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YETTER-DRINFELD MODULES AND GALOIS EXTENSIONS OVER COQUASI-HOPF ALGEBRAS

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În această lucrare se defineşte categoria Yetter–Drinfeld modulelor peste o coquasi-bialgebră şi se arată că este monoidală, pre–braided şi coincide cu centrul slab stâng al categoriei comodulelor. În particular, pentru o extensie Galois $B \subseteq A$ peste o coquasi-Hopf algebră, centralizatorul A^B este o algebră comutativă în categoria braided $\mathcal{YD}_H^H$ şi centrul categoriei (H, A)-bimodulelor Hopf este chiar categoria Yetter-Drinfeld modulelor, generalizând rezultatele obţinute de Schauenburg în cazul algebrelor Hopf.

In this paper we define Yetter–Drinfeld modules over a coquasi-bialgebra and prove that the category of Yetter–Drinfeld modules is monoidal, pre–braided and coincides with the left weak center of the comodule category. In particular, given a Galois extension $B \subseteq A$ over a coquasi-Hopf algebra, the centralizer A^B is a commutative algebra in the braided category $\mathcal{YD}_H^H$ and the center of the category of Hopf (H, A)-bimodules is equivalent to the category of Yetter-Drinfeld modules, generalizing the results obtained by Schauenburg in the Hopf algebra case.

Keywords: coquasi-Hopf algebra, monoidal category, Yetter-Drinfeld module, Galois extension.

1. Introduction

The definition of a coquasi-bialgebra H ensures that the category of right H-comodules $\mathcal{M}^H$ is monoidal, with usual tensor product over the base field. The first example of (the dual of) such an object was provided by Drinfeld in [1], in connection with the Knizhnik-Zamolodchikov system of partial differential equations. A coquasi-bialgebra is a coalgebra endowed with two coalgebra maps, a multiplication and a unit, which verifies the usual axioms of a bialgebra up to conjugation by an invertible element $\omega \in (H \otimes H \otimes H)^*$ (the reassociator). Consequently, the difference between a coquasi-bialgebra and an ordinary bialgebra is that the associativity of tensor product in the monoidal category of comodules does not coincide with the usual associativity of tensor product in the category of vector spaces. But is this main feature of coquasi-bialgebras, namely the monoidality of corepresentations, which made possible generalizations of major properties from Hopf algebras

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(the existence and uniqueness of integrals ([2]), the structure of Hopf bimodules ([3]), construction of the Drinfeld double ([4]), etc.). Hence it seems natural to define Yetter-Drinfeld modules over a coquasi-bialgebra as right H-comodules endowed with some special right action of H, satisfying some compatibility relations. The same construction, but for the dual concept of quasi-bialgebra has been performed in [5]. We show that the category of Yetter-Drinfeld modules is pre-braided (braided if H has an antipode) and the functor to the category of right H-comodules forgetting the H-action is monoidal.

From experience with Hopf algebras and various generalizations of them, it is reasonable to expect that the category of Yetter–Drinfeld modules over a coquasi-bialgebra is isomorphic with the (weak) center of the category of comodules. For example, when working over a quasi-bialgebra (the dual notion of a coquasi-bialgebra), this was obtained by Majid in [5]. We show that the same result holds also for coquasi-bialgebras.

Yetter-Drinfeld modules over Hopf algebras are known to be intimately related with Galois extensions ([6], [7]). For coquasi-Hopf algebras, a similar notion has been proposed by the author in a previous paper ([8]). It should be noticed that this definition of Galois extensions works only for coquasi-Hopf algebras, as it involves the presence of the antipode. Although this may look restrictive, it allowed us to recover all principal results from the classical Hopf-Galois theory (theorems of Kreimer-Takeuchi ([9]), Doi-Takeuchi ([10]) and Blattner -Montgomery ([11]), Schneider ([12])). Therefore it seems natural to search for a similar connection with Yetter-Drinfeld modules for coquasi-Hopf algebras. We shall see that for a Galois extension $B \subseteq A$ over a coquasi-Hopf algebra H, the centralizer A^B is a commutative algebra in the braided category $\mathcal{YD}_H^H$ and find a monoidal equivalence between the category of Yetter-Drinfeld modules $\mathcal{YD}_H^H$ and the center of the category of Hopf (A, H)-bimodules, generalizing the results obtained by Schauenburg in the Hopf algebra case ([7]).

We start by recalling some definitions and fixing notations. Throughout the paper we work over some base field $\Bbbk$. Tensor products, algebras, linear spaces, etc. will be over $\Bbbk$. Unadorned $\otimes$ means $\otimes_{\Bbbk}$. An introduction to the study of coquasi-bialgebras and coquasi-Hopf algebras can be found in [13]. A good reference for monoidal categories and functors is [14]. We shall use Sweedler's notation for the coproduct: $\Delta(h) = h_1 \otimes h_2 \in H \otimes H$ (summation understood), for $h \in H$ where H is a coalgebra and extend this notation also for comodules: $\rho_V(v) = v_0 \otimes v_1 \in V \otimes H$, for $v \in V$, where (V, ρ_V) is right H-comodule. The category of right comodules over a coalgebra H will be denoted $\mathcal{M}^H$ and the set of right H-colinear morphisms by $Hom^H(V, W)$, for $V, W \in \mathcal{M}^H$. For any vector spaces V, W, the set of $\Bbbk$-linear maps from V to W will be $Hom(V, W)$.

2. Coquasi-Hopf algebras

A coquasi-bialgebra $(H, m, u, \omega, \Delta, \varepsilon)$ is a $\Bbbk$-coalgebra (H, Δ, ε), together with two coalgebra morphisms: the multiplication $m : H \otimes H \longrightarrow H$ (denoted $m(h \otimes g) = hg$), the unit $u : \Bbbk \longrightarrow H$ (denoted $u(1) = 1_H$), and a convolution invertible element $\omega \in (H \otimes H \otimes H)^*$ such that:

$$
\begin{aligned}
h_1(g_1k_1)\omega(h_2, g_2, k_2) &= \omega(h_1, g_1, k_1)(h_2g_2)k_2 && (2.1)\\
1_H h &= h1_H = h && (2.2)\\
\omega(h_1, g_1, k_1l_1)\omega(h_2g_2, k_2, l_2) &= \omega(g_1, k_1, l_1)\omega(h_1, g_2k_2, l_2) && (2.3)\\
&\quad\ \omega(h_2, g_3, k_3) &&\\
\omega(h, 1_H, g) &= \varepsilon(h)\varepsilon(g) && (2.4)
\end{aligned}
$$

hold for all $h, g, k, l \in H$. As a consequence, we have also

$$\omega(1_H, h, g) = \omega(h, g, 1_H) = \varepsilon(h)\varepsilon(g)$$

for each $g, h \in H$.

