel measures on T. This operator is itself a contraction and it has an unique fixed point, called fractal measure (or Hutchinson measure). It is important to note that the support of this measure is exactly the attractor of the I.F.S. (see [9]). In [4], using [1], [2], [3] it was proved the existence of a fractal measure in the case of vector measures (taking values in a normed vector space).

One can ask the question: given a sequence of I.F.S., with the associated fractal measures μ_n, such that it converges to an I.F.S. with the associated fractal measure μ, it is true that μ_n converges to μ?

This paper gives the answer to these questions. We have to mention that the idea for this research was given by [9] (section 3.4.). Other papers in which similar topics were studied are [5] and [7].

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2. Preliminary facts

2.1. The Hausdorff-Pompeiu metric. The attractor associated to an iterated function system. In this section, we will present without proofs some results that we will use throughout the paper. We mention that this results were briefly presented in [7] and [8]. For more details and for some of the proofs of the results, one can consult [1], [2], [3], [4], [6] and [9].

Let (T,d) be a metric space. We denote by $\mathcal{P}^*(T)$ the family of non-empty and bounded subsets of T. For any $x \in T$ and $A \in \mathcal{P}^*(T)$ we will denote: $d(x,A) = \inf\limits_{y\in A} d(x,y)$. If $A, B \in \mathcal{P}^*(T)$ we define $d(A,B) = \sup\limits_{x\in A} d(x,B)$. In a similar way, we define $d(B,A) = \sup\limits_{y\in B} d(y,A)$. Now, we denote:
$\delta(A,B) = \max\{d(A,B), d(B,A)\}$. Let us define

$$\mathcal{K}^*(T) = \{K \subset T | K \text{ is compact and non-empty}\}.$$

Proposition 2.1. *i)* $\delta : \mathcal{K}^*(T) \times \mathcal{K}^*(T) \to [0,\infty)$ *is a metric on* $\mathcal{K}^*(T)$ *called* the Hausdorff-Pompeiu metric.*;*

ii) If $\omega : T \to T$ *is a Lipschitz function , then* $\delta(\omega(A), \omega(B)) \leq L \cdot \delta(A,B)$*,* L *being the Lipschitz constant of* ω*;*

iii) if $(A_i)_{1\leq i\leq n}, (B_i)_{1\leq i\leq n} \subset \mathcal{K}^*(T)$*, then* $\delta\left(\bigcup\limits_{i=1}^{n} A_i, \bigcup\limits_{i=1}^{n} B_i\right) \leq \max\limits_{1\leq i\leq n} \delta(A_i, B_i)$.

Proposition 2.2. *i) If* (T,d) *is complete, then* $(\mathcal{K}^*(T), \delta)$ *is also complete;*

ii) If (T,d) *is compact,* $(\mathcal{K}^*(T), \delta)$ *is also compact.*

Definition 2.1. *Let* (T,d) *be a complete metric space and* $(\omega_i)_{1\leq i\leq n}$*,*
$\omega_i : T \to T$*,* $i = \overline{1,n}$ *such that any* ω_i *is a contraction of ratio* $r_i \in [0,1)$*. The family* $(\omega_i)_{1\leq i\leq n}$ *is called* iterated function system *(I.F.S.).*

Definition 2.2. *If* $(\omega_i)_{1\leq i\leq n}$ *is a I.F.S. on the complete metric space* (T,d)*, we define* $S : \mathcal{K}^*(T) \to \mathcal{K}^*(T), S(E) = \bigcup\limits_{i=1}^{n} \omega_i(E), \forall E \in \mathcal{K}^*(T)$.

Proposition 2.3. *The function* S *above defined is a contraction of ratio* $r \leq \max\limits_{1\leq i\leq n} r_i$*. Hence, using the contraction principle, we deduce that there is an unique set* $K \in \mathcal{K}^*(T)$ *such that* $K = S(K)$*.*

Definition 2.3. *The set* K *introduced by Proposition* **2***.3 is called* the attractor *(or:* the fractal*) associated to the I.F.S.* $(\omega_i)_{1\leq i\leq n}$.

2.2. The Hutchinson metric on the positive normalized Borel measures. The Markov operator and the Hutchinson measure. Let (T,d) be a compact metric space and $\mathcal{B}$ its Borel subsets.

