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# GENERALIZED $\eta$-RICCI SOLITONS ON LP-KENMOTSU MANIFOLDS ASSOCIATED TO THE SCHOUTEN-VAN KAMPEN CONNECTION

Shahroud Azami[1]

*In this paper, we investigate LP-Kenmotsu manifolds admitting generalized $\eta$-Ricci solitons associated to the Schouten-van Kampen connection. We provide two examples of generalized $\eta$-Ricci solitons on a LP-Kenmotsu manifolds to prove our results.*

**Keywords:** LP-Kenmotsu manifolds, generalized $\eta$-Ricci soliton, Schouten-van Kampen

**MSC2020:** 53C25, 53D15, 53E20.

## 1. **Introduction**

The almost para contact Riemannian manifold was introduced by Sato [23] in 1976. Then, the notion of a para-Sasakian and SP para-Sasakian manifolds have been defined and studied by Adati and Matsumoto [1] as a class of almost contact Riemannian manifolds. The Kenmotsu manifold was introduced by Kenmotsu [16] in 1972 as a new class of almost contact metric manifolds. Kenmotsu manifolds are very closely related to the warped product manifolds. Sinha and Prasad [26] studied para Kenmotsu manifolds as a class of almost para contact metric manifolds. In 1989, Matsumoto introduced [17] defined and studied Lorentzian para-Sasakian manifolds. Mihai and Rosca [19] also added some remarks on Lorentzian para-Sasakian manifolds. In 2018, Haseeb and Prasad defined and investigated a class of Lorentzian almost para contact metric manifolds namely Lorentzian para-Kenmotsu (briefly LP-Kenmotsu manifolds) manifolds [14]. Devi et al. [12] studied certain curvature connections on Lorentzian para-Kenmotsu manifolds.

In 1982, Hamilton [13] introduced the notion of Ricci soliton as a generalization of Einstein metrics and a special solution to Ricci flow on a Riemannian manifold. A Ricci soliton [6] is a triplet $(g, V, \lambda)$ on a pseudo-Riemannian manifold $M$ such that

$$\mathcal{L}_V g + 2S + 2\lambda g = 0, \tag{1}$$

where $\mathcal{L}_V$ is the Lie derivative along the potential vector field $V$, $S$ is the Ricci tensor, and $\lambda$ is a real constant. The Ricci soliton is said to be shrinking, steady and expanding according as $\lambda < 0$ , $\lambda = 0$ and $\lambda > 0$, respectively. If the vector field $V$ is the gradient of a potential function $\psi$, then $g$ is called a gradient Ricci soliton. Prasad et al. [22] studied Ricci solitons on $\phi$-semi-symmetric LP-Kenmotsu manifolds with a quarter-symmetric non-metric connection. In 2016, Nurowski and Randall [20] introduced the notion of generalized Ricci soliton as follows

$$\mathcal{L}_V g + 2\mu V^\flat \otimes V^\flat - 2\alpha S - 2\lambda g = 0, \tag{2}$$

where $V^\flat$ is the canonical 1-form associated to $V$. Also, as a generalization of Ricci soliton, the notion of $\eta$-Ricci soliton was introduced by Cho and Kimura [10] which it is a 4-tuple

[1]Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran, e-mail: azami@sci.ikiu.ac.ir

$(g, V, \lambda, \rho)$, where $V$ is a vector field on $M$, $\lambda$ and $\rho$ are constants, and $g$ is a pseudo-Riemannian metric satisfying the equation

$$\mathcal{L}_V g + 2S + 2\lambda g + 2\rho\eta \otimes \eta = 0, \tag{3}$$

where $S$ is the Ricci tensor associated to $g$. Many authors studied the $\eta$-Ricci solitons [5, 11, 15, 21]. In particular, if $\rho = 0$, then the $\eta$-Ricci soliton equation reduces to the Ricci soliton equation. Motivated by the above studies M. D. Siddiqi [25] introduced the notion of generalized $\eta$-Ricci soliton as follows

$$\mathcal{L}_V g + 2\mu V^\flat \otimes V^\flat + 2S + 2\lambda g + 2\rho\eta \otimes \eta = 0. \tag{4}$$

Motivated by [3, 7, 18] and the above works, we study generalized $\eta$-Ricci solitons on LP-Kenmotsu manifolds assoicated the Schouten-van Kampen connection. We give an example of generalized $\eta$-Ricci soliton on a LP-Kenmotsu manifold assoicated the Schouten-van Kampen connection.

The paper is organized as follows. In Section 2, we recall some necessary and fundamental concepts and formulas on LP-Kenmotsu manifolds which be used throughout the paper. In Section 3, we give the main results and their proofs. In Section 4, we give two examples of LP-Kenmotsu manifolds admit in generalized $\eta$-Ricci soliton with respect to the Schouten-van Kampen connection.