A coquasi-Hopf algebra is a coquasi-bialgebra H endowed with a coalgebra antihomomorphism $S : H \longrightarrow H$ (the antipode) and with elements $\alpha, \beta \in H^*$ satisfying

$$
\begin{aligned}
S(h_1)\alpha(h_2)h_3 &= \alpha(h)1_H && (2.5)\\
h_1\beta(h_2)S(h_3) &= \beta(h)1_H && (2.6)\\
\omega(h_1\beta(h_2), S(h_3), \alpha(h_4)h_5) &= \varepsilon(h) && (2.7)\\
&= \omega^{-1}(S(h_1), \alpha(h_2)h_3\beta(h_4), S(h_5)) &&
\end{aligned}
$$

for all $h \in H$. These relations imply $S(1_H) = 1_H$ and $\alpha(1_H)\beta(1_H) = 1$, so by rescaling α and β, we may assume that $\alpha(1_H) = 1$ and $\beta(1_H) = 1$.

For H a coquasi-bialgebra, the category of right H-comodules $\mathcal{M}^H$ is monoidal: the tensor product is over the base field and the comodule structure of the tensor product is the codiagonal one (i.e. $\rho_{V\otimes W}(v \otimes w) = v_0 \otimes w_0 \otimes v_1w_1$, for $v \in V$, $w \in W$ with $V, W \in \mathcal{M}^H$). The reassociator map is

$$
\begin{aligned}
\Phi_{U,V,W} &: (U \otimes V) \otimes W \longrightarrow U \otimes (V \otimes W)\\
\Phi_{U,V,W}((u \otimes v) \otimes w) &= u_0 \otimes (v_0 \otimes w_0)\omega(u_1, v_1, w_1)
\end{aligned}
$$

for $u \in U$, $v \in V$, $w \in W$ and $U, V, W \in \mathcal{M}^H$. The unit object in the monoidal category is the base field $\Bbbk$ with trivial coaction.

Together with a coquasi-bialgebra $H = (H, \Delta, \varepsilon, m, 1_H, \omega)$, we also have H^{op}, H^{cop}, and $H^{op,cop}$ as coquasi-bialgebras, where "op" means opposite multiplication and "cop" means opposite comultiplication. The coquasi-structures are obtained by putting $\omega_{cop} = \omega^{-1}$, $\omega_{op} = (\omega^{-1})^{321}$, $\omega_{op,cop} = \omega^{321}$, where $\omega^{321}(h,g,k) = \omega(k,g,h)$. If H is a coquasi-Hopf algebra with bijective antipode (S, α, β), then so are H^{op}, H^{cop}, and $H^{op,cop}$ with $S_{op} = S_{cop} = (S_{op,cop})^{-1} = S^{-1}$, $\alpha_{cop} = \beta S^{-1}$, $\alpha_{op} = \alpha S^{-1}$, $\alpha_{op,cop} = \beta$, $\beta_{cop} = \alpha S^{-1}$, $\beta_{op} = \beta S^{-1}$ and $\beta_{op,cop} = \alpha$.

In order to simplify notations, we shall introduce the following linear maps: p_R, $p_L, q_R, q_L \in (H \otimes H)^*$, by the formulas:

$$\begin{aligned} p_R(h,g) &= \omega^{-1}(h, g_1\beta(g_2), S(g_3)) \\ p_L(h,g) &= \omega(S^{-1}(h_3)\beta S^{-1}(h_2), h_1, g) \\ q_R(h,g) &= \omega^{-1}(h, g_3, \alpha S^{-1}(g_2)S^{-1}(g_1)) \\ q_L(h,g) &= \omega^{-1}(S(h_1)\alpha(h_2), h_3, g) \end{aligned}$$

for any $h, g \in H$, where H is a coquasi-Hopf algebra with bijective antipode. Similar elements were also defined for quasi-Hopf algebras, from where we have borrowed the notation ([15]). Dualizing the results in the quasi-Hopf algebra case, we have the following properties:

$$\begin{aligned} (h_1g_1)S(g_3)p_R(h_2,g_2) &= h_2p_R(h_1,g) && (2.8) \\ S^{-1}(h_3)(h_1g_1)p_L(h_2,g_2) &= g_2p_L(h,g_1) && (2.9) \\ (h_2g_3)S^{-1}(g_1)q_R(h_1,g_2) &= h_1q_R(h_2,g) && (2.10) \\ S(h_1)(h_3g_2)q_L(h_2,g_1) &= g_1q_L(h,g_2) && (2.11) \end{aligned}$$

and

$$\begin{aligned} q_R(h_1g_1, S(g_3))p_R(h_2,g_2) &= \varepsilon(h)\varepsilon(g) && (2.12) \\ q_L(S^{-1}(h_3), h_1g_1)p_L(h_2,g_2) &= \varepsilon(h)\varepsilon(g) && (2.13) \\ p_R(h_2g_3, S^{-1}(g_1))q_R(h_1,g_2) &= \varepsilon(h)\varepsilon(g) && (2.14) \\ p_L(S(h_1), h_3g_2)q_L(h_2,g_1) &= \varepsilon(h)\varepsilon(g) && (2.15) \end{aligned}$$

for all elements $h, g \in H$.

In [16], it was constructed a twist $f \in (H \otimes H)^*$ which controls how far is S from a anti-algebra morphism:

$$\mathbf{f}(h_1,g_1)S(h_2g_2) = S(g_1)S(h_1)\mathbf{f}(h_2,g_2) \qquad (2.16)$$

for any $h, g \in H$. If we denote

$$\begin{aligned} \gamma(h,g) &= \omega(S(g_2), S(h_2), h_4)\omega^{-1}(S(g_1)S(h_1), h_5, g_4)\alpha(h_3)\alpha(g_3) && (2.17) \\ \delta(h,g) &= \omega(h_1g_1, S(g_5), S(h_4))\omega^{-1}(h_2, g_2, S(g_4))\beta(h_3)\beta(g_3) && (2.18) \end{aligned}$$

then the twist $\mathbf{f}$ and its convolution inverse are given by

$$\begin{aligned}\mathbf{f}(h,g) &= \omega^{-1}(S(g_1)S(h_1), h_3g_3, S(h_5g_5))\gamma(h_2,g_2)\beta(h_4g_4)\\ \mathbf{f}^{-1}(h,g) &= \omega^{-1}(S(h_1g_1), h_3g_3, S(g_5)S(h_5))\delta(h_4,g_4)\alpha(h_4g_4)\end{aligned}$$

We have also that

$$\mathbf{f}(h_1,g_1)\alpha(h_2g_2) = \gamma(h,g) \tag{2.19}$$

$$\beta(h_1g_1)\mathbf{f}^{(-1)}(h_2,g_2) = \delta(h,g) \tag{2.20}$$

for any $h, g \in H$, according to [17].