$\mathbb{B} = \{\mu : \mathcal{B} \to [0,1] | \mu(T) = 1\}$ (the family of *positive normalized Borel measures*). Let also define $\mathrm{Lip}_1(T) = \{f : T \to \mathbb{R} | |f(x) - f(y)| \leq d(x,y)\}$ and

$$d_H : \mathbb{B} \times \mathbb{B} \to \mathbb{R}, d_H(\mu,\nu) = \sup_{f\in \mathrm{Lip}_1(T)} \left\{ \left| \int f d\mu - \int f d\nu \right| \right\}.$$

Proposition 2.4. *i)* d_H *is a metric on* $\mathbb{B}$ *called* the Hutchinson metric*;*

ii) $(\mathbb{B}, d_H)$ *is a compact metric space.*

We consider an I.F.S. $(\omega_i)_{1\leq i\leq n}$ *and the positive numbers* $(p_i)_{1\leq i\leq n}$ *such that* $\sum\limits_{i=1}^{n} p_i = 1$*. Now, we can define the Markov operator* M *associated to the I.F.S.* $(\omega_i)_{1\leq i\leq n}$*, acting via* $\mu \mapsto M(\mu)$*, where* $M(\mu)(A) = \sum\limits_{i=1}^{n} p_i \mu(\omega_i^{-1}(A))$*, for any* $\mu \in \mathbb{B}$ *and for any* $A \in \mathcal{B}$*.*

Proposition 2.5. *i) For any $\mu \in \mathbb{B}$, $M(\mu) \in \mathbb{B}$;*

ii) $M : \mathbb{B} \to \mathbb{B}$ is a contraction of ratio $r \leq \max_i r_i$ (r_i being the ratio of the contraction ω_i);

iii) There exists an unique measure $\mu^* \in \mathbb{B}$ such that $M(\mu^*) = \mu^*$;

iv) The support of μ^* is the attractor associated to the I.F.S. $(\omega_i)_{1\leq i\leq n}$.

Definition 2.4. *The measure μ^* defined by Proposition* **2***.5 is called* the Hutchinson measure *(or* the fractal measure*) associated to the I.F.S. $(\omega_i)_{1\leq i\leq n}$ and to the numbers $(p_i)_{1\leq i\leq n}$.*

2.3. **Fractal vector measures.** Let $(X, \langle\,,\rangle)$ be a Hilbert space, (T, d) a compact metric space and we denote, as before, by $\mathcal{B}$ the Borel subsets of T. We consider a σ-additive measure $\mu : \mathcal{B} \to X$. For any Borel set $A \subset T$, *the variation of μ on A* is: $|\mu|(A) \overset{def}{=} \sup\left\{\sum_i \|\mu(A_i)\|\right\}$, where the supremum is computed with respect to all the partitions of A with finite families of Borel sets.

If $|\mu|(T) < \infty$ we say that *the measure has bounded variation.* In this case, denoting $\|\mu\| = |\mu|(T)$, the application $\|\cdot\|$ is a norm on the vector space:

$$\mathrm{cabv}(X) = \{\mu : \mathcal{B} \to X | \mu \text{ is } \sigma\text{-additive and } |\mu|(T) < \infty\}.$$

It can be proved that $(\mathrm{cabv}(X), \|\cdot\|)$ is a Banach space.

Now, we denote $\mathcal{L}(X) = \{R : X \to X | R \text{ is linear and continuous}\}$. Let $N \in \mathbb{N}^*$; for any $i \in \{1, \ldots, N\}$, we consider the contraction $\omega_i : T \to T$, with its ratio r_i and $R_i \in \mathcal{L}(X)$. One can define the following operator, denoted by H, via: $H(\mu) = \sum_{i=1}^{N} R_i(\mu(\omega_i^{-1}))$, (this means: $H(\mu)(A) = \sum_{i=1}^{N} R_i(\mu(\omega_i^{-1}(A)))$,for any $A \in B$ and $\mu \in \mathrm{cabv}(X)$).

It can be proved that for any $\mu \in \mathrm{cabv}(X), H(\mu) \in \mathrm{cabv}(X)$ and $\|H\| \leq \sum_{i=1}^{N} \|R_i\|_o$ ($\|\cdot\|_o$ being the operatorial norm on $\mathcal{L}(X)$).

We recall now the following notations, definitions and results:

1) $C(X) = \{f : T \to X | f \text{ is continuous}\}$;
2) $S(X) = \{f : T \to X | f \text{is a simple function}\}$;
3) For any $a > 0$, fixed, $B_a(X) = \{\mu \in \mathrm{cabv}(X) | \|\mu\| \leq a\}$;
4) $BL(X) = \{f : T \to X | f \text{ is a Lipschitz function}\}$;
5) $TM(X) = \{f : T \to X \mid f$ is totally measurable, that is, $\exists (f_n)_{n\geq 1} \subset S(X)$ such that $f_n \xrightarrow{u} f\}$.