## 2. **Preliminaries**

A $n$-dimensional Lorentzian metric manifold $(M, g)$ is said to be a Lorentzian almost para-contact manifold [2] with an almost contact structure $(\phi, \xi, \eta, g)$, if there exist a $(1, 1)$-tensor field $\phi$, a vector field $\xi$ and a 1-form $\eta$ such that

$$\phi^2(X) = X + \eta(X)\xi, \eta(\xi) = -1, \tag{5}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{6}$$

for all vector fields $X, Y$ on $M$. In this case, we have $\phi\xi = 0$, $\eta \circ \phi = 0$, and $\eta(X) = g(X, \xi)$. The fundamental 2-form $\Phi$ of $M$ is determined by $\Phi(X, Y) = g(X, \phi Y) = g(\phi X, Y)$, for all vector fields $X, Y$ on $M$. A Lorentzian almost para-contact manifold $M$ is called Lorentzian para-Kenmotsu manifold [14] if

$$(\nabla_X \phi)Y = -g(\phi X, Y)\xi - \eta(Y)\phi X \tag{7}$$

for all vector fields $X, Y$ on $M$. In a LP-Kenmotsu manifold, we have

$$\nabla_X \xi = -X - \eta(X)\xi, \tag{8}$$

$$(\nabla_X \eta)Y = -g(X, Y) - \eta(X)\eta(Y), \tag{9}$$

where $\nabla$ is the Levi-Civita connection with respect to the Lorentzian metric $g$. Using (18) and (9), we find

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{10}$$

$$R(X, \xi)Y = -g(X, Y)\xi + \eta(Y)X, \tag{11}$$

$$\eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \tag{12}$$

for all vector fields $X, Y, Z$, where $R$ is the Riemannian curvature tensor. The Ricci tensor $S$ of a LP-Kenmotsu manifold $M$ is defined by $S(X, Y) = \sum_{i=1}^{n} \epsilon_i g(R(e_i, X)Y, e_i)$ and we have

$$S(X, \xi) = (n - 1)\eta(X), \tag{13}$$

for all vector field $X$ on $M$.

Let $M$ be an almost contact metric manifold and $TM$ be the tangent bundle of $M$. We have two naturally defined distribution on tangent bundle $TM$ as $H = \ker\eta$ and $\hat{H} = \mathrm{span}\{\xi\}$, thus we get $TM = H \oplus \hat{H}$. Therefore, by this composition we can define the

Schouten-van Kampen connection $\bar{\nabla}$ [4, 27] on $M$ with respect to Levi-Civita connection $\nabla$ as follows

$$\bar{\nabla}_X Y = \nabla_X Y - \eta(Y)\nabla_X \xi + ((\nabla_X \eta)(Y))\xi \tag{14}$$

for all vector fields $X, Y$ on $M$. On LP-Kenmotsu manifolds, using (8), (9), and (14) we obtain

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi \tag{15}$$

for all vector fields $X, Y$ on $M$. Let $\bar{R}$ and $\bar{S}$ be the curvature tensors and the Ricci tensors of the connection $\bar{\nabla}$, respectively, that is,

$$\bar{R}(X,Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z, \qquad \bar{S}(X,Y) = \sum_{i=1}^{n} \epsilon_i g(\bar{R}(e_i, X)Y, e_i).$$

On LP-Kenmotsu manifolds, applying (15) and the above relation we have

$$\begin{aligned} \bar{R}(X,Y)Z &= R(X,Y)Z + 3g(Y,Z)X - 3g(X,Z)Y + 2g(Y,Z)\eta(X)\xi \\ &\quad -2g(X,Z)\eta(Y)\xi + 2\eta(Y)\eta(Z)X - 2\eta(X)\eta(Z)Y, \end{aligned} \tag{16}$$

and

$$\bar{S}(X,Y) = S(X,Y) + (3n-7)g(X,Y) + 2n\eta(X)\eta(Y), \tag{17}$$

for all vector fields $X, Y, Z$ on $M$, where $S$ denotes the Ricci tensor of the connection $\nabla$. Using (17), the Ricci operator $\bar{Q}$ of the connection $\bar{\nabla}$ is determined by

$$\bar{Q}X = QX + (3n-7)X + 2n\eta(X)\xi. \tag{18}$$

Let $r$ and $\bar{r}$ be the scalar curvature of the Levi-Civita connection $\nabla$ and the Schouten-van Kampen connection $\bar{\nabla}$. The equation (17) yields

$$\bar{r} = r + 3n^2 - 9n. \tag{19}$$

The generalized $\eta$-Ricci soliton associated to the Schouten-van Kampen connection is defined by

$$\alpha\bar{S} + \frac{\beta}{2}\overline{\mathcal{L}}_V g + \mu V^\flat \otimes V^\flat + \rho\eta\otimes\eta + \lambda g = 0, \tag{20}$$

where $\bar{S}$ denotes the Ricci tensor of the connection $\bar{\nabla}$,

$$(\overline{\mathcal{L}}_V g)(Y,Z) := g(\tilde{\nabla}_Y V, Z) + g(Y, \tilde{\nabla}_Z V),$$

$V^\flat$ is the canonical 1-form associated to $V$ that is $V^\flat(X) = g(V,X)$ for all vector field $X$, $\lambda$ is a smooth function on $M$, and $\alpha, \beta, \mu, \rho$ are real constant such that $(\alpha, \beta, \mu) \neq (0,0,0)$. The generalized $\eta$-Ricci soliton equation reduces to

(1) the $\eta$-Ricci soliton equation when $\alpha = 1$ and $\mu = 0$,
(2) the Ricci soliton equation when $\alpha = 1$, $\mu = 0$, and $\rho = 0$,
(3) the generalized Ricci soliton equation when $\rho = 0$.

