A REMARK ON THE LIMIT SHADOWING PROPERTY FOR ITERATED FUNCTION SYSTEMS

Xinxing Wu, Shudi Liang, Yang Luo, Ma Xin, Xu Zhang*

This paper shows that a class of iterated function systems (IFSs) defined on the circle have the limit shadowing property, where this type of IFSs is generated by two maps, one is a constant map, another is a diffeomorphism with one attracting fixed point and one repelling fixed point. This result answers affirmatively a question whether or not the IFS system constructed in Example 4.6 of [M.F. Nia and S.A. Ahmadi, U.P.B. Sci. Bull., Series A, ***80*** *(2018), 145-154] has the limit shadowing property.*

Keywords: Iterated function system (IFS), limit shadowing property.

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1. Introduction

An *iterated function system* (IFS) $\mathcal{F}$ is a family of continuous self-maps $f_\lambda : X \to X$ $(\lambda \in \Lambda)$ defined on a metric space X, where Λ is a nonempty index set and X has a metric d, denoted by (X, d) [4]. For convenience, an IFS is denoted by $\mathcal{F} = \{X; f_\lambda |\ \lambda \in \Lambda\}$. Barnsley [1] used IFSs as a unified way for generating a broad class of fractals, which have been applied to many fields, such as image compression and image processing, and so on.

The theory of shadowing provides tools for fitting real trajectories near to approximate trajectories. The motivation comes from computer simulations, where we always have a numerical error when calculating a trajectory, no matter how careful we are, but at the same time we want to be sure that what we see on the computer screen is a good approximation of the genuine orbit of the system. Shadowing is a classical notion, which originated from the works of Anosov, Bowen and others (see [9, 11] for historical remarks and more recent advances). For more results on the IFS and shadowing properties, one

(X. Wu) School of Sciences, Southwest Petroleum University, Chengdu 610500, China e-mail: wuxinxing5201314@163.com

(S. Liang) School of Sciences, Southwest Petroleum University, Chengdu 610500, China

(Y. Luo) School of Sciences, Southwest Petroleum University, Chengdu 610500, China

(M. Xin) School of Science, Southwest University of Science and Technology, Mianyang 621010, China e-mail: cauchy7203@gmail.com

(X. Zhang) Department of Mathematics, Shandong University, Weihai 264209, China, e-mail: xu_zhang_sdu@mail.sdu.edu.cn

*Corresponding author.

is referred to [2, 3, 5, 6, 8, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21] and references therein.

Throughout this paper, let $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$.

For any $\sigma = (\lambda_0, \lambda_1, \lambda_2, \ldots) \in \Lambda^{\mathbb{Z}^+}$, denote

$$\mathcal{F}_0 = \mathrm{id}_X \text{ and } \mathcal{F}_{\sigma_n} = f_{\lambda_n} \circ \cdots \circ f_{\lambda_0} \text{ for any } n \in \mathbb{Z}^+.$$

A sequence $\{x_i\}_{i=0}^{+\infty} \subset X$ is called

(1) an *orbit* of $\mathcal{F}$ if there exists $\sigma = (\lambda_0, \lambda_1, \lambda_2, \ldots) \in \Lambda^{\mathbb{Z}^+}$ such that $f_{\lambda_i}(x_i) = x_{i+1}$ for all $i \in \mathbb{Z}^+$, i.e., $\mathcal{F}_{\sigma_i}(x_0) = x_{i+1}$ for all $i \in \mathbb{Z}^+$;