Lemma 2.1. *Let $f \in BL(X)$; we denote by $\|f\|_{BL} = \|f\|_\infty + \|f\|_L$, where $\|f\|_L$ is the Lipschitz constant of f. Then, $\|\cdot\|_{BL}$ is a norm on $BL(X)$.*

Now, we can define

6) $BL_1(X) = \{f \in BL(X) \mid \|f\|_{BL} \leq 1\}$;

Definition 2.5. *Let $f \in S(X)$, $f = \sum_{i=1}^{m} \varphi_{A_i} x_i$, where $(A_i)_{1\leq i\leq m}$ is a partition of T with Borel sets and φ_{A_i} is the characteristic function of $A_i, x_i \in X$. The number $\sum_{i=1}^{m} \langle x_i, \mu(A_i)\rangle$ is called* the integral of f with respect to μ *and is denoted by $\int f d\mu$ (it is easy to prove that the value of the integral doesn't depend on the representation of f).*

Definition 2.6. *If $f \in TM(X)$, we define $\int f d\mu = \lim_{n\to\infty} \left(\int f_n d\mu\right), (f_n)_{n\geq 1}$ being a sequence of simple functions which converges uniformly to f (one can prove that this integral doesn't depend on the sequence $(f_n)_{n\geq 1}$, uniformly convergent to f).*

Lemma 2.2. *a) The application* $\|\cdot\|_{MK}$: cabv$(X) \to [0,\infty)$ *defined by* $\|\mu\|_{MK} = \sup\{|\int f d\mu|, f \in BL_1(X)\}$ *is a norm on* cabv(X)*, called* the Monge-Kantorovich type norm*;*

b) Let $a > 0$*. If* $X = K^n$ *(where* $K = \mathbb{R}$ *or* $\mathbb{C}$*), then the topology generated on* $B_a(K^n)$ *by* $\|\cdot\|_{MK}$ *is the same with the weak-∗ topology;*

c) $B_a(K^n)$ *equipped with the metric generated by* $\|\cdot\|_{MK}$ *is a compact metric space.*

Lemma 2.3 (Change of variable formula)**.** *For any* $f \in C(X)$ *and* H *as before, we have:* $\int f dH(\mu) = \int g d\mu$*, where* $g = \sum_{i=1}^{N} R_i^* \circ f \circ \omega_i$ *(*R_i^* *being the adjoint of* R_i*).*

Theorem 2.1. *a) Let us consider* $f \in BL_1(K^n)$ *and* $g = \sum_{i=1}^{N} R_i^* \circ f \circ \omega_i$*, as in Lemma* **2***.3. Then* g *is a Lipschitz function and* $\|g\|_L \le \sum_{i=1}^{N} \|R_i\|_o r_i$*;*

b) Let H *be as before. We consider the normed vector space* (cabv(K^n), $\|\cdot\|_{MK}$)*. Then,* $H \in \mathcal{L}(\text{cabv}(K^n))$ *and* $\|H\|_o \le \sum_{i=1}^{N} \|R_i\|(1 + r_i)$*.*

Theorem 2.2. *Let us suppose that the hypothesis of Theorem* **2***.1 are fulfilled and* $\sum_{i=1}^{N} \|R_i\|_o(1+r_i) < 1$*. Let* $a > 0$*,* $\mu^0 \in$ cabv(K^n)*; we define* $P :$ cabv$(K^n) \to$ cabv(K^n)*,* $P(\mu) = H(\mu)+\mu^0$*. Let, also,* $A \subset B_a(K^n)$*, non-empty, weak-∗ close, such that* $P(A) \subset A$*. We denote by* P_0 *the restriction of* P *to* A*. Then there is a unique measure* $\mu^* \in A$ *such that* $P_0(\mu^*) = \mu^*$*. If* $\mu^0 = 0$ *(the zero-measure) then* $\mu^* = 0$*.*

Definition 2.7. *The measure* μ^* *introduced by Theorem* **2***.2 is called* the Hutchinson vector measure *(or the fractal vector measure).*

Remark 2.1. *For the proofs of the results in sections 2.1 and 2.2. one can see* [9]*. For the proofs of lemmas* **2***.1 and* **2***.2 one can consult* [2] *(respectively* [3]*) and for the proofs of lemma* **2***.3 (respectively theorems* **2***.1 and* **2***.2) one can consult* [6] *(respectively* [4] *and* [8]*). For more details about the definitions* **2***.5 and* **2***.6 consult the paper* [2].

3. Results

3.1. The general framework. Let X, Y Banach spaces and $\omega : Y \to X$ a contraction of ratio r. Let, also, $(T_n)_{n\ge1} \subset \mathcal{L}(X, Y)$ such that $\alpha \overset{not}{=} \sup_{n\ge1} \|T_n\|_o < \frac{1}{r}$. For any $n \ge 1$ we consider the operators $U_n : Y \to Y, U_n \overset{def}{=} T_n \circ \omega$.