3. Yetter-Drinfeld modules

Let H a coquasi-bialgebra. Dualizing the results obtained in [5] for quasi-bialgebras, we may define the category of right-right Yetter-Drinfeld modules in the following way:

Definition 3.1 *A (right-right) Yetter-Drinfeld module over H is a right H-comodule V, endowed with a linear map $\triangleleft : V \otimes H \longrightarrow H$ satisfying the following properties:*

$$v \triangleleft 1_H = v \tag{3.1}$$

$$(v \triangleleft h_2)_0 \otimes h_1(v \triangleleft h_2)_1 = v_0 \triangleleft h_1 \otimes v_1h_2 \tag{3.2}$$

$$\begin{aligned}(v \triangleleft h_2)_0 \triangleleft g_1\omega(h_1, (v \triangleleft h_2)_1, g_2) &= (v_0 \triangleleft (h_2g_2))_0\\ &\quad \omega(h_1, g_1, (v_0 \triangleleft (h_2g_2))_1)\omega(v_1, h_3, g_3)\end{aligned} \tag{3.3}$$

for all $v \in V$, $h \in H$.

We denote by $\mathcal{YD}_H^H$ the category of Yetter-Drinfeld modules with morphisms the right H-colinear maps respecting the right H-action $\triangleleft$. We have obviously a canonical functor $\mathcal{U} : \mathcal{YD}_H^H \longrightarrow \mathcal{M}^H$, which forgets the right H-action $\triangleleft$. It would had been natural to call these objects Yetter-Drinfeld comodules rather than modules. But as in the associative setting a Yetter-Drinfeld module is simultaneously a module and a comodule subject to the compatibility condition (3.2), we preferred to keep the usual terminology. In particular, Yetter-Drinfeld modules specialize to ordinary Yetter-Drinfeld modules if H is an ordinary bialgebra.

Remark 3.2 *1. Similarly we may define the notions of a left-left, right-left and left-right Yetter-Drinfeld module over a coquasi-bialgebra by ${}^H_H\mathcal{YD} = \mathcal{YD}^{H^{op,cop}}_{H^{op,cop}}$, ${}^H\mathcal{YD}_H = \mathcal{YD}^{H^{cop}}_{H^{cop}}$, respectively ${}_H\mathcal{YD}^H = \mathcal{YD}^{H^{op}}_{H^{op}}$.*

2. *If H is a coquasi-Hopf algebra with bijective antipode, then axiom (3.2) is equivalent to*

$$(v\triangleleft h)_0\otimes(v\triangleleft h)_1 = (v_0\triangleleft h_4)_0\otimes S(h_2)(v_1h_5)p_L(S(h_1), v_2h_6)q_L(h_3, (v_0\triangleleft h_4)_1) \tag{3.4}$$

Proof. Suppose $V \in \mathcal{M}^H$ endowed with a linear map $\triangleleft : V \otimes H \longrightarrow H$ satisfying (3.2). For any $v \in V$, $h \in H$, we compute that:

$$\begin{aligned}
(v \triangleleft h)_0 \otimes (v \triangleleft h)_1 &= (v \triangleleft h_2)_0 \otimes (v \triangleleft h_2)_1\varepsilon(h_1)\varepsilon((v \triangleleft h_2)_2) \\
\overset{(2.15)}{=} &\ (v \triangleleft h_2)_0 \otimes (v \triangleleft h_4)_1 p_L(S(h_1), h_3(v \triangleleft h_4)_3) \\
&\quad q_L(h_2, (v \triangleleft h_4)_2) \\
\overset{(2.11)}{=} &\ (v \triangleleft h_6)_0 \otimes S(h_2)[h_4(v \triangleleft h_6)_2]p_L(S(h_1), h_5(v \triangleleft h_6)_3) \\
&\quad q_L(h_3, (v \triangleleft h_6)_1) \\
\overset{(3.2)}{=} &\ (v_0 \triangleleft h_4)_0 \otimes S(h_2)(v_1h_5)p_L(S(h_1), v_2h_6) \\
&\quad q_L(h_3, (v_0 \triangleleft h_4)_1)
\end{aligned}$$

Conversely, assume that (3.4) holds. In particular, we have

$$v\triangleleft h = (v\triangleleft h)_0\varepsilon((v\triangleleft h)_1) = (v_0\triangleleft h_3)_0 p_L(S(h_1), v_1h_4)q_L(h_2, (v_0\triangleleft h_3)_1) \tag{3.5}$$

then

$$\begin{aligned}
(v \triangleleft h_2)_0 \otimes h_1(v \triangleleft h_2)_1 &= (v_0 \triangleleft h_5)_0 \otimes h_1[S(h_3)(v_1h_6)]p_L(S(h_2), v_2h_7) \\
&\quad q_L(h_4, (v_0 \triangleleft h_5)_1) \quad \text{by (3.4)} \\
\overset{(2.9)}{=} &\ (v_0 \triangleleft h_3)_0 \otimes v_2h_5p_L(S(h_1), v_1h_4) \\
&\quad q_L(h_2, (v_0 \triangleleft h_3)_1) \\
\overset{(3.5)}{=} &\ v_0 \triangleleft h_1 \otimes v_1h_2
\end{aligned}$$

■

Theorem 3.3 *Let H be a coquasi-bialgebra. Then:*

1. *The category of right-right Yetter-Drinfeld modules is monoidal, with tensor product and associator inherited from the category of right H-comodules.*

2. *It is pre-braided, with braiding*

$$c_{V,W} : V \otimes W \longrightarrow W \otimes V, v \otimes w \longrightarrow w_0 \otimes v \triangleleft w_1 \tag{3.6}$$

for V, $W \in \mathcal{YD}_H^H$. If H has a bijective antipode, then $\mathcal{YD}_H^H$ is even braided, with inverse of the braiding given by

$$\begin{aligned} c_{V,W}^{-1}(w \otimes v) &= (v_0 \lhd S^{-1}(w_4))_0 \otimes w_0 q_L(S^{-1}(w_5), (v_0 \lhd S^{-1}(w_4))_1) \\ &\quad \omega^{-1}(w_6, v_1 S^{-1}(w_3), w_1) p_R(v_2, S^{-1}(w_2)) \end{aligned}$$

3. *The forgetful functor $\mathcal{U} : \mathcal{YD}_H^H \longrightarrow \mathcal{M}^H$ is monoidal.*

Proof. The proof follows closely the one in [5] and it is based exclusively on computations that we leave to the reader. We shall only give the missing formula, namely for the right H-action on the tensor product

$$\begin{aligned} (v \otimes w) \lhd h &= (v_0 \lhd h_2)_0 \otimes (w_0 \lhd h_4)_0 \omega(h_1, (v_0 \lhd h_2)_1, (w_0 \lhd h_4)_1) \\ &\quad \omega^{-1}(v_1, h_3, (w_0 \lhd h_4)_2) \omega(v_2, w_1, h_5) \end{aligned}$$

For later use, notice that the formula (3.6) makes sense also for W just a right H-comodule (not necessarily an Yetter-Drinfeld module) and defines a natural transformation $c_{V,-} : V \otimes - \longrightarrow - \otimes V$, where $V \otimes -$ and $- \otimes V$ are considered as endofunctors of $\mathcal{M}^H$. ■

From experience with Hopf algebras and quasi-Hopf algebras, it is natural to ask if the category of Yetter–Drinfeld modules over a coquasi-bialgebra is related to the (weak) center of the category of comodules. The reader is referred to [14] for the center construction, and to [18] for the notion of weak center of a monoidal category. For quasi-bialgebras, it is known that the category of Yetter–Drinfeld modules is equivalent to the monoidal weak center of the category of modules (see [5]). But when dealing with the monoidal category of comodules, it seems to be none explicit proof in the literature, except for a short notice of Schauenburg ([7]). We shall present here the definition of the weak center in the context of the comodule category over a coquasi-bialgebra (although it is applicable to any monoidal category).