(2) an *asymptotic pseudo-orbit* of $\mathcal{F}$ if there exists $\sigma = (\lambda_0, \lambda_1, \lambda_2, \ldots) \in \Lambda^{\mathbb{Z}^+}$ such that

$$\lim_{n\to+\infty} d(f_{\lambda_n}(x_n), x_{n+1}) = 0.$$

Definition 1.1. [7] An IFS $\mathcal{F}$ has the *limit shadowing property* if every asymptotic pseudo-orbit $\{x_i\}_{i=0}^{+\infty}$ is asymptotically shadowed by an orbit of some point $z \in X$, i.e., there exists $\sigma = (\lambda_0, \lambda_1, \lambda_2, \ldots) \in \Lambda^{\mathbb{Z}^+}$ such that

$$\lim_{n\to+\infty} d(\mathcal{F}_{\sigma_n}(z), x_{n+1}) = 0.$$

Consider the quotient space of the interval $[0, 1]$ with 0 and 1 as a common point. For convenience, this quotient space is denoted by $[0, 1]/\{0, 1\}$, or simply $[0, 1)$. Note that there exists an order for numbers in $[0, 1)$. There exists a natural homeomorphism between this quotient space $[0, 1)$ and the unit circle $\mathbb{S}^1$ contained in the complex plane $\mathbb{C}$ by the exponential map $e^{2\pi \mathbf{i} x}$ for $x \in [0, 1)$. For any sequence $\{z_i\}_{i=0}^{+\infty} \subset \mathbb{S}^1$, there exists a sequence $\{x_i\}_{i=0}^{+\infty} \subset [0, 1)$ such that $z_i = e^{2\pi \mathbf{i} x_i}$.

Example 1.1. [7, Example 4.6] Consider the unit circle $\mathbb{S}^1$ with coordinate $x \in [0, 1)$ and denote by d the distance on $\mathbb{S}^1$ induced by the usual distance on the real line, i.e., $d(e^{2\pi \mathbf{i} x}, e^{2\pi \mathbf{i} y}) = \min\{|x - y|, 1 - |x - y|\}$ for any $x, y \in [0, 1)$. For any $c \in [0, 1)$, let $\phi_1 : \mathbb{S}^1 \to \mathbb{S}^1$ be a dynamical system generated by the map $f_1 : [0, 1) \to [0, 1)$ defined by

$$f_1(x) = x - x^2(x - \frac{1}{2})(x - 1)^2, \ x \in [0, 1),$$

and let $\phi_2 : \mathbb{S}^1 \to \mathbb{S}^1$ be a dynamical system generated by the map $f_2 : [0, 1) \to [0, 1)$ defined by

$$f_2(x) \equiv c, \ x \in [0, 1).$$

It is evident that ϕ_1, ϕ_2 are continuous self-maps defined on $(\mathbb{S}^1, d)$. *Let IFS* $\mathcal{G}^{(c)} = \{\mathbb{S}^1; \phi_1, \phi_2\}$.

Nia [5] obtained that every uniformly contracting IFS has the average shadowing property and showed that Sierpinski IFS has the average shadowing property. Recently, we proved that every IFS with the (asymptotic) average shadowing property is chain mixing [16]. Nia and Ahmadi [7] proved that every uniformly contracting or expanding IFS has the (exponential) limits shadowing property and the IFS $\mathcal{G}^{(\frac{1}{4})}$ does not have the exponential limit shadowing property and proposed the following question:

Question 1. [7] Does the IFS $\mathcal{G}^{(\frac{1}{4})}$ in Example 1.1 have the limit shadowing property?

This paper gives a positive answer to Question 1, that is, the IFS $\mathcal{G}^{(\frac{1}{4})}$ has the limit shadowing property (see Corollary 2.1).

2. IFSs with the limit shadowing property

Theorem 2.1. *Let $\phi_1 : \mathbb{S}^1 \to \mathbb{S}^1$ be a dynamical system generated by a continuously differentiable function $f_1 : [0,1] \to [0,1]$ satisfying that*

(1) f_1 has only three fixed points $0, p, 1$ for some $p \in (0,1)$;
(2) f_1 is a monotonically increasing function;
(3) $\min\{f'(0), f'(1)\} > 1$ and $f'(p) < 1$;
(3') $\max\{f'(0), f'(1)\} < 1$ and $f'(p) > 1$;

and let $\phi_2 : \mathbb{S}^1 \to \mathbb{S}^1$ be a dynamical system generated by a continuous function $f_2 : [0,1) \to [0,1)$ defined by

$$f_2(x) \equiv c \text{ for some } c \in [0,1).$$

If $c \in (0,1)$, then the IFS $\mathcal{K}^{(c)} = \{\mathbb{S}^1; \phi_1, \phi_2\}$ has the limit shadowing property.