The following two lemmas were proved in [7].

Lemma 3.1. *For any* n*,* U_n *is a contraction of ratio less or equal to* $\alpha \cdot r$*.*

Lemma 3.2. *Let us suppose that there exists* $T \in \mathcal{L}(X, Y)$ *such that* $T_n \xrightarrow{\|\cdot\|_o} T$*. Then, for any* $K \in \mathcal{K}^*(Y), U_n(K) \xrightarrow{\delta} U(K)$*, where we denoted:* $U = T \circ \omega$*.*

Remark 3.1. *In the proof of lemma* **3***.2, for an arbitrarily and fixed* $\varepsilon > 0$*, we find a rank* N_0 *such that for any* $n \ge N_0$*,* $\delta(U_n(K), U(K)) \le \varepsilon$*. This rank depends not only on* ε*, but also on* K*. However, if we take* $Y_0 \subset Y$*, compact, such that* $U_n(Y_0) \subset Y_0$ *and* $U(Y_0) \subset Y_0$*, denoting again by* U_n *and* U *the restrictions of these functions on* Y_0*, it is easy to prove (see* [7]*) that* N_0 *depends only on* ε*.*

Let now $(\omega_j)_{1\le j\le m}$, $\omega_j : Y_0 \to X$ be contractions of ratio r_j, Y_0 being a compact and non-empty subset of a Banach space Y. We denote $r = \max_i r_i$. Let us consider T_n, $T \in$

$\mathcal{L}(X,Y)$ such that $\alpha \overset{not}{=} \sup_n \|T_n\|_o < \frac{1}{r}$ and $T_n \xrightarrow{\|\cdot\|_o} T$. We denote $U_j^n = T_n \circ \omega_j$, $U_j = T \circ \omega_j$ and we will suppose, as before, that $U_j^n(Y_0) \subset Y_0$, $U_j(Y_0) \subset Y_0$. Using lemma **3**.1, we have that the functions $U_j^n : Y_0 \to Y_0$ and $U_j : Y_0 \to Y_0$ are contractions of ratios less or equal by αr. [Here, if Y is finite dimensional, we can take $Y_0 = B[0,R], R \geq \frac{\alpha\beta}{1-\alpha r}, \beta = \max_j \|\omega_j(0)\|$]. We can deduce that $(U_j^n)_j$ is an I.F.S. on $\mathcal{K}^*(Y_0)$. Y_0 being compact in the Banach space Y, it results that Y_0 is a complete metric space (with respect to the metric given by the restriction on Y_0 of the norm on Y). Consequently, (proposition **2**.2), $\mathcal{K}^*(Y_0)$ is complete. Hence (proposition **2**.3) there exists an unique set $K_n \in \mathcal{K}^*(Y_0)$ such that $K_n = \bigcup_{j=1}^m U_j^n(K_n)$ (the attractor associated to the I.F.S. $(U_j^n)_j$. Similar, $(U_j)_j$ is an I.F.S. with its attractor $K = \bigcup_{j=1}^m U_j(K)$.

The following result was also proved in [7].

Theorem 3.1. *We suppose that the above conditions are fulfilled. Then, we have:* $K_n \xrightarrow{\delta} K$.

3.2. **Convergences properties for positive Hutchinson measures.** Let us denote $\mathcal{B}$ the Borel subsets of Y_0 and $\mathbb{B} = \{\mu : \mathcal{B} \to [0,\infty) | \mu(Y_0) = 1\}$. Let, also, $p_1, \ldots, p_n \in (0,1)$ such that $\sum_{j=1}^m p_j = 1$. We define (as in the section 2.2) the Markov operator $M(\mu)$, via $M(\mu)(A) = \sum_{j=1}^m p_j \mu(\omega_j^{-1}(A))$, for any $\mu \in \mathbb{B}$ and $A \in \mathcal{B}$.

We will use the following result:

Lemma 3.3. *With the above notations, for any continuous function* $f : Y_0 \to \mathbb{R}$ *we have:*

$$\int f dM(\mu) = \sum_{j=1}^m p_j \int f \circ \omega_j d\mu.$$

(for the proof one can consult [9]*). It is also easy to prove that:*

Lemma 3.4. *For any* $\varepsilon > 0$*, there exists* $N_0 \in \mathbb{N}$ *such that for any* $n \geq N_0, x \in Y_0$ *and* $j \in \{1, \ldots, m\}$ *we have:* $\|U_j^n(x) - U_j(x)\| < \varepsilon$.