Note that

$$\begin{aligned} (\overline{\mathcal{L}}_V g)(X,Y) &= g(\tilde{\nabla}_Y V, Z) + g(Y, \tilde{\nabla}_Z V) \\ &= \mathcal{L}_V g(X,Y) + 2\eta(V)g(X,Y) - g(X,V)\eta(Y) - g(Y,V)\eta(X). \end{aligned} \tag{21}$$

## 3. Main results and their proofs

A LP-Kenmotsu manifold is said to $\eta$-Einstein with respect to the Schouten-van Kampen connection if its Ricci tensor $\bar{S}$ is of the form $\bar{S} = ag + b\eta \otimes \eta$, where $a$ and $b$ are smooth functions on manifold. Let $M$ be a LP-Kenmotsu manifold. Now, we consider $M$ satisfies the generalized $\eta$-Ricci soliton (20) associated to the Schouten-van Kampen connection and the potential vector field $V$ is a pointwise collinear vector field with the structure vector field $\xi$, that is, $V = f\xi$ for some function $f$ on $M$. Using (18) we get

$$\mathcal{L}_{f\xi}g(X,Y) = (Xf)\eta(Y) + (Yf)\eta(X) - 2f(g(X,Y) + \eta(X)\eta(Y)),$$

hence

$$\overline{\mathcal{L}}_{f\xi}g(X,Y) = (Xf)\eta(Y) + (Yf)\eta(X) - 4f(g(X,Y) + \eta(X)\eta(Y))$$

for all vector fields $X, Y$ on $M$. Also, we have

$$\xi^\flat \otimes \xi^\flat(X,Y) = \eta(X)\eta(Y), \tag{22}$$

for all vector fields $X, Y$. Applying $V = f\xi$, (17) and (22) in the equation (20) we infer

$$\alpha\bar{S}(X,Y) + \frac{\beta}{2}\left[(Xf)\eta(Y) + (Yf)\eta(X) - 4f(g(X,Y) + \eta(X)\eta(Y))\right]$$
$$+(\mu f^2 + \rho)\eta(X)\eta(Y) + \lambda g(X,Y) = 0 \tag{23}$$

for all vector fields $X, Y$ on $M$. We plug $Y = \xi$ in the above equation and using (13) and (17) to yield

$$-\frac{\beta}{2}Xf + \frac{\beta}{2}(\xi f)\eta(X) + ((2n-8)\alpha - \mu f^2 - \rho + \lambda)\eta(X) = 0 \tag{24}$$

for all vector fields $X$ on $M$. Taking $X = \xi$ in (24) gives

$$\beta\xi f = -((2n-8)\alpha - \mu f^2 - \rho + \lambda). \tag{25}$$

Inserting (25) in (24), we conclude

$$\beta Xf = ((2n-8)\alpha - \mu f^2 - \rho + \lambda)\eta(X), \tag{26}$$

which yields

$$\beta df = ((2n-8)\alpha - \mu f^2 - \rho + \lambda)\eta. \tag{27}$$

Applying (27) in (23) we obtain

$$\alpha\bar{S}(X,Y) = (2\beta f - \lambda)g(X,Y) - ((2n-8)\alpha + \lambda - 2\beta f)\eta(X)\eta(Y), \tag{28}$$

which implies $\alpha\bar{r} = 2\beta f(1-n) + \lambda(n+1) + (2n-8)\alpha$. Therefore, this leads to the following:

**Theorem 3.1.** *Let $(M, g, \phi, \xi, \eta)$ be a LP-Kenmotsu manifold. If $M$ admits a generalized $\eta$-Ricci soliton $(g, V, \alpha, \beta, \mu, \rho, \lambda)$ with respect to the Schouten-van Kampen connection such that $\alpha \neq 0$ and $V = f\xi$ for some smooth function $f$ on $M$, then $M$ is an $\eta$-Einstein manifold with respect to the Schouten-van Kampen connection.*

From (28) we also have the following:

**Corollary 3.1.** *Let $(M, g, \phi, \xi, \eta)$ be a LP-Kenmotsu manifold. If $M$ admits a generalized $\eta$-Ricci soliton $(g, V, \alpha, \beta, \mu, \rho, \lambda)$ with respect to the Schouten-van Kampen connection such that $V = f\xi$ for some smooth function $f$ on $M$, then $\alpha\bar{r} = 2\beta f(1-n) + \lambda(n+1) + (2n-8)\alpha$.*

Now, let $M$ be an $\eta$-Einstein LP-Kenmotsu manifold with respect to the Schouten-van Kampen connection and $V = \xi$. Then we get $\bar{S} = ag + b\eta \otimes \eta$ for some functions $a$ and $b$ on $M$. From (21) we have

$$\overline{\mathcal{L}}_{\xi}g(X,Y) = -4(g(X,Y) + \eta(X)\eta(Y)),$$

for all vector fields $X, Y$. Therefore,

$$\alpha\bar{S} + \frac{\beta}{2}\overline{\mathcal{L}}_{\xi}g + \mu\xi^{\flat}\otimes\xi^{\flat} + \rho\eta\otimes\eta + \lambda g$$
$$= (a\alpha + \lambda - 2\beta)g + (b\alpha + \mu + \rho - 2\beta)\eta\otimes\eta.$$