Definition 3.4 *For a coquasi-bialgebra H, the (left) weak center of the category of right H-comodules $\mathcal{W}_l(\mathcal{M}^H)$ has objects $(V, c_{V,-})$, where $V \in \mathcal{M}^H$ and $c_{V,W} : V \otimes W \longrightarrow W \otimes V$ is a natural transformation in $W \in \mathcal{M}^H$ that satisfies the commutative diagram*

$$\begin{array}{ccccc} (V \otimes W) \otimes X & \xrightarrow{c_{V,W} \otimes I_X} & (W \otimes V) \otimes X & \xrightarrow{\Phi_{V,W,X}} & W \otimes (V \otimes X) \\ \downarrow \Phi_{V,W,X} & & & & \downarrow I_W \otimes c_{V,X} \\ V \otimes (W \otimes X) & \xrightarrow{c_{V,W \otimes X}} & (W \otimes X) \otimes V & \xrightarrow{\Phi_{W,X,V}} & W \otimes (X \otimes V) \end{array} \qquad (3.7)$$

for all $W, X \in \mathcal{M}^H$ and $c_{V,\Bbbk} = I_V$. A morphism $f : (V, c_{V,-}) \longrightarrow (V', c_{V',-})$ in the (weak) center is a right H-colinear map satisfying

$$
\begin{array}{ccc}
V \otimes W & \stackrel{f\otimes I_W}{\longrightarrow} & V' \otimes W \\
\downarrow c_{V,W} & & \downarrow c_{V',W} \\
W \otimes V & \stackrel{I_W\otimes f}{\longrightarrow} & W \otimes V'
\end{array}
$$

for all $W \in \mathcal{M}^H$. The category $\mathcal{W}_l(\mathcal{M}^H)$ is monoidal and pre–braided with monoidal tensor product $(V, c_{V,-}) \otimes (W, c_{W',-}) = (V \otimes W, c_{V\otimes W,-})$, where

$$
\begin{aligned}
c_{V\otimes W,X} \quad : \quad & (V \otimes W) \otimes X \stackrel{\Phi_{V,W,X}}{\longrightarrow} V \otimes (W \otimes X) \stackrel{I_V\otimes c_{W,X}}{\longrightarrow} V \otimes (X \otimes W) \\
& \stackrel{\Phi^{-1}_{V,X,W}}{\longrightarrow} (V \otimes X) \otimes W \stackrel{c_{V,X}\otimes I_W}{\longrightarrow} (X \otimes V) \otimes W \stackrel{\Phi_{X,V,W}}{\longrightarrow} X \otimes (V \otimes W)
\end{aligned}
$$

The center of the category of right H-comodules $\mathcal{Z}(\mathcal{M}^H)$ is formed with the objects $(V, c_{V,-})$ of $\mathcal{W}_l(\mathcal{M}^H)$ for which $c_{V,-}$ is a natural isomorphism.

Then we have:

Theorem 3.5 *The category of right-right Yetter-Drinfeld modules is (monoidal braided) isomorphic to the weak left center of the category of right H-comodules $\mathcal{W}_l(\mathcal{M}^H)$.*

Proof. For a Yetter-Drinfeld module V and a right H-comodule W, use relation (3.6) to define natural transformation $c_{V,W}$. Then by (3.1)-(3.3) one can easily check that $(V, c_{V,-}) \in \mathcal{W}_l(\mathcal{M}^H)$. ■

As for the reverse direction, we can associate to every object $(V, c_{V,-})$ of $\mathcal{W}_l(\mathcal{M}^H)$ a right action of H as follows:

$$
\triangleleft : V \otimes H \longrightarrow H, \quad v \triangleleft h = (\varepsilon \otimes I_V)c_{V,H}(v \otimes h) \tag{3.8}
$$

Let $W \in \mathcal{M}^H$ and $w^* \in W^*$. Define the map $f_{w^*} : W \longrightarrow H$, $f(w) = w_1 w^*(w_0)$. Then f_{w^*} is H-colinear. By naturality of $c_{V,-}$, the following diagram commutes:

$$
\begin{array}{ccc}
V \otimes W & \stackrel{I_V\otimes f_{w^*}}{\longrightarrow} & V \otimes H \\
\downarrow c_{V,W} & & \downarrow c_{V,H} \\
W \otimes V & \stackrel{f_{w^*}\otimes I_V}{\longrightarrow} & H \otimes V
\end{array}
$$

Then $(f_{w^*} \otimes I_V)c_{V,W}(v \otimes w) = w^*(w_0)c_{V,H}(v \otimes w_1)$. Applying $\varepsilon \otimes I_V$ to both sides leads to $(w^* \otimes I_V)c_{V,W}(v \otimes w) = w^*(w_0)v \triangleleft w_1$, for all $w^* \in W^*$. This implies

$$\begin{aligned} c_{V,W}(v \otimes w) &= w_0 \otimes (\varepsilon \otimes I_V)c_{V,H}(v \otimes w_1) \\ &= w_0 \otimes v \triangleleft w_1 \end{aligned} \tag{3.9}$$

for all objects $W \in \mathcal{M}^H$.

Relation (3.1) for $\triangleleft$ is a consequence of $c_{V,\Bbbk} = I_V$ and of the fact that the unit map $u : \Bbbk \longrightarrow H$ is a right comodule map, where $\Bbbk$ has trivial comodule structure and H is comodule via Δ. Next, (3.2) follows from the colinearity of $c_{V,H}$ and from (3.9), while relation (3.3) is a consequence of (3.7) applied to $W = X = H$. Then $(V, \triangleleft)$ becomes a right-right Yetter–Drinfeld module. It is easy to see now that (3.8) and (3.9) are inverse correspondences, hence $\mathcal{YD}_H^H$ and $\mathcal{W}_l(\mathcal{M}^H)$ are isomorphic categories.