Consequence. Let N_0 given by lemma **3**.4. We have: $\max_j \sup_{x \in Y_0} \|U_j^n(x) - U_j(x)\| \leq \varepsilon, \forall n \geq N_0$.

Now let us consider $p_1, \ldots, p_m \in (0,1)$ with $\sum_{j=1}^m p_j = 1$, $M_n, M : \mathbb{B} \to \mathbb{B}, M_n(\mu) = \sum_{j=1}^m p_j \cdot \mu\left((U_j^n)^{-1}\right)$ respectively $M(\mu) = \sum_{j=1}^m p_j \cdot \mu\left((U_j)^{-1}\right)$ (the Markov operators associated to the I.F.S. $(U_j^n)_j$, respectively $(U_j)_j$). We also denote by μ_n, respectively μ the Hutchinson measures associated to M_n, respectively M.

Theorem 3.2. *With the above notations, we have:* $\lim_{n\to\infty} d_H(\mu_n, \mu) = 0$.

Proof. Let $\varepsilon > 0$, arbitrarily, fixed. We have:

$$d_H(\mu_n, \mu) = d_H(M_n(\mu_n), M(\mu)) \leq d_H(M_n(\mu_n), M_n(\mu)) + d_H(M_n(\mu), M(\mu)) \qquad (\mathbf{3}.1)$$

According proposition **2**.5 (ii), we have:

$$d_H(M_n(\mu_n), M_n(\mu)) \leq \alpha r d_H(\mu_n, \mu) \qquad (\mathbf{3}.2)$$

From (**3**.1) and (**3**.2) we deduce:

$$(1 - r\alpha) d_H(\mu_n, \mu) \leq d_H(M_n(\mu), M(\mu)) \qquad (\mathbf{3}.3)$$

For any $f : Y_0 \to \mathbb{R}$, $f \in \mathrm{Lip}_1(Y_0)$, we can write (according to lemma **3**.3):

$$\left|\int f dM_n(\mu) - \int f dM(\mu)\right| \le \sum_{j=1}^{m} p_j \left|\int (f\circ U_j^n - f\circ U_j) d\mu\right| \le$$

$$\underset{(\mu(Y_0)=1)}{\le} \sum_{i=1}^{m} p_j \sup_{x\in Y_0} |f(U_j^n(x)) - f(U_j(x))| \le \sum_{j=1}^{m} p_j \sup_{x\in Y_0} \|U_j^n(x) - U_j(x)\| \le$$

$$\le \max_j \sup_{x\in Y_0} \|U_j^n(x) - U_j(x)\| \cdot \underbrace{\left(\sum_{j=1}^{m} p_j\right)}_{=1} \le \varepsilon$$

for n big enough, using lemma **3**.4 and its consequence. Hence,

$$\sup_{f\in \mathrm{Lip}_1(Y_0)} \left|\int f dM_n(\mu) - \int f dM(\mu)\right| \le \varepsilon$$

for n big enough, that is $d_H(M_n(\mu), M(\mu)) \le \varepsilon$. It means: $\lim\limits_{n\to\infty} d_H(\mu_n, \mu) = 0$. □

3.3. **Convergence properties for fractal vector measures.** In this section, K will be $\mathbb{R}$ or $\mathbb{C}$, $N \in \mathbb{N}$. We will use now the section 2.3 and the notations from section 3.1. We denote by $\mathcal{B}$ the Borel subsets of Y_0, $\mathrm{cabv}(K^N) = \{\mu : \mathcal{B} \to K^N | \mu \text{ is } \sigma\text{-additive and } |\mu|(Y_0) < \infty\}$. Let $(R_j)_{1\le j\le m} \subset \mathcal{L}(K^N)$,

$$H^n : \mathrm{cabv}(K^N) \to \mathrm{cabv}(K^N), H^n(\mu) = \sum_{j=1}^{m} R_j \circ \mu \circ (U_j^n)^{-1},$$

that means: $H^n(\mu)(A) = \sum\limits_{j=1}^{m} R_j(\mu(U_j^n)^{-1}(A))$, for any Borel subset of K^N. Similar, let $H(\mu) = \sum\limits_{j=1}^{m} R_j \circ \mu \circ U_j^{-1}$. We consider $a > 0$, arbitrarily, fixed and $B_a(K^N) = \{\mu \in \mathrm{cabv}(K^N) | \|\mu\| \le a\}$. Let

$$P^n : \mathrm{cabv}(K^N) \to \mathrm{cabv}(K^N), P^n(\mu) = H^n(\mu) + \mu^0,$$

and

$$P : \mathrm{cabv}(K^N) \to \mathrm{cabv}(K^N), P(\mu) = H(\mu) + \mu^0,$$

where $\mu^0 \in \mathrm{cabv}(K^N)$ is arbitrarily and fixed. We will suppose that there exists $A \subset B_a(K^N)$, weak-$*$ close such that $P^n(A) \subset A$ and $P(A) \subset A$. For example, if $\|H\|_o \le \left(\sum\limits_{j=1}^{m} \|R_j\|_o\right)(1+\alpha r) < 1$ ($r = \max\limits_j r_j, \alpha = \sup \|T_n\|_o$) and $\|\mu_0\| \le a\left[1 - \sum\limits_{j=1}^{n} \|R_j\|_o(1+\alpha r)\right]$, then we can take $A = B_a\left(K^N\right)$.