From the above equation $M$ admits a generalized $\eta$-Ricci soliton $(g, \xi, \alpha, \beta, \mu, \rho, \lambda)$ with respect to the Schouten-van Kampen connection if $\lambda = -a\alpha + 2\beta$ and $\rho = -b\alpha - \mu + 2\beta$. Hence, we can state the following theorem:

**Theorem 3.2.** *Suppose that $M$ is a $\eta$-Einstein LP-Kenmotsu manifold with respect to the Schouten-van Kampen connection such that $\bar{S} = ag + b\eta\otimes\eta$ for some function $a$ and constant $b$ on $M$. Then manifold $M$ satisfies a generalized $\eta$-Ricci soliton $(g, \xi, \alpha, \beta, \mu, -b\alpha - \mu + 2\beta, -a\alpha + 2\beta)$ with respect to the Schouten-van Kampen connection.*

Now assume that a LP-Kenmotsu manifold with respect to the Schouten-van Kampen connection satisfying the condition $\bar{R}(X, Y).\bar{S} = 0$ for all vector fields $X, Y$ on $M$. Then we have

$$\bar{S}(\bar{R}(X, Y)Z, W) + \bar{S}(Z, \bar{R}(X, Y)W) = 0,$$

for all vector fields $X, Y, Z, W$ on $M$. Replacing $X$ by $\xi$ in the above equation, we conclude

$$\bar{S}(\bar{R}(\xi, Y)Z, W) + \bar{S}(Z, \bar{R}(\xi, Y)W) = 0,$$

for all vector fields $Y, Z, W$ on $M$. Using (16) we can write

$$\bar{S}(2g(Y, Z)\xi - \eta(Z)Y, W) + \bar{S}(Z, 2g(Y, W) - \eta(W)Y) = 0,$$

which yields

$$2g(Y, Z)\bar{S}(\xi, W) - \eta(Z)\bar{S}(Y, W) + 2g(Y, W)\bar{S}(Z, \xi) - \eta(W)\bar{S}(Z, Y) = 0.$$

Putting $Z = \xi$ in the above equation gives

$$\bar{S}(Y, W) = 2(2n - 8)g(Y, W) - (2n - 8)\eta(Y)\eta(W).$$

Thus we have the following theorem.

**Theorem 3.3.** *Let $M$ be a LP-Kenmotsu manifold with the Schouten-van Kampen connection satisfy the condition $\bar{R}.\bar{S} = 0$. Then manifold $M$ satisfies a generalized $\eta$-Ricci soliton $(g, \xi, \alpha, \beta, \mu, (2n-8)\alpha - \mu + 2\beta, -2(2n-8)\alpha + 2\beta)$ with respect to the Schouten-van Kampen connection.*

Now assume that a LP-Kenmotsu manifold with respect to the Schouten-van Kampen connection satisfying the condition $(\bar{S}(X, Y).\bar{R})(U, W)Z = 0$ for all vector fields $X, Y, Z, U, W$ on $M$. Let $(X \wedge_{\bar{S}} Y)Z = \bar{S}(Y, Z)X - \bar{S}(X, Z)Y$, then we have

$$(X \wedge_{\bar{S}} Y)\bar{R}(U, W)Z + \bar{R}((X \wedge_{\bar{S}} Y)U, W)Z + \bar{R}(U, (X \wedge_{\bar{S}} Y)W)Z$$
$$+\bar{R}(U, W)(X \wedge_{\bar{S}} Y)Z = 0.$$

Putting $Y = \xi$ in the above equation we obtain

$$(2n - 8)\left[\eta(\bar{R}(U, W)Z)\eta(X) + \eta(\bar{R}(X, W)Z)\eta(U) + \eta(\bar{R}(U, X)Z)\eta(W)\right.$$
$$\left.+\eta(\bar{R}(U, W)X)\eta(Z)\right] + \bar{S}(X, \bar{R}(U, W)Z) - \bar{S}(X, U)\eta(\bar{R}(\xi, W)Z)$$
$$-\bar{S}(X, W)\eta(\bar{R}(U, \xi)Z) - \bar{S}(X, Z)\eta(\bar{R}(U, W)\xi) = 0.$$

Replacing $U$ and $Z$ by $\xi$ in the last equation we deduce

$$\bar{S}(X, W) = -(2n - 8)g(X, W) - 2(2n - 8)\eta(X)\eta(W).$$

Thus we have the following theorem.

**Theorem 3.4.** *Let $M$ be a LP-Kenmotsu manifold with the Schouten-van Kampen connection satisfy the condition $\bar{S}.\bar{R}=0$. Then manifold $M$ satisfies a generalized $\eta$-Ricci soliton $(g,\xi,\alpha,\beta,\mu,2(2n-8)\alpha-\mu+2\beta,(2n-8)\alpha+2\beta)$ with respect to the Schouten-van Kampen connection.*

**Definition 3.1.** *Let $M$ be a LP-Kenmotsu manifold with the Schouten-van Kampen connection $\bar{\nabla}$. The concircular curvature tensor $\bar{C}$ with respect to the Schouten-van Kampen connection on $M$ is defined by*

$$\bar{C}(X,Y)Z=\bar{R}(X,Y)Z-\frac{\bar{r}}{n(n-1)}\left(g(Y,Z)X-g(X,Z)Y\right), \tag{29}$$