4. Galois extensions over coquasi-Hopf algebras

Let H be a coquasi-bialgebra. Recall that the category of right H-comodules is monoidal. A right H-comodule algebra A is an algebra in the monoidal category $\mathcal{M}^H$. This means (A, ρ_A) is a right H-comodule, we have a multiplication map $\mu_A : A \otimes A \longrightarrow A$, denoted $\mu_A(a \otimes b) = ab$, for $a, b \in A$, and a unit map $u_A : \Bbbk \longrightarrow A$, where we put $u_A(1) = 1_A$, which are both H-colinear, such that

$$(ab)c = a_0(b_0c_0)\omega(a_1, b_1, c_1) \tag{4.1}$$

holds for any $a, b, c \in A$. Notice that the multiplication on A does not need to be associative. Consider the space of coinvariants

$$B = A^{coH} = \{a \in A \,|\, \rho_A(a) = a \otimes 1_H\}$$

It is immediate that this is an associative $\Bbbk$-algebra (unlike A!) with unit and multiplication induced by the unit and the multiplication of A.

If H is a coquasi-Hopf algebra, we say that the extension $B \subseteq A$ is Galois ([8], Definition 8) if the map $can : A \otimes_B A \longrightarrow A \otimes H$, given by

$$a \otimes_B b \longrightarrow a_0b_0 \otimes b_4\omega^{-1}(a_1, b_1\beta(b_2), S(b_3)) \tag{4.2}$$

is bijective. If the extension is Galois, denote $can^{-1}(1_A \otimes h) = \sum_i l_i(h) \otimes_B r_i(h)$ for $h \in H$. We have the obvious relation $\sum_i l_i(1_H) \otimes_B r_i(1_H) = 1_A \otimes_B 1_A$. Moreover, in [8], Proposition 23, it was shown that the following formulas hold

$$\begin{aligned}
\sum_i l_i(h)\otimes_B r_i(h)_0\otimes r_i(h)_1 &= \sum_i l_i(h_1)\otimes_B r_i(h_1)\otimes h_2 && (4.3)\\
\sum_i l_i(h)_0\otimes_B r_i(h)\otimes l_i(h)_1 &= \sum_i l_i(h_2)\otimes_B r_i(h_2)\otimes S(h_1) && (4.4)\\
\sum_i l_i(h)r_i(h) &= \alpha(h)1_A && (4.5)\\
\sum_i a_0\beta(a_1)l_i(a_2)\otimes_B r_i(a_2) &= 1_A\otimes_B a && (4.6)\\
\sum_i l_i(h)\otimes_B 1_A\otimes_B r_i(h) &= \sum_{i,j} l_i(h_1)\otimes_B r_i(h_1)\beta(h_2)l_j(h_3)\otimes && (4.7)\\
&\quad r_j(h_3)\\
\sum_i l_i(hg)\otimes_B r_i(hg) &= \sum_{i,j}\mathbf{f}^{-1}(h_1,g_1)l_i(g_2)l_j(h_2)\otimes_B && (4.8)\\
&\quad r_j(h_2)r_i(g_2)
\end{aligned}$$

for all $h, g \in H$, $a \in A$, generalizing the Hopf algebra case. Also from the definition of the Galois map *can* we can easily obtain that

$$\sum_i bl_i(h)\otimes_B r_i(h) = \sum_i l_i(h)\otimes_B r_i(h)b \tag{4.9}$$

for all $h \in H$, $b \in B$.

For A a right H-comodule algebra, we may define the notion of right module over A in the monoidal category $\mathcal{M}^H$. Explicitly, this is a right H-comodule (M, ρ_M), endowed with a right A-action $r_M : M\otimes A \longrightarrow M$, denoted $r_M(m, a) = ma$, such that

$$\begin{aligned}
(ma)b &= m_0(a_0b_0)\omega(m_1, a_1, b_1)\\
m1_A &= m\\
\rho_M(ma) &= m_0a_0\otimes m_1a_1
\end{aligned}$$

hold for all $m \in M$, $a, b \in A$. The category of such objects, with morphisms the right H-colinear maps which respect the A-action, is called the category of relative right (H, A)-Hopf modules and denoted $\mathcal{M}_A^H$. Similarly we can define left A-modules and A-bimodules within $\mathcal{M}^H$, with corresponding categories of left (H, A)-Hopf modules ${}_A\mathcal{M}^H$, respectively (H, A)-Hopf bimodules ${}_A\mathcal{M}_A^H$. For later use, notice that for an A-bimodule M in $\mathcal{M}^H$ we have two linear maps $l_M : a\otimes m \longrightarrow am$ and $r_M : m\otimes a \longrightarrow ma$ with properties as above which have also to satisfy the compatibility condition $(am)b = a_0(m_0b_0)\omega(a_1, m_1, b_1)$ for any $a, b \in A$ and $m \in M$.

For $M \in \mathcal{M}_A^H$ and $M' \in {}_A\mathcal{M}^H$, their tensor product $\bigcirc_A$ over A within $\mathcal{M}^H$ is defined as the coequalizer of the colinear maps

$$(M \otimes A) \otimes M' \underset{(I_M \otimes l_{M'})\Phi_{M,A,M'}}{\overset{r_M \otimes I_{M'}}{\rightrightarrows}} M \otimes M' \longrightarrow M \bigcirc_A M' \longrightarrow 0$$

in the category of comodules (which is abelian, as we are working over a commutative field), where we have denoted by r_M, $l_{M'}$ the right, respectively the left multiplication maps with elements of A (we refer to [19] for details about tensor product over an algebra in a monoidal category). The comodule structure on $M \bigcirc_A M'$ is induced by the codiagonal one on $M \otimes M'$. If M is a Hopf bimodule, then $M \bigcirc_A M'$ becomes a left (H, A)-Hopf module with left A-multiplication induced by the first tensorand. If M' is a Hopf bimodule, there is a similar result. In particular, for any two Hopf bimodules M, M', their tensor product $M \bigcirc_A M'$ is again a Hopf bimodule, hence the category of Hopf bimodules is monoidal, as for ordinary bimodules, except for the presence of the reassociator, which is induced by Φ, with A as unit object. For any right comodule V, the map $(V \otimes A) \otimes M \overset{(I_V \otimes l_M)\Phi_{V,A,M}}{\longrightarrow} V \otimes M$ induces a natural isomorphism $(V \otimes A) \bigcirc_A M \simeq V \otimes M$ in $\mathcal{M}_A^H$. If V is also left (H, A)-Hopf module, then the previous isomorphism holds in ${}_A\mathcal{M}_A^H$. Similarly we have $M \bigcirc_A (A \otimes V) \simeq M \otimes V$ in ${}_A\mathcal{M}^H$, for any comodule V and left (H, A)-module M.