Indeed, for $\mu \in B_a\left(K^N\right)$ we have:

$$\|P(\mu)\| \le \|H(\mu)\| + \|\mu^0\| \le \|\mu^0\| + \left(\sum_{j=1}^{m} \|R_j\|_o(1+r\alpha)\right) \underbrace{\|\mu\|}_{\le a} \le$$

$$\le a\left[1 - \sum_{j=1}^{m} \|R_j\|_o(1+\alpha r)\right] + \left(\sum_{j=1}^{m} \|R_j\|_o(1+\alpha r)\right) \cdot a = a$$

$\Longrightarrow P(\mu) \in B_a\left(K^N\right)$. Similar, if for any $n \in \mathbb{N}^*$, $\|H^n\|_o \le \left(\sum\limits_{j=1}^{m} \|R_j\|_o\right)(1+\alpha r)$ we obtain $P^n(\mu) \in B_a\left(K^N\right)$.

Let us suppose that all these conditions are fulfilled. Then, according theorem **2**.2, there exists $\mu^* \in B_a\left(K^N\right)$ such that $P(\mu^*) = \mu^*$, and, for any $n \geq 1$, there exists $\mu_n^* \in B_a\left(K^N\right)$ such that $P^n(\mu_n^*) = \mu_n^*$ (the fractal vector measures associated to P, respectively P^n).

We will denote by q_n the ratio of P^n.

Theorem 3.3. *We suppose that all the above hypotheses are fulfilled. Then* $\lim\limits_{n\to\infty} \|\mu_n^* - \mu^*\|_{MK} = 0$ *($\|\cdot\|_{MK}$ is the Monge-Kantorovich type norm (see lemma* **2**.*2).*

Proof.

$$\begin{aligned}\|\mu_n^* - \mu^*\|_{MK} = \|P^n(\mu_n^*) - P(\mu^*)\|_{MK} \leq \|P^n(\mu_n^*) - P^n(\mu^*)\|_{MK} + \\ + \|P^n(\mu^*) - P^n(\mu^*)\|_{MK} \leq q_n\|\mu_n^* - \mu^*\|_{MK} + \|P^n(\mu^*) - P(\mu^*)\|_{MK}\end{aligned} \quad (\mathbf{3}.4)$$

We have, obviously: $q_n \leq \|H^n\|_o \leq \left(\sum\limits_{j=1}^m \|R_j\|_o\right)(1+\alpha r) < 1$.

Hence $q \overset{def}{=} \sup\limits_n q_n < 1$. According (**3**.4), we can write:

$$\begin{aligned}\|\mu_n^* - \mu^*\|_{MK} \leq q\|\mu_n^* - \mu^*\|_{MK} + \|P^n(\mu^*) - P(\mu^*)\|_{MK} \Longrightarrow \\ \Longrightarrow (1-q)\|\mu_n^* - \mu^*\|_{MK} \leq \|P^n(\mu^*) - P(\mu^*)\|.\end{aligned} \quad (\mathbf{3}.5)$$

Let now $\varepsilon > 0$ arbitrarily fixed and $f \in BL_1\left(K^N\right)$. We can write:

$$\underbrace{\left|\sum_{j=1}^m \left(\int R_j^* \circ f \circ U_j^n d\mu^* - \int R_j^* \circ f \circ U_j d\mu^*\right)\right|}_{\overset{\text{lemma } \mathbf{2}.2}{=} \left|\int f dH^n(\mu^*) - \int f dH(\mu^*)\right|} =$$

$$= \left|\sum_{j=1}^m \int R_j^* \circ (f \circ U_j^n - f \circ U_j) d\mu^*\right| \leq \sum_{j=1}^m \left|\int R_j^* \circ (f \circ U_j^n - f \circ U_j) d\mu^*\right| \leq$$

$$\leq \sum_{j=1}^m \int \|R_j^* \circ (f \circ U_j^n - f \circ U_j)\| d|\mu^*| \leq \sum_{j=1}^m \int \|R_j^*\|_o \|f \circ U_j^n - f \circ U_j\| d|\mu^*| \leq$$

$$\leq \sum_{j=1}^m \|R_j^*\|_o \int \|U_j^n(x) - U_j(x)\| d|\mu^*|(x) \leq \max_j \sup_{x\in Y_0} \|U_j^n(x) - U_j(x)\| \cdot |\mu^*|(Y_0) < \varepsilon,$$

for n big enough, according the consequence of lemma **3**.4 and using the fact that $\sum\limits_{j=1}^m \|R_j^*\|_o < 1$.