*for all vector fields $X,Y,Z$ on $M$.*

Now consider a LP-Kenmotsu manifold $M$ is concircularly flat with respect to the Schouten-van Kampen connection, that is, $\bar{C}(X,Y)Z=0$ for all vector fields $X,Y,Z$ on $M$. Hence we get

$$\bar{R}(X,Y)Z=\frac{\bar{r}}{n(n-1)}\left(g(Y,Z)X-g(X,Z)Y\right), \tag{30}$$

and

$$g(\bar{R}(X,Y)Z,\xi)=\frac{\bar{r}}{n(n-1)}\left(g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\right). \tag{31}$$

From (12) and (16) we ahve

$$g(\bar{R}(X,Y)Z,\xi)=2\left(g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\right). \tag{32}$$

Applying (32) in (31), we infer

$$\frac{\bar{r}-2n^2+2n}{n(n-1)}\left(g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\right)=0. \tag{33}$$

If $\bar{r}\neq 2n^2-2n$ then

$$g(Y,Z)\eta(X)=g(X,Z)\eta(Y). \tag{34}$$

Replacing $Y$ and $X$ by $\xi$ and $QX$, respectively, we conclude

$$S(X,Z)=(1-n)\eta(X)\eta(Y) \tag{35}$$

and

$$\bar{S}(X,Z)=(3n-7)g(X,Z)+(1+n)\eta(X)\eta(Y) \tag{36}$$

for all vector fields $X,Z$ on $M$. Therefore, we have the following theorem.

**Theorem 3.5.** *Let $M$ be a concircularly flat LP-Kenmotsu manifold with respect to the Schouten-van Kampen connection satisfy the condition $\bar{r}\neq 2n^2-2n$. Then manifold $M$ satisfies a generalized $\eta$-Ricci soliton $(g,\xi,\alpha,\beta,\mu,-(1+n)\alpha-\mu+2\beta,-(3n-7)\alpha+2\beta)$ with respect to the Schouten-van Kampen connection.*

**Definition 3.2.** *A LP-Kenmotsu manifold with the Schouten-van Kampen connection $\bar{\nabla}$ is called $\xi$-concircularly flat with respect to the Schouten-van Kampen connection if $\bar{C}(X,Y)\xi=0$ for all vector fields $X,Y$ on $M$.*

Now assume that $M$ is a $\xi$-concircularly flat LP-Kenmotsu manifold with the Schouten-van Kampen connection. In this case, we have

$$\bar{R}(X,Y)\xi=\frac{\bar{r}}{n(n-1)}\left(\eta(Y)X-\eta(X)Y\right). \tag{37}$$

From (10) and (16) we have

$$\bar{R}(X,Y)\xi=2\left(\eta(Y)X-\eta(X)Y\right). \tag{38}$$

Applying (38) in (37), we infer

$$\frac{\bar{r} - 2n^2 + 2n}{n(n-1)} (\eta(Y)X - \eta(X)Y) = 0. \tag{39}$$

If $\bar{r} \neq 2n^2 - 2n$ then $\eta(Y)X - \eta(X)Y = 0$. Replacing $Y$ by $\xi$ we conclude $-X - \eta(X)\xi = 0$, and taking inner product with vector field $W$ we get $g(X, W) = -\eta(X)\eta(W)$ and replacing $X$ by $QX$, we obtain $S(X, W) = (1 - n)\eta(X)\eta(W)$, hence,

$$\bar{S}(X, W) = (3n - 7)g(X, W) + (1 + n)\eta(X)\eta(W). \tag{40}$$

for all vector fields $X, W$ on $M$. Therefore, we have the following theorem.

**Theorem 3.6.** *Let $M$ be a $\xi$-concircularly flat LP-Kenmotsu manifold with respect to the Schouten-van Kampen connection satisfy the condition $\bar{r} \neq 2n^2 - 2n$. Then manifold $M$ satisfies a generalized $\eta$-Ricci soliton $(g, \xi, \alpha, \beta, \mu, -(1+n)\alpha - \mu + 2\beta, -(3n-7)\alpha + 2\beta)$ with respect to the Schouten-van Kampen connection.*

**Definition 3.3.** *A LP-Kenmotsu manifold with the Schouten-van Kampen connection $\bar{\nabla}$ is called pseudo-concircularly flat with respect to the Schouten-van Kampen connection if $g(\bar{C}(\phi X, Y)Z, \phi W)) = 0$ for all vector fields $X, Y, Z, W$ on $M$.*

Now assume that $M$ is a pseudo-concircularly flat LP-Kenmotsu manifold with the Schouten-van Kampen connection. In this case, we have

$$g(\bar{R}(\phi X, Y)Z, \phi W) = \frac{\bar{r}}{n(n-1)} \left(g(Y, Z)g(\phi X, \phi W) - g(\phi X, Z)g(Y, \phi W)\right). \tag{41}$$

Insetring $Y = Z = e_i$ and summing for $i$, we obtain

$$S(\phi X, \phi W) = \left[-3n + 7 + \frac{\bar{r}(n+3)}{n(n-1)}\right] g(\phi X, \phi W). \tag{42}$$

Using (42) and

$$S(\phi X, \phi W) = S(X, W) + (n-1)\eta(X)\eta(W) \tag{43}$$

we have

$$S(X, W) = \left[-3n + 7 + \frac{\bar{r}(n+3)}{n(n-1)}\right] g(X, W) + \left[-4n + 8 + \frac{\bar{r}(n+3)}{n(n-1)}\right] \eta(X)\eta(W). \tag{44}$$

Hence,

$$\bar{S}(X, W) = \frac{\bar{r}(n+3)}{n(n-1)} g(X, W) + \left[-2n + 8 + \frac{\bar{r}(n+3)}{n(n-1)}\right] \eta(X)\eta(W). \tag{45}$$

for all vector fields $X, W$ on $M$. Therefore, we have the following theorem.