Proof. It suffices to prove that f_1 satisfies (1), (2), and (3), as the rest for (3') can be proved similarly. First, it is easy to observe the following facts:

Fact (1) for any $x \in (0,p)$, $f_1(x) > x$;

Fact (2) for any $x \in (p,1)$, $f_1(x) < x$;

Fact (3) for any $x \in (0,1)$, $\lim_{n\to+\infty} f_1^n(x) = p$;

Fact (4) there exist an interval $[a,b] \subset (0,1)$ and $0 < \alpha < 1$ such that $p \in (a,b)$, $f_1([a,b]) \subset [a,b]$, and for any $x \in [a,b]$, $0 < f'(x) < \alpha$.

Given any fixed asymptotic pseudo-orbit $\{e^{2\pi \mathbf{i} x_i}\}_{i=0}^{+\infty} \subset \mathbb{S}^1$ of $\mathcal{K}^{(c)}$, where $x_i \in [0,1)$, there exists $\sigma = (\lambda_0, \lambda_1, \lambda_2, \ldots) \in \{1,2\}^{\mathbb{Z}^+}$ such that

$$\lim_{i\to+\infty} d(\phi_{\lambda_i}(e^{2\pi \mathbf{i} x_i}), e^{2\pi \mathbf{i} x_{i+1}}) = 0.$$

Let $\mathscr{N}(\sigma) = \{i \in \mathbb{Z}^+ : \lambda_i = 2\}$ and consider the following two cases:

`Case 1.` If $\mathscr{N}(\sigma)$ is a finite set, then there exists some $K \in \mathbb{N}$ such that for any $n \geq K$, $\lambda_n = 1$, implying that

$$\lim_{i\to+\infty} d(\phi_1(e^{2\pi \mathbf{i} x_{i+K}}), e^{2\pi \mathbf{i} x_{i+K+1}}) = 0,$$

i.e., $\{e^{2\pi \mathbf{i} x_{i+K}}\}_{i=0}^{+\infty}$ is an asymptotic pseudo-orbit of ϕ_1. Applying [10, Theorem 3.1.2] yields that $(\mathbb{S}^1, \phi_1)$ has the limit shadowing property. Thus, there exists $z \in \mathbb{S}^1$ such that

$$\lim_{i\to+\infty} d(\phi_1^i(z), e^{2\pi \mathbf{i} x_{i+K}}) = 0.$$

Take $z^* = \phi_1^{-K}(z)$ and $\sigma^* = (1, 1, 1, \ldots) \in \{1, 2\}^{\mathbb{Z}^+}$. Hence,

$$\lim_{i\to+\infty} d(\mathcal{K}^{(c)}_{\sigma^*_{n-1}}(z^*), e^{2\pi \mathbf{i} x_i}) = \lim_{i\to+\infty} d(\phi_1^i(z^*), e^{2\pi \mathbf{i} x_i}) = 0.$$

`Case 2.` If $\mathscr{N}(\sigma)$ is an infinite set, suppose $\mathscr{N}(\sigma) = \{k_1, k_2, k_3, \ldots\}$ with $k_1 < k_2 < k_3 < \cdots$. From Fact (3), it follows that there exists $K \in \mathbb{N}$ such that $f_1^K(c) > a$. Take $\zeta = \max\{f'(x) : x \in [0, 1)\} > 1$. For any $0 < \varepsilon < \min\left\{\frac{1-\alpha}{100}, \frac{1}{4}(f_1^K(c) - a), \frac{1}{4}(b - p)\right\}$, choose $\delta > 0$ such that $\left[(K + 2)\zeta^K + \frac{1}{1+\alpha}\right] \delta < \frac{\varepsilon}{2}$. From $\lim_{n\to+\infty} |f_{\lambda_n}(x_n) - x_{n+1}| = 0$, it follows that there exists $N \in \mathbb{N}$ such that for any $n \geq N$, $|f_{\lambda_n}(x_n) - x_{n+1}| < \delta$, implying that

$$|f_{\lambda_{k_n}}(x_{k_n}) - x_{k_n+1}| = |c - x_{k_n+1}| < \delta.$$

For any $n \geq N$, consider the following two subcases:

(a) If $k_{n+1} - k_n \leq K + 1$, then for any $i \in [k_n + 1, k_{n+1}]$,

$$\begin{aligned}
& |x_i - f_{\lambda_{i-1}} \circ \cdots \circ f_{\lambda_{k_n+1}}(c)| \\
\leq\ & |x_i - f_{\lambda_{i-1}}(x_{i-1})| + |f_{\lambda_{i-1}}(x_{i-1}) - f_{\lambda_{i-1}}(f_{\lambda_{i-2}}(x_{i-2}))| + \cdots \\
& + |f_{\lambda_{i-1}} \circ \cdots \circ f_{\lambda_{k_n+1}}(x_{k_n+1}) - f_{\lambda_{i-1}} \circ \cdots \circ f_{\lambda_{k_n+1}}(c)| \\
=\ & |x_i - f_1(x_{i-1})| + |f_1(x_{i-1}) - f_1^2(x_{i-2})| + \cdots \\
& + |f_1^{i-1-k_n}(x_{k_n+1}) - f_1^{i-1-k_n}(c)| \\
\leq\ & \delta + \zeta \cdot \delta + \cdots + \zeta^{i-1-k_n} \cdot \delta \leq (K + 1) \cdot \zeta^K \cdot \delta < \frac{\varepsilon}{2}.
\end{aligned}$$

(b) If $k_{n+1} - k_n > K+1$, from (a), it follows that for any $i \in [k_n+1, k_n+K+1]$,

$$|x_i - f_{\lambda_{i-1}} \circ \cdots \circ f_{\lambda_{k_n+1}}(c)| < \frac{\varepsilon}{2},$$

implying that

$$\begin{aligned}
& p + \frac{\varepsilon}{2} > f_1^K(c) + \frac{\varepsilon}{2} = f_{\lambda_{k_n+K}} \circ \cdots \circ f_{\lambda_{k_n+1}}(c) + \frac{\varepsilon}{2} \\
>\ & x_{k_n+K+1} > f_{\lambda_{k_n+K}} \circ \cdots \circ f_{\lambda_{k_n+1}}(c) - \frac{\varepsilon}{2} = f_1^K(c) - \frac{\varepsilon}{2}.
\end{aligned}$$

Applying mathematical induction, it follows from this, Fact (4), and

$$\begin{aligned} & |x_i - f_1^{i-(k_n+K+1)}(x_{k_n+K+1})| \\ \leq\ & |x_i - f_{\lambda_{i-1}}(x_{i-1})| + |f_{\lambda_{i-1}}(x_{i-1}) - f_{\lambda_{i-1}} \circ f_{\lambda_{i-2}}(x_{i-2})| + \cdots \\ & +|f_{\lambda_{i-1}} \circ \cdots \circ f_{\lambda_{k_n+K+2}}(x_{k_n+K+2}) - f_1^{i-(k_n+K+1)}(x_{k_n+K+1})| \\ =\ & |x_i - f_1(x_{i-1})| + |f_1(x_{i-1}) - f_1^2(x_{i-2})| + \cdots \\ & +|f_1^{i-(k_n+K+2)}(x_{k_n+K+2}) - f_1^{i-(k_n+K+1)}(x_{k_n+K+1})| \end{aligned}$$