We shall suppose from now on that H is a coquasi-Hopf algebra and A is a right H-comodule algebra with coinvariants $B = A^{coH}$ such that $B \subseteq A$ is Galois. For any Hopf bimodule $M \in {}_A\mathcal{M}_A^H$, define the centralizer of B in M by $M^B = \{m \in M \mid bm = mb \; \forall b \in B\}$. Note that M is indeed a B-bimodule by restricting scalars, hence it make sense to speak about the centralizer M^B. Then we have the following generalization of the Miyashita-Ulbrich action for coquasi-Hopf algebras:

Proposition 4.1 *Under the above assumptions, M^B becomes an Yetter-Drinfeld module with coaction inherited from the one on M, and right H-action $\triangleleft : M^B \otimes H \longrightarrow M^B$, given by*

$$m \triangleleft h = \sum_i l_i(h_4)[m_0 r_i(h_4)]\omega(h_1\beta(h_2), S(h_3), m_1h_5) \tag{4.10}$$

Proof. First we have to show that M^B with the restricted coaction is a H-subcomodule of M. For any $m \in M^B$ and $b \in B$ we have

$$bm_0 \otimes m_1 = (bm)_0 \otimes (bm)_1 = (mb)_0 \otimes (mb)_1 = m_0b \otimes m_1$$

Next, the right H-action is well defined:

$$\begin{aligned}
b(m \triangleleft h) &= b\sum_i l_i(h_4)[m_0 r_i(h_4)]\omega(h_1\beta(h_2), S(h_3), m_1h_5) \\
(4.9) \quad &= \sum_i l_i(h_4)[m_0 r_i(h_4)b]\omega(h_1\beta(h_2), S(h_3), m_1h_5) \\
&= [\sum_i l_i(h_4)[m_0 r_i(h_4)]\omega(h_1\beta(h_2), S(h_3), m_1h_5)]b \\
&= (m \triangleleft h)b
\end{aligned}$$

for any $m \in M^B$, $h \in H$, and $b \in B$, where in the third line we have used the fact that $b \in B$ is H-coinvariant. Now it is obvious that $m \triangleleft 1_H = m$. In order to show relation (3.2), we compute for $m \in M^B$ and $h \in H$ that

$$\begin{aligned}
(m \triangleleft h_2)_0 \otimes h_1(m \triangleleft h_2)_1 &= \sum_i l_i(h_5)_0[m_0 r_i(h_5)_0]\omega(h_2\beta(h_3), S(h_4), m_2h_6) \\
&\quad \otimes h_1[l_i(h_5)_1(m_1 r_i(h_5)_1)] \\
(4.4, 4.3) \quad &= \sum_i l_i(h_6)[m_0 r_i(h_6)]\omega(h_2\beta(h_3), S(h_4), m_2h_8) \\
&\quad \otimes h_1[S(h_5)(m_1h_7)] \\
(2.1), (2.6) \quad &= \sum_i l_i(h_4)[m_0 r_i(h_4)]\omega(h_1\beta(h_2), S(h_3), m_1h_5) \\
&\quad \otimes m_2h_6 \\
(4.10) \quad &= m_0 \triangleleft h_1 \otimes m_1h_2
\end{aligned}$$

Finally, for $m \in M^B$ and $h, g \in H$ we have

$$\begin{aligned}
&(m_0 \triangleleft (h_2g_2))_0\omega(h_1, g_1, (m_0 \triangleleft (h_2g_2))_1)\omega(m_1, h_3, g_3) = \\
&= \sum_i l_i(h_5g_5)_0[m_0 r_i(h_5g_5)_0] \\
&\quad \omega(h_2g_2\beta(h_3g_3), S(h_4g_4), m_2(h_6g_6)) \\
&\quad \omega(h_1, g_1, l_i(h_5g_5)_1[m_1 r_i(h_5g_5)_1])\omega(m_3, h_7, g_7) \\
(4.4), (4.3), (4.8) \quad &= \sum_{i,j}[l_i(g_7)l_j(h_7)]\{m_0[r_j(h_7)r_i(g_7)]\}\beta(h_3g_3) \\
&\quad \mathbf{f}^{-1}(h_6, g_6)\omega(h_1, g_1, S(h_5g_5)[m_1(h_8g_8)]) \\
&\quad \omega(h_2g_2, S(h_4g_4), m_2(h_9g_9))\omega(m_3, h_{10}, g_{10}) \\
(2.16), (2.20) \quad &= \sum_{i,j}[l_i(g_{10})l_j(h_9)]\{m_0[r_j(h_9)r_i(g_{10})]\}\beta(h_5)\beta(g_5) \\
&\quad \omega(h_1, g_1, [S(g_9)S(h_8)][m_1(h_{10}g_{11})]) \\
&\quad \omega(h_2g_2, S(g_8)S(h_7), m_2(h_{11}g_{12}))\omega(m_3, h_{12}, g_{13})
\end{aligned}$$

$$
\begin{aligned}
& && \omega(h_3g_3, S(g_7), S(h_6))\omega^{-1}(h_4, g_4, S(g_6)) \\
(2.1), (4.1) &=&& \sum_{i,j}[l_i(g_{10})l_j(h_9)]\{[m_0r_j(h_9)]r_i(g_{10})\}\beta(h_5)\beta(g_5) \\
& && \omega(h_1, g_1, [S(g_9)S(h_8)][(m_1h_{10})g_{11}]) \\
& && \omega(h_2g_2, S(g_8)S(h_7), (m_2h_{11})g_{12}) \\
& && \omega(h_3g_3, S(g_7), S(h_6))\omega^{-1}(h_4, g_4, S(g_6)) \\
(2.3), (2.3), (2.6) &=&& \sum_{i,j}[l_i(g_8)l_j(h_6)]\{[m_0r_j(h_6)]r_i(g_8)\}\beta(h_2)\beta(g_4) \\
& && \omega(g_1, S(g_7)S(h_5), (m_1h_7)g_9) \\
& && \omega(h_1, g_2[S(g_6)S(h_4)], (m_2h_8)g_{10})\omega(g_3, S(g_5), S(h_3)) \\
(2.1), (2.6) &=&& \sum_{i,j}[l_i(g_6)l_j(h_6)]\{[m_0r_j(h_6)]r_i(g_6)\}\beta(h_2)\beta(g_3) \\
& && \omega(g_1, S(g_5)S(h_5), (m_1h_7)g_7)\omega(g_2, S(g_4), S(h_4)) \\
& && \omega(h_1, S(h_3), (m_2h_8)g_8) \\
(4.1), (4.1) &=&& \sum_{i,j} l_i(g_7)\{[l_j(h_8)(m_0r_j(h_8))]r_i(g_7)\}\beta(h_2)\beta(g_3) \\
& && \omega^{-1}(S(h_7), m_1h_9, g_8)\omega(S(g_6), S(h_6), (m_2h_{10})g_9) \\
& && \omega(g_1, S(g_5)S(h_5), (m_3h_{11})g_{10})\omega(g_2, S(g_4), S(h_4)) \\
& && \omega(h_1, S(h_3), (m_4h_{12})g_{11}) \\
(2.3), (2.6), (2.3), (2.6) &=&& \sum_{i,j} l_j(g_4)\{[l_i(h_7)(m_0r_i(h_7))]r_j(g_4)\} \\
& && \omega(g_1\beta(g_2), S(g_3), [S(h_6)(m_1h_8)]g_5) \\
& && \omega(h_2\beta(h_3), S(h_4), m_3h_{10})\omega(h_1, S(h_5)(m_2h_9), g_6) \\
(4.10) &=&& (m \lhd h_2)_0 \lhd g_1\omega(h_1, (m \lhd h_2)_1, g_2)
\end{aligned}
$$

■

Corollary 4.2 *We have obtained a functor* ${}_A\mathcal{M}_A^H \xrightarrow{(-)^B} \mathcal{YD}_H^H$.