Hence, $\sup\limits_{f\in BL_1(K^N)} \left|\int f dH^n(\mu^*) - \int f dH(\mu^*)\right| \leq \varepsilon$;

$$\Longrightarrow \|H^n(\mu^*) - H(\mu^*)\|_{MK} \leq \varepsilon \Longrightarrow \|P^n(\mu^*) - P(\mu^*)\|_{MK} \leq \varepsilon$$

$$\overset{(\mathbf{3}.5)}{\Longrightarrow} (1-q)\|\mu_n^* - \mu^*\|_{MK} \leq \varepsilon, \text{ for } n \text{ large enough.}$$

We deduce that $\mu_n^* \xrightarrow{\|\cdot\|_{MK}} \mu^*$. □

Example 4.1. We consider $X = Y = L^2([0,1]) \overset{not}{=} L^2$. For any $j \in \mathbb{N}, j \geq 1$ let $K_j : [0,1]\times[0,1] \to \mathbb{R}$, continuous, such that $\max\limits_{(x,y)} K_j(x,y) = \frac{1}{j+1}$ (for example, we can take $K_j(x,y) = \frac{x^j y^j}{j+1}$). We define for any $f \in L^2$, $\omega_j(f)(x) = \int\limits_{[0,1]} K_j(x,y)\sin|f(y)|d\lambda(y)$, λ being the Lebesgue measure.

i) We prove that $\omega_j(f) \in L^2 : \forall x \in [0,1]$, we have:

$$|\omega_j(f)(x)|^2 = \left| \int\limits_{[0,1]} K_j(x,y) \underbrace{\sin|f(y)|}_{\leq |f(y)|} d\lambda(y) \right|^2 \leq \left(\int\limits_{[0,1]} |K_j(x,y)||f(y)| d\lambda(y) \right)^2$$

$$\leq \frac{1}{(j+1)^2} \left(\int\limits_{[0,1]} |f(y)| d\lambda(y) \right)^2$$

$$\leq \frac{1}{(j+1)^2} \left(\int\limits_{[0,1]} d\lambda(y) \right) \cdot \int\limits_{[0,1]} |f(y)|^2 d\lambda(y) = \frac{1}{(j+1)^2} \|f\|_2^2$$

(we denoted by $\|\cdot\|_2$ the norm on $L^2([0,1])$); Hence, we can write: $\int\limits_{[0,1]} |\omega_j(f)(x)|^2 d\lambda(x) \leq \int\limits_{[0,1]} \frac{1}{(j+1)^2} \|f\|_2^2 d\lambda(x) = \frac{1}{(j+1)^2} \|f\|_2^2$.

We can deduce that $\omega_j(f) \in L^2$ and $\|\omega_j(f)\| \leq \frac{1}{j+1} \|f\|_2$.

ii) We prove that the functions ω_i are contractions: for any f, $g \in L^2$ we have:

$$|\omega_j(f)(x) - \omega_j(g)(x)|^2 \leq \left(\int\limits_{[0,1]} |K_j(x,y)|| \sin|f(y)| - \sin|g(y)|| d\lambda(y) \right)^2 \leq$$

$$\leq \frac{1}{(j+1)^2} \left(\int\limits_{[0,1]} d\lambda(y) \right) \cdot \int\limits_{[0,1]} \left(\sin|f(y)| - \sin|g(y)|\right)^2 d\lambda(y) =$$

$$= \frac{4}{(j+1)^2} \int\limits_{[0,1]} \sin^2 \frac{||f(y)| - |g(y)||}{2} \cos^2 \frac{||f(y)| + |g(y)||}{2} d\lambda(y) \leq$$

$$\leq \frac{1}{(j+1)^2} \int\limits_{[0,1]} ||f(y)| - |g(y)||^2 d\lambda(y) \leq \frac{1}{(j+1)^2} \int\limits_{[0,1]} |f(y) - g(y)|^2 d\lambda(y) =$$

$$= \frac{1}{(j+1)^2} \|f - g\|_2^2; \text{ it results that: } \int\limits_{[0,1]} |\omega_i(f)(x) - \omega_j(f)(x)|^2 d\lambda(x) \leq$$

$$\leq \int\limits_{[0,1]} \frac{1}{(j+1)^2} \|f - g\|_2^2 d\lambda(y) = \frac{1}{(j+1)^2} \|f - g\|_2^2 \Longrightarrow \|\omega_j(f) - \omega_j(g)\| \leq \frac{1}{j+1} \|f - g\|_2.$$

Hence, for $m \in \mathbb{N}^*$, fixed, we have the contractions $(\omega_j)_{1 \leq j \leq m}$ of fixed ratio $r_j \leq \frac{1}{j+1}$.