**Theorem 3.7.** *Let $M$ be a pseudo-concircularly flat LP-Kenmotsu manifold with respect to the Schouten-van Kampen connection and has constant scalar curvatur. Then manifold $M$ satisfies a generalized $\eta$-Ricci soliton $(g, \xi, \alpha, \beta, \mu, -b\alpha - \mu + 2\beta, -a\alpha + 2\beta)$ with respect to the Schouten-van Kampen connection where $a = \frac{\bar{r}(n+3)}{n(n-1)}$ and $b = -2n + 8 + \frac{\bar{r}(n+3)}{n(n-1)}$.*

**Definition 3.4.** *A vector field $V$ is said to a conformal Killing vector field if*

$$(\mathcal{L}_V g)(X, Y) = 2hg(X, Y), \tag{46}$$

*for all vector fields $X, Y$, where $h$ is some function on $M$. The conformal Killing vector field $V$ is called*

- *proper when $h$ is not constant,*
- *homothetic vector field when $h$ is a constant,*
- *Killing vector field when $h = 0$.*

Let vectoe field $V$ is a conformal Killing vector field with respect to the Schouten-van Kampen connection and satisfies in $(\overline{\mathcal{L}}_V g)(X,Y) = 2hg(X,Y)$. By (17) and (20) we have

$$\alpha\bar{S}(X,Y) + \beta hg(X,Y) + \mu V^\flat(X)V^\flat(Y) + \rho\eta(X)\eta(Y) + \lambda g(X,Y) = 0. \quad (47)$$

for all vector fields $X, Y$. By inserting $Y = \xi$ in the above equation we get

$$g((2n-8)\alpha\xi + \beta h\xi + \mu\eta(V)V + \rho\xi + \lambda\xi, X) = 0. \quad (48)$$

Since $X$ is arbitrary vector field we have the following theorem.

**Theorem 3.8.** *If the metric $g$ of a LP-Kenmotsu manifold satisfies the generalized $\eta$-Ricci soliton $(g, V, \alpha, \beta, \mu, \rho, \lambda)$ where $V$ is and conformally Killing vector field with respect to the Schouten-van Kampen connection, that is $\overline{\mathcal{L}}_V g = 2hg$ then*

$$((2n-8)\alpha + \beta h + \rho + \lambda)\xi + \mu\eta(V)V = 0. \quad (49)$$

**Definition 3.5.** *A nonvanishing vector field $V$ on pseudo-Riemannian manifold $(M, g)$ is called torse-forming* [29] *if*

$$\nabla_X V = fX + \omega(X)V, \quad (50)$$

*for all vector field $X$, where $\nabla$ is the Levi-Civita connection of $g$, $f$ is a smooth function and $\omega$ is a 1-form. The vector field $V$ is called*

- *concircular* [9, 28] *whenever in the equation (50) the 1-form $\omega$ vanishes identically,*
- *concurrent* [24, 30] *if in equation (50) the 1-form $\omega$ vanishes identically and $f = 1$,*
- *parallel vector field if in equation (50) $f = \omega = 0$,*
- *torqued vector field* [8] *if in equation (50) $\omega(V) = 0$.*

Let $(g, V, \alpha, \beta, \mu, \rho, \lambda)$ be a generalized $\eta$-Ricci soliton on a LP-Kenmotsu manifold where $V$ is a torse-forming vector filed and satisfied in (50). Then

$$\begin{aligned} &\alpha\bar{S}(X,Y) + \frac{\beta}{2}\left[(\mathcal{L}_V g)(X,Y) + 2\eta(V)g(X,Y) - g(X,V)\eta(Y) - g(Y,V)\eta(X)\right] \\ &+\mu V^\flat(X)V^\flat(Y) + \rho\eta(X)\eta(Y) + \lambda g(X,Y) = 0, \end{aligned} \quad (51)$$

for all vector fields $X, Y$. On the other hand,

$$(\mathcal{L}_V g)(X,Y) = 2fg(X,Y) + \omega(X)g(V,Y) + \omega(Y)g(Y,X), \quad (52)$$

for all vector fields $X, Y$. Applying (52) into (51) we arrive at

$$\begin{aligned} &\alpha\bar{S}(X,Y) + \frac{\beta}{2}\left[2fg(X,Y) + \omega(X)g(V,Y) + \omega(Y)g(Y,X)\right. \\ &\left.+2\eta(V)g(X,Y) - g(X,V)\eta(Y) - g(Y,V)\eta(X)\right] + \mu V^\flat(X)V^\flat(Y) \\ &+\rho\eta(X)\eta(Y) + \lambda g(X,Y) = 0. \end{aligned} \quad (53)$$

We take contraction of the above equation over $X$ and $Y$ to obtain

$$\alpha\bar{r} + n\left[\beta f + \lambda\right] - \rho + \beta\omega(V) + (n-1)\beta\eta(V) + \mu|V|^2 = 0. \quad (54)$$

Therefore we have the following theorem.