Proof. By the previous proposition, it remains to check only the behavior of $(-)^B$ on morphisms: let $f \in Hom_{A-A}^H(M, M')$. The A-bilinearity of f ensures that f sends M^B to M'^B, the H-colinearity follows by definition. All that is left is to verify that f is invariant under the generalized Miyashita-Ulbrich action:

$$
\begin{aligned}
f(m \lhd h) &= f(\sum_i l_i(h_4)[m_0r_i(h_4)]\omega(h_1\beta(h_2), S(h_3), m_1h_5)) \\
&= \sum_i l_i(h_4)[f(m_0)r_i(h_4)]\omega(h_1\beta(h_2), S(h_3), m_1h_5)
\end{aligned}
$$

$$(4.10) \quad = \quad f(m) \triangleleft h$$

for all $m \in M^B$ and $h \in H$, by the A-bilinearity and H-colinearity of f. ■

In particular, A is an object in ${}_A\mathcal{M}_A^H$, therefore A^B is a Yetter-Drinfeld module. But the product of two elements of A^B lies in A^B as it can easily be seen and it is natural to ask if A^B becomes a braided commutative algebra in $\mathcal{YD}_H^H$, as in the classical Hopf algebraic case. The answer is positive, by the following

Proposition 4.3 *A^B is a braided commutative algebra in the monoidal category of right-right Yetter-Drinfeld modules.*

Proof. First, we need to check that the induced multiplication $\mu_{A^B} : A^B \otimes A^B \longrightarrow A^B$ is a morphism of Yetter-Drinfeld modules: it is colinear, as the multiplication on A is. For the right H-linearity, we compute

$$\begin{aligned}
(a \otimes b) \otimes h \quad & \longrightarrow \quad (a \otimes b) \triangleleft h \longrightarrow (a_0 \triangleleft h_2)_0 (b_0 \triangleleft h_4)_0 \\
& \qquad \omega(h_1, (a_0 \triangleleft h_2)_1, (b_0 \triangleleft h_4)_1) \\
& \qquad \omega^{-1}(a_1, h_3, (b_0 \triangleleft h_4)_2)\omega(a_2, b_1, h_5) \\
& = \quad \sum_{i,j} \{l_i(h_6)[a_0 r_i(h_6)]\}\{l_j(h_{15})[b_0 r_j(h_{15})]\}\beta(h_3)\beta(h_{11}) \\
& \qquad \omega(h_1, S(h_5)[a_1 h_7], S(h_{14})[b_1 h_{16}])\omega(h_2, S(h_4), a_2 h_8) \\
& \qquad \omega^{-1}(a_3, h_9, S(h_{13})[b_2 h_{17}])\omega(h_{10}, S(h_{12}), b_3 h_{18}) \\
& \qquad \omega(a_4, b_4, h_{19}) \\
(4.1), (4.1), (4.1) \quad & = \quad \sum_{i,j} [l_i(h_8)a_0]\{[r_i(h_8)l_j(h_{22})][b_0 r_j(h_{22})]\}\beta(h_3)\beta(h_{16}) \\
& \qquad \omega^{-1}(h_9, S(h_{21}), b_1 h_{23})\omega(S(h_7)a_1, h_{10}, S(h_{20})(b_2 h_{24})) \\
& \qquad \omega(h_1, [S(h_6)a_2]h_{11}, S(h_{19})(b_3 h_{25}))\omega^{-1}(S(h_5), a_3, h_{12}) \\
& \qquad \omega(h_2, S(h_4), a_4 h_{13})\omega^{-1}(a_5, h_{14}, S(h_{18})(b_4 h_{26})) \\
& \qquad \omega(h_{15}, S(h_{17}), b_5 h_{27})\omega(a_6, b_6, h_{28}) \\
(2.1), (2.3), (2.6) \quad & = \quad \sum_{i,j} [l_i(h_7)a_0]\{[r_i(h_7)l_j(h_{21})][b_0 r_j(h_{21})]\}\beta(h_2)\beta(h_{14}) \\
& \qquad \omega^{-1}(h_9, S(h_{20}), b_1 h_{22})\omega(S(h_6)a_1, h_8, S(h_{19})(b_2 h_{23})) \\
& \qquad \omega^{-1}(S(h_5), a_2, h_9)\omega^{-1}(S(h_4), a_3 h_{10}, S(h_{18})(b_3 h_{24})) \\
& \qquad \omega(h_1, S(h_3), (a_4 h_{11})[S(h_{17})(b_4 h_{25})]) \\
& \qquad \omega^{-1}(a_5, h_{12}, S(h_{16})(b_5 h_{26}))\omega(h_{13}, S(h_{15}), b_6 h_{27}) \\
(2.1), (2.1), (2.6) \quad & = \quad \sum_{i,j} [l_i(h_7)a_0]\{[r_i(h_7)l_j(h_{20})][b_0 r_j(h_{20})]\}\beta(h_2)\beta(h_{14}) \\
& \qquad \omega^{-1}(h_8, S(h_{19}), b_1 h_{21})\omega(S(h_6)a_1, h_9, S(h_{18})(b_2 h_{22}))
\end{aligned}$$

$$
\begin{aligned}
& & & \omega^{-1}(S(h_5), a_2, h_{10})\omega^{-1}(S(h_4), a_3h_{11}, S(h_{17})(b_3h_{23}))\\
& & & \omega^{-1}(a_4, h_{12}, S(h_{16})(b_4h_{24}))\\
& & & \omega(h_{13}, S(h_{15}), b_5h_{25})\omega(h_1, S(h_3), a_5(b_6h_{26}))\\
(2.1), (2.6) & & = & \sum_{i,j}[l_i(h_5)a_0]\{[r_i(h_5)\beta(h_6)l_j(h_7)][b_0r_j(h_7)]\}\}\\
& & & \omega^{-1}(S(h_4), a_1, b_1h_8)\omega(h_1\beta(h_2), S(h_3), (a_2b_2)h_9)\\
& & & \omega(a_3, b_3, h_{10})\\
(4.7) & & = & \sum_i[l_i(h_5)a_0][b_0r_i(h_5)]\omega^{-1}(S(h_4), a_1, b_1h_6)\omega(a_2, b_2, h_7)\\
& & & \omega(h_1\beta(h_2), S(h_3), (a_3b_3)h_8)\\
& & = & \sum_i l_i(h_4)[a_0(b_0r_i(h_4))]\omega(a_1, b_1, h_5)\\
& & & \omega(h_1\beta(h_2), S(h_3), (a_2b_2)h_6)\\
(4.1), (4.1) & & = & \sum_i l_i(h_4)[(a_0b_0)r_i(h_4)]\omega(h_1\beta(h_2), S(h_3), (a_1b_1)h_5)\\
& & = & (ab) \lhd h
\end{aligned}
$$