that for any $i \in (k_n + K + 1, k_{n+1}]$,

$$\{x_{k_n+K+1}, \ldots, x_i\} \subset [a, b]\,.$$

This, together with Fact (4), implies that

$$\begin{aligned} & |x_i - f_1^{i-(k_n+K+1)}(x_{k_n+K+1})| \\ \leq\ & |x_i - f_{\lambda_{i-1}}(x_{i-1})| + |f_{\lambda_{i-1}}(x_{i-1}) - f_{\lambda_{i-1}} \circ f_{\lambda_{i-2}}(x_{i-2})| + \cdots \\ & +|f_{\lambda_{i-1}} \circ \cdots \circ f_{\lambda_{k_n+K+2}}(x_{k_n+K+2}) - f_{\lambda_{i-1}} \circ \cdots \circ f_{\lambda_{k_n+K+1}}(x_{k_n+K+1})| \\ =\ & |x_i - f_1(x_{i-1})| + |f_1(x_{i-1}) - f_1(f_1(x_{i-2}))| + \cdots \\ & +|f_1^{i-(k_n+K+2)}(x_{k_n+K+2}) - f_1^{i-(k_n+K+2)}(f_1(x_{k_n+K+1}))| \\ \leq\ & \delta + \alpha \cdot \delta + \cdots + \alpha^{i-(k_n+K+2)} \cdot \delta < \frac{1}{1-\alpha}\delta. \end{aligned}$$

Meanwhile,

$$\begin{aligned} & |f_1^{i-(k_n+K+1)}(x_{k_n+K+1}) - f_1^{i-(k_n+K+1)}(f_1^K(c))| \\ \leq\ & \alpha^{i-(k_n+K+1)}|x_{k_n+K+1} - f_1^K(c)| \\ <\ & \alpha^{i-(k_n+K+1)} \cdot \frac{\varepsilon}{2} < \frac{\varepsilon}{2}. \end{aligned}$$

Then,

$$|x_i - f_1^{i-k_n-1}(c)| < \frac{1}{1-\alpha}\delta + \frac{\varepsilon}{2} < \varepsilon.$$

Note that for any $z \in [0, 1)$ and any $n \in \mathbb{N}$, $\mathcal{K}^{(c)}_{\sigma_{k_n}}(e^{2\pi \mathbf{i} z}) = e^{\frac{\pi}{2}\mathbf{i}}$. This, together with (a) and (b), implies that for any $z \in [0, 1)$ and any $i \in [k_n + 1, k_{n+1}]$ $(n \geq N)$,

$$d(\mathcal{K}^{(c)}_{\sigma_{i-1}}(e^{2\pi \mathbf{i} z}), e^{2\pi \mathbf{i} x_i}) < \varepsilon.$$

Therefore, by the arbitrariness of ε, one has

$$\lim_{i\to+\infty} d(\mathcal{K}^{(c)}_{\sigma_{i-1}}(e^{2\pi \mathbf{i} z}), e^{2\pi \mathbf{i} x_i}) = 0.$$

Summing up Cases 1 and 2 yields that $\mathcal{K}^{(c)}$ has the limit shadowing property. □

Corollary 2.1. *The IFS $\mathcal{G}^{(c)}$ in Example 1.1 has the limit shadowing property for any $c \in (0, 1)$. In particular, $\mathcal{G}^{(\frac{1}{4})}$ has the limit shadowing property.*

Proof. Applying Theorem 2.1, this holds trivially. □

Remark 2.1. $\mathcal{K}^{(0)}$ in Theorem 2.1 does not have the limit shadowing property. In fact, it follows from Fact 3) that for any $k \geq 2$, there exists $n_k > 2^k$ such that $\frac{p}{2} < f_1^{n_k}(\frac{1}{k}) < p$. Let $L_1 = 0$ and $L_k = 2(n_2 + \cdots + n_k)$ for all $k \geq 2$. Choose a sequence $\xi = \{x_i\}_{i=0}^{+\infty} \subset [0, \frac{1}{2})$ as following:

$$x_i = \begin{cases} 0, & i \in L_k \text{ for some } k \in \mathbb{N}, \\ f_1^{i-L_k-1}(\frac{1}{k+1}), & i \in [L_k + 1, L_{k+1}) \text{ for some } k \in \mathbb{N}, \end{cases}$$

and take $\sigma = (\lambda_0, \lambda_1, \lambda_2, \dots) \in \{1, 2\}^{\mathbb{Z}^+}$ with

$$\lambda_i = \begin{cases} 2, & i \in \{L_k - 1 : k \geq 2\}, \\ 1, & \text{otherwise.} \end{cases}$$