**Theorem 3.9.** *If the metric $g$ of a LP-Kenmotsu manifold satisfies the generalized $\eta$-Ricci soliton $(g, V, \alpha, \beta, \mu, \rho, \lambda)$ where $V$ is the torse-forming vector filed and satisfied in (50), then*

$$\lambda = -\frac{1}{n}\left[\alpha(r + 3n^2 - 9n) - \rho + \beta\omega(V) + (n-1)\beta\eta(V) + \mu|V|^2\right] - \beta f. \quad (55)$$

4. **Examples**

In this section, we give two examples of LP-Kenmotsu manifolds with respect to the Schouten-van Kampen connection.

**Example 4.1.** *Let $(x, y, z)$ be the standard coordinates in $\mathbb{R}^3$ and $M = \{(x, y, z) \in \mathbb{R}^3 | z > 0\}$. We consider the linearly independent vector fields $e_1 = z\frac{\partial}{\partial x}$, $e_2 = z\frac{\partial}{\partial y}$, $e_3 = z\frac{\partial}{\partial z}$. We define the metric $g$ by*

$$g(e_i, e_j) = \begin{cases} 1, & \text{if } i = j \text{ and } i, j \in \{1, 2\}, \\ -1, & \text{if } i = j = 3 \\ 0, & \text{otherwise.} \end{cases}$$

*We define an almost contact structure $(\phi, \xi, \eta)$ on $M$ by*

$$\xi = e_3, \quad \eta(X) = g(X, e_3), \quad \phi = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

*for all vector field $X$. Note the relations $\phi^2(X) = X + \eta(X)\xi$, $\eta(\xi) = -1$, and $g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$ hold. Thus $(M, \phi, \xi, \eta, g)$ defines an almost contact structure on $M$. We obtain*

| $[,]$ | $e_1$ | $e_2$ | $e_3$ |
|---|---|---|---|
| $e_1$ | *0* | *0* | $-e_1$ |
| $e_2$ | *0* | *0* | $-e_2$ |
| $e_3$ | $e_1$ | $e_2$ | *0* |

*The Levi-Civita connection $\nabla$ of $M$ is give by*

$$\nabla_{e_i} e_j = \begin{pmatrix} -e_3 & 0 & -e_1 \\ 0 & -e_3 & -e_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

*Hence the structure $(\phi, \xi, \eta)$ satisfies the formula $\nabla_X \xi = -X - \eta(X)\xi$ and $(\nabla_X \phi)Y = -g(\phi X, Y) - \eta(Y)\phi X$, thus $(M, \phi, \xi, \eta, g)$ becomes a LP-Kenmotsu manifold. Now, using (16) we get the Schouten-van- Kampen connection on $M$ as follows*

$$\bar{\nabla}_{e_i} e_j = \begin{pmatrix} -2e_3 & 0 & -2e_1 \\ 0 & -2e_3 & -2e_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

*The nonvanishing components of curvature tensor with respect to the Schouten-van Kampen connection are:*

$$\bar{R}(e_1, e_2)e_1 = -4e_2, \ \bar{R}(e_1, e_2)e_2 = 4e_1, \ \bar{R}(e_1, e_3)e_1 = -4e_3,$$
$$\bar{R}(e_1, e_3)e_3 = -2e_1, \ \bar{R}(e_2, e_3)e_2 = -2e_3, \ \bar{R}(e_2, e_3)e_3 = -2e_2,$$

*Thus, we get $\bar{S} = 4g + 6\eta \otimes \eta$. If we assume that $V = \xi$ then $\overline{\mathcal{L}}_V g = -4(g + \eta \otimes \eta)$. Then $(g, \xi, \alpha, \beta, \mu, \rho = -6\alpha + 2\beta - \mu, \lambda = -4\alpha + 2\beta)$ is a generalized $\eta$-Ricci soliton on manifold $M$ with respect to the Schouten-van Kampen connection.*

**Example 4.2.** *Let $(x, y, z, u, v)$ be the standard coordinates in $\mathbb{R}^5$ and $M = \{(x, y, z, u, v) \in \mathbb{R}^3 | v > 0\}$. We consider the linearly independent vector fields $e_1 = z\frac{\partial}{\partial x}, e_2 = z\frac{\partial}{\partial y}, e_3 = z\frac{\partial}{\partial z}, e_4 = z\frac{\partial}{\partial u}, e_5 = z\frac{\partial}{\partial v}$. We define the metric $g$ by*

$$g(e_i, e_j) = \begin{cases} 1, & \text{if } i = j \text{ and } i, j \in \{1, 2, 3, 4\}, \\ -1, & \text{if } i = j = 5 \\ 0, & \text{otherwise.} \end{cases}$$

*We define an almost contact structure $(\phi, \xi, \eta)$ on $M$ by*

$$\xi = e_5, \quad \eta(X) = g(X, e_5), \quad \phi = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