Example 4.2. For any $n \in \mathbb{N}, n \geq 1$, we define:

$$T_n(f)(x) = \int\limits_{[0,1]} n \sin \frac{xy}{n} f(y) d\lambda(y), \forall f \in L^2.$$

Obviously, T_n is a linear operator.

i) We prove that $T_n(f) \in L^2$:

$$|T_n(f)(x)|^2 \leq \left(\int_{[0,1]} n \underbrace{|\sin \frac{xy}{n}|}_{\leq \frac{xy}{n} \leq \frac{1}{n}} |f(y)| d\lambda(y) \right)^2 \leq \left(\int_{[0,1]} |f(y)| d\lambda(y) \right)^2$$
$$\leq \int_{[0,1]} |f(y)|^2 d\lambda(y) = \|f\|_2^2;$$

we deduce that $\int_{[0,1]} |T_n(f)(x)|^2 d\lambda(x) \leq \|f\|_2^2$ and this inequality shows that $T_n \in \mathcal{L}(L^2)$ and $\|T_n\|_o \leq 1$.
We consider now $T(f)(x) = \int_{[0,1]} xyf(y)d\lambda(y)$. One can easily prove that $T \in \mathcal{L}(L^2)$ (in the same way as for T_n). We shall prove that
$\|(T_n - T)(f)\|_2 \to 0$ for any $f \in L^2$ with $\|f\|_2 \leq 1$. We have (for $x \in (0,1]$):

$$|T_n(f)(x) - T(f)(x)|^2 \leq \left(\int_{[0,1]} \left| n \sin \frac{xy}{n} - xy \right| |f(y)| d\lambda(y) \right)^2 \leq$$
$$\leq \int_{[0,1]} f^2(y) \left[xy \left(\frac{\sin \frac{xy}{n}}{\frac{xy}{n}} - 1 \right) \right]^2 d\lambda(y); \qquad (\mathbf{3}.6)$$

let $f_n : [0,1] \to \mathbb{R}, f_n(y) = f^2(y) \cdot x^2 y^2 \cdot \left(\frac{\sin \frac{xy}{n}}{\frac{xy}{n}} - 1 \right)^2, \forall y \in (0,1], f_n(0) =$
$= 0$. Obviously, the functions f_n are λ-measurable and $|f_n| \leq f^2$, for any $y \in (0,1]$, $f_n(y) \to 0$. Using the Lebesgue theorem of dominate convergence, we obtain that $\lim_{n\to\infty} \int_{(0,1]} f_n(y) d\lambda(y) = \int_{(0,1]} 0 d\lambda(y) = 0$ and, from ($\mathbf{3}$.6),
$\lim_{n\to\infty} \left| [T_n(f) - T(f)](x) \right| = 0, \forall x \in [0,1]$ (for $x = 0$ the equality is obvious).
But, for $f \in L^2$ such that $\|f\|_2 \leq 1$, we have:

$$|(T_n - T)(f)(x)|^2 \leq \left(\int_{[0,1]} |f(y)|^2 d\lambda(y) \right)^2 = \|f\|_2^4 \leq 1,$$

$(T_n - T)(f)(x) \to 0, \forall x \in [0,1]$ and using again the dominate convergence theorem we deduce:

$$\int_{[0,1]} |(T_n(f) - T(f))(x)|^2 d\lambda(x) \to \int_{[0,1]} 0 d\lambda(x) = 0,$$

that means : $\|(T_n - T)(f)\|_2 \to 0, \forall f \in L^2$, with $\|f\|_2 \leq 1$.

Remark 3.2. *Using 4.1 and 4.2 together, we can write:*

$$T_n(\omega_j(f))(x) = \int_{[0,1]} n \sin \frac{xy}{n} \left(\int_{[0,1]} K_j(y,z) \sin |f(z)| d\lambda(z) \right) d\lambda(y);$$

$$T_n(\omega_j(f))(x) = \int_{[0,1]} xy \left(\int_{[0,1]} K_j(y,z) \sin |f(z)| d\lambda(z) \right) d\lambda(y).$$

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