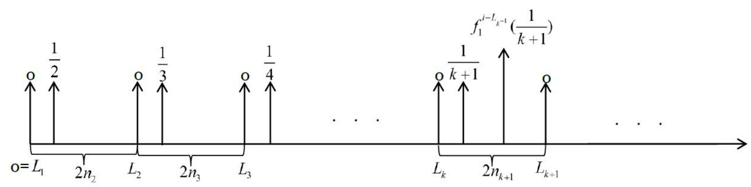

FIGURE 1. The illustration diagram of the construction of the sequence $\xi = \{x_i\}_{i=0}^{+\infty}$.

It is easy to see that

$$\lim_{i \to +\infty} |f_{\lambda_i}(x_i) - x_{i+1}| = 0,$$

i.e., $\{e^{2\pi \mathbf{i} x_i}\}_{i=0}^{+\infty}$ is an asymptotic pseudo-orbit of $\mathcal{K}^{(0)}$. For any $z = e^{2\pi \mathbf{i} x} \in \mathbb{S}^1$ with $x \in [0, 1)$ and any $\sigma' = (\lambda'_0, \lambda'_1, \lambda'_2, \dots) \in \{1, 2\}^{\mathbb{Z}^+}$, it can be verified that

(1) if there exists some $i \in \mathbb{Z}^+$ such that $\lambda'_i = 2$ or $x = 0$, then for any $k \geq i$,

$$d(\mathcal{K}^{(0)}_{\sigma'_{L_k+n_{k+1}}}(z), e^{2\pi \mathbf{i} x_{L_k+n_{k+1}+1}}) = d(1, e^{2\pi \mathbf{i} f_1^{n_{k+1}}(\frac{1}{k+1})}) \geq \min\left\{\frac{p}{2}, 1-p\right\};$$

(2) if for any $i \in \mathbb{Z}^+$, $\lambda'_i = 1$ and $x \in (0, 1)$, then $\lim_{i \to +\infty} \mathcal{K}^{(0)}_{\sigma'_i}(z) = e^{\pi \mathbf{i}}$, implying that

$$\lim_{k \to +\infty} d(\mathcal{K}^{(0)}_{\sigma'_{L_k-1}}(z), x_{L_k}) = d(e^{\pi \mathbf{i}}, 1) = \frac{1}{2} > 0.$$

Then, $\{e^{2\pi \mathbf{i} x_i}\}_{i=0}^{+\infty}$ can not be asymptotically shadowed by $\{z, \mathcal{K}^{(0)}_{\sigma'_0}(z), \mathcal{K}^{(0)}_{\sigma'_1}(z), \dots\}$. Therefore, $\mathcal{K}^{(0)}$ does not have the limit shadowing property due to the arbitrariness of z and σ'.

The main reasons ensuring that the IFS $\mathcal{K}^{(c)}$ has the limit shadowing property for any $c \in (0, 1)$ are that all points in $(0, 1)$ are attracted to p after iterations by using f_1 and that p is a hyperbolic fixed point of f_1. However, we do not know whether the following IFS $\mathcal{H}^{(c)}$ has the limit shadowing property for any $c \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, when almost all points of f_1 are attracted to non-hyperbolic fixed points.

For any $c \in [0, 1)$, let $\psi_1 : \mathbb{S}^1 \to \mathbb{S}^1$ be a dynamical system generated by the map $f_1 : [0, 1) \to [0, 1)$ defined by

$$f_1(x) = x + x^2(x - \frac{1}{2})(x - 1)^2,$$

and let $\psi_2 : \mathbb{S}^1 \to \mathbb{S}^1$ be a dynamical system generated by the map $f_2 : [0, 1) \to [0, 1)$ defined by

$$f_2(x) \equiv c.$$

Let IFS $\mathcal{H}^{(c)} = \{\mathbb{S}^1; \psi_1, \psi_2\}$. Clearly, $\mathcal{H}^{(\frac{1}{2})}$ does not have the limit shadowing property.

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