*for all vector field $X$. Note the relations $\phi^2(X) = X + \eta(X)\xi$, $\eta(\xi) = -1$, and $g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$ hold. Thus $(M, \phi, \xi, \eta, g)$ defines an almost contact structure on $M$. We have*

| $[,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ |
|---|---|---|---|---|---|
| $e_1$ | 0 | 0 | 0 | 0 | $-e_1$ |
| $e_2$ | 0 | 0 | 0 | 0 | $-e_2$ |
| $e_3$ | 0 | | 0 | 0 | $-e_3$ |
| $e_4$ | 0 | 0 | 0 | 0 | $-e_4$ |
| $e_5$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | 0 |

*The Levi-Civita connection $\nabla$ of $M$ is determined by*

$$\nabla_{e_i} e_j = \begin{pmatrix} -e_5 & 0 & 0 & 0 & -e_1 \\ 0 & -e_5 & 0 & 0 & -e_2 \\ 0 & 0 & -e_5 & 0 & -e_3 \\ 0 & 0 & 0 & -e_5 & -e_4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

*We see that the structure $(\phi, \xi, \eta)$ satisfies the formula $\nabla_X \xi = -X - \eta(X)\xi$ and $(\nabla_X \phi)Y = -g(\phi X, Y) - \eta(Y)\phi X$, thus $(M, \phi, \xi, \eta, g)$ becomes a LP-Kenmotsu manifold. Now, using (16) we get the Schouten-van- Kampen connection on $M$ as follows*

$$\bar{\nabla}_{e_i} e_j = \begin{pmatrix} -2e_5 & 0 & 0 & 0 & -2e_1 \\ 0 & -2e_5 & 0 & 0 & -2e_2 \\ 0 & 0 & -2e_5 & 0 & -2e_3 \\ 0 & 0 & 0 & -2e_5 & -2e_4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

*The nonvanishing components of curvature tensor with respect to the Schouten-van Kampen connection are:*

$$\begin{aligned}
&\bar{R}(e_1, e_2)e_1 = -4e_2, \ \bar{R}(e_1, e_2)e_2 = 4e_1, \ \bar{R}(e_1, e_3)e_1 = -4e_3, \\
&\bar{R}(e_1, e_3)e_3 = 4e_1, \ \bar{R}(e_1, e_4)e_1 = -4e_4, \ \bar{R}(e_1, e_4)e_4 = 4e_1, \\
&\bar{R}(e_1, e_5)e_1 = -2e_5, \ \bar{R}(e_1, e_5)e_5 = -2e_1, \ \bar{R}(e_2, e_3)e_2 = -4e_3, \\
&\bar{R}(e_2, e_3)e_3 = 4e_2, \ \bar{R}(e_2, e_4)e_2 = -4e_4, \ \bar{R}(e_2, e_4)e_4 = 4e_2, \\
&\bar{R}(e_2, e_5)e_2 = -2e_5, \ \bar{R}(e_2, e_5)e_5 = -2e_2, \ \bar{R}(e_3, e_4)e_3 = -4e_4, \\
&\bar{R}(e_3, e_4)e_4 = 4e_3, \ \bar{R}(e_3, e_5)e_3 = -2e_5, \ \bar{R}(e_3, e_5)e_5 = -2e_3, \\
&\bar{R}(e_4, e_5)e_4 = -2e_5, \ \bar{R}(e_4, e_5)e_5 = -2e_4.
\end{aligned}$$

*Hence, we obtain $\bar{S} = 10g + 2\eta \otimes \eta$. If we consider $V = \xi$ then $\overline{\mathcal{L}}_V g = -4(g + \eta \otimes \eta)$. Therefore $(g, \xi, \alpha, \beta, \mu, \rho = -2\alpha + 2\beta - \mu, \lambda = -10\alpha + 2\beta)$ is a generalized $\eta$-Ricci soliton on manifold $M$ with respect to the Schouten-van Kampen connection.*

## 5. **Conclusions**

The main study of the paper is to obtain geometrical conditions and characteristics of generalized $\eta$-Ricci solitons with respect to the Schouten-van Kampen connection to apply their existence in a LP-Kenmotsu manifold. We first assume that $(g, f\xi, \alpha, \beta, \mu, \lambda)$ satisfies

in a generalized $\eta$-Ricci soliton and we show that in this case, maifold is a $\eta$-Einstein LP-Kenmotsu manifold. Then, we show that any $\eta$-Einstein LP-Kenmotsu manifold satisfies in a generalized $\eta$-Ricci soliton. Also, we give some geometric conditions on LP-Kenmotsu manifolds which under these condition manifolds satisfy in generalized $\eta$-Ricci solitons. Then we study a generalized $\eta$-Ricci soliton $(g, V, \alpha, \beta, \mu, \lambda)$ on LP-Kenmotsu manifolds when vector field $V$ is a conformal Killing vector field or a torse-forming vector field. There are some quastions that have arisen from our article and are a potential study for further research.
(1) Are the results of this paper also hold in other classes of Riemannian and Lorentzian almost contact manifolds?
(2) Are the results of this paper also are true if we consider $*$-Ricci tensor instead of Ricci tensor, that is, we consider generalized $*$-$\eta$-Ricci solitons?
(3) Are the results of this paper also hold when we consider another connection instead of Shouten-van Kampen connection?

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