This can be rewritten as

$$
\begin{aligned}
(ab) \lhd h &= (a_0 \lhd h_2)_0(b_0 \lhd h_4)_0\omega(h_1, (a_0 \lhd h_2)_1, (b_0 \lhd h_4)_1) \\
& \quad \omega^{-1}(a_1, h_3, (b_0 \lhd h_4)_2)\omega(a_2, b_1, h_5)
\end{aligned} \tag{4.11}
$$

for all $a, b \in A^B$ and $h \in H$. The associativity is simply the one in $\mathcal{M}^H$, while for the (braided) commutativity we find that

$$
\begin{aligned}
b_0(a \lhd b_1) &= \sum_i \beta(b_2)b_0\{l_i(b_4)[a_0r_i(b_4)]\}\omega(b_1, S(b_3), a_1b_5))\\
(4.1) \quad &= \sum_i \beta(b_1)[b_0l_i(b_2)][ar_i(b_2)]\\
(4.6) \quad &= ab
\end{aligned}
$$

for all $a, b \in A^B$. ■

Notice that for each Hopf bimodule $M \in {}_A\mathcal{M}_A^H$, the centralizer M^B inherits a natural structure of A^B-bimodule within $\mathcal{YD}_H^H$ by restricting the scalars. The proof is similar to the one in the previous proposition, where instead of multiplication of A^B one should consider the left (or right) A^B-action on M^B. Therefore the above functor factorizes to ${}_A\mathcal{M}_A^H \xrightarrow{(-)^B} {}_{A^B}(\mathcal{YD}_H^H)_{A^B}$.

In the Hopf algebra case, the categories of Hopf bimodules and Yetter-Drinfeld modules are intimately related by the fact that the center of the first mentioned category is precisely the category of Yetter-Drinfeld modules. This was shown by

Schauenburg in [7], under the assumption of a flat Galois extension. As coquasi-Hopf algebras naturally generalize Hopf algebras in terms of monoidal categories, it is not surprising to see that this results still holds in the coquasi-case. First, restating Definition 3.4 but this time for the monoidal category $({}_A\mathcal{M}_A^H, \bigcirc_A)$, gives us the notion of the center of the category of Hopf bimodules $\mathcal{W}_l({}_A\mathcal{M}_A^H)$. We shall denote the braiding on $\mathcal{W}_l({}_A\mathcal{M}_A^H)$ by $\sigma_{-,-}$ in order to avoid confusions with the natural transformation $c_{-,-}$ giving the braid on $\mathcal{YD}_H^H$. Then we have the following:

Theorem 4.4 *Let H a coquasi-Hopf algebra and $B \subseteq A$ a right H-Galois which is flat as left B-module. Then there is a monoidal equivalence*

$$\mathcal{YD}_H^H \longrightarrow \mathcal{W}_l({}_A\mathcal{M}_A^H)$$

which assigns the vector space $A \otimes V$ to each Yetter-Drinfeld module V. This is an (H, A)-Hopf bimodule with codiagonal coaction and left induced A-action from the first tensorand, while the right A-action comes from

$$(A\otimes V)\otimes A \xrightarrow{\Phi_{A,V,A}} A\otimes(V\otimes A) \xrightarrow{I_A\otimes c_{V,A}} A\otimes(A\otimes V) \xrightarrow{\Phi^{-1}_{A,V,A}} (A\otimes A)\otimes V \xrightarrow{\mu_A\otimes I_V} A\otimes V$$

It is an object in the weak left center via the natural transformation $\sigma_{A\otimes V,W} : (A \otimes V) \bigcirc_A W \longrightarrow W \bigcirc_A (A \otimes V)$ induced by

$$\begin{aligned}&(A \otimes V) \otimes W \xrightarrow{\Phi_{A,V,W}} A \otimes (V \otimes W) \xrightarrow{I_A\otimes c_{V,W}} A \otimes (W \otimes V)\\ &\xrightarrow{\Phi^{-1}_{A,V,W}} (A \otimes W) \otimes V \xrightarrow{l_W\otimes I_V} W \otimes V \simeq W \bigcirc_A (A \otimes V)\end{aligned}$$

The monoidal structure of this functor is given by

$$(A \otimes V) \bigcirc_A (A \otimes V') \simeq (A \otimes V) \otimes V' \xrightarrow{\Phi_{A,V,V'}} A \otimes (V \otimes W)$$

Proof. The proof follows the same ideas as in Theorem 4.3 from [7]. Therefore we leave the details to the reader, with the mention that this is rather technical, due to coquasi-structures involved . We just give the construction of the inverse functor of the equivalence. Let $(M, \sigma_{M,-})$ be an object in the weak left center of the category of Hopf bimodules. But $A \otimes A$ is a Hopf bimodule with natural structure maps

$$\begin{aligned}\rho_{A\otimes A}(a \otimes b) &= a_0 \otimes b_0 \otimes a_1 b_1\\ a(b \otimes c) &= a_0 b_0 \otimes c_0 \omega^{-1}(a_1, b_1, c_1)\\ (a \otimes b)c &= a_0 \otimes b_0 c_0 \omega(a_1, b_1, c_1)\end{aligned}$$

for $a, b, c \in A$, therefore it induces a left A-linear right H-colinear morphism $\theta : M \longrightarrow A \otimes M$ by

$$M \overset{I_M \otimes u_A}{\longrightarrow} M \otimes A \simeq M \bigcirc_A (A \otimes A) \overset{\sigma_{M,A\otimes A}}{\longrightarrow} (A \otimes A) \bigcirc_A M \simeq A \otimes M$$

Denote ${}^{\theta}M = \{m \in M \mid \theta(m) = 1_A \otimes m\}$. Then ${}^{\theta}M$ is a subcomodule of M (more precisely an Yetter-Drinfeld submodule of M^B) and we obtain a functor $\mathcal{W}_l({}_A\mathcal{M}_A^H) \longrightarrow \mathcal{YD}_H^H$, by $M \longrightarrow {}^{\theta}M$ on objects and by restriction on morphisms. This will turn out to be the requested quasi-inverse of the equivalence. ■

5. Conclusions

The natural generalization from bialgebras to coquasi-bialgebras by means of monoidal categories opens a wide space of possible research. For example, in the context of the present paper, there are many subjects still to be investigated, as the connection of Yetter-Drinfeld modules with the Drinfeld double, the category of two-sided two-cosided Hopf bimodules, the coquasi-triangular structures on H, the bosonisation process and so on. On the other hand, Galois extensions provide particular examples of braided commutative algebras in $\mathcal{YD}_H^H$, and it is reasonable to expect that a theorem similar to the one of Brzezinski and Militaru ([20]) holds in an appropriate coquasi-algebraic context.

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