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ON THE INFLUENCE OF THE TIRE LATERAL ELASTICITY ON THE HEAVE VIBRATIONS OF AN AUTOMOBILE

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In lucrare se studiază oscilațiile de săltare ale autombilului cu suspensie pendulară. Se foloseşte un model al acestui tip de suspensie fără a se presupune că deplasările în sistem sunt mici. Se ajunge la un sistem de ecuații diferențiale neliniare, potrivit pentru studiul oscilaților la trecerea roților peste neregularități mari ale căii. Se fac aprecieri cu privire la precizia determinării rigidității şi amortizării suspensiei când automobilul nu se deplasează. Pentru ecuațiile liniarizate se compară rezultatele obținute cu cele corespunzătoare modelului uzual din literatură. Se constată că deformațiile laterale ale pneurilor determinate de variația ecartamentului roților afectează confortul la oscilații şi ținuta de drum ale automobilului.

In the present paper the heave vibrations of an automobile are investigated in the case of a swing axle suspension. A model of this suspension type is used without considering the displacements as small. We obtain a non-linear system of the differential equations, which is used to study the vibrations during automobile passing over surface with high undulations. Considerations are made about the determination accuracy of the suspension stiffness and suspension damping by testing when the automobile does not move. The motion equations are linearised and the obtained results are compared with those corresponding to the usual ride quarter-car model. It has been found that the tire lateral deformations produced by the track width variation deteriorate the ride comfort and the road holding of an automobile.

Keywords: automobile, differential equations, heave vibrations, stiffness, swing axle suspension, transfer function

1. Introduction

Usually, to study the heavy vibrations of automobiles, equivalent suspension models are employed which are composed of elastic elements with certain stiffnesses corresponding to the vertical displacement of the different masses. These masses are connected between themselves by the agency of these elastic elements. In practice, in the case of the independent suspensions, during heave displacement of the automobile body the alteration of the track width takes place, leading inevitably the lateral deformations of the tires. These generate the

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lateral forces which influence the suspension dynamics depending on the features of the suspension guide mechanism. The influence of different suspension guide mechanism types on the automobile vibrations taking into account the lateral deformations of the tires has been investigated in literature [1, 2, 3]. For this purpose a linear differential equation system has been established based on some approximations in conjunction with the displacements considered as being small.

In the paper, the heave vibrations of an automobile are investigated in the case of a swing axle suspension employing a model without considering displacements as small. Thus, a non-linear differential equation system is established and the influence of the lateral deformations of the tires on the heave (bounce) vibrations of an automobile are investigated. Also, assessments are made regarding the determination of the suspension stiffness and damping when the automobile no longer moves. Finally, the linearised equations have been established and the comparisons to the riding quarter –car model are made.

2. Equations of motions

We consider the rectilinear motion of an automobile which travels over an undulating surface. It is assumed that surface profile for two wheels of the swing axle are identical. Also, the pitch motion is not considered and, in fact, the model of a half of the automobile corresponding to the examined axle is studied.

We consider the fixed reference system Oyz attached to the road. The y axis is located into a horizontal plane. The elevation of the surface profile is h_r (Figure 1) and the ordinate of the gravity center of the sprung mass is z_g. We suppose that the gravity center of the unsprung mass corresponding to the axle looking to travel direction (the same with the direction of the Ox axis) coincides with the center of the wheel C (therefore the masses of the arm, the spring and the shock absorber are neglected).

During oscillations and automobile passing over road irregularities the tire is deflected into vertical and the lateral directions with ς_t and η_t, respectively (these quantities are zero when on the tire does not act any force). So, the normal tire reaction Z_w and the lateral tire (wheel) reaction Y_w are developed. When the tire is not loaded the height of the cross section is H_0 and, in Fig. 1, point D represents the lowest point of the wheel rim diameter. Therefore $|CD|=r_{rm}$ represents the radius of the wheel rim.

The following notations are introduced:

a_e[m]-distance from the upper fastening point of the spring E to the vertical line that passes through point A;

h_e [m]-distance from point E to the horizontal line that passes through the point A;

h_0 [m]-distance from the gravity center of the sprung mass to the horizontal line that passes through the point A;

l=|AC| [m]-length of the swing arm of the suspension;
l_e [m]-length of the spring (elastic element);
l_{ef} [m]-spring length in free state;
m [kg]-half sprung mass; m_w [kg]-wheel mass;
I_{wx} [kg.m^2]-mass moment of inertia of the wheel about an axis which passes through wheel center and is perpendicular to the wheel axis;
g [m.s^{-2}]-acceleration due to gravity;
$\vec{F}_e, \vec{F}_{sh}$ [N]-force developed by the elastic element (spring), force developed by the shock absorber;
c_{sh} [N.s.m^{-1}]-damping coefficient of the shock absorber;
k [N/rad]-cornering stiffness of a tire;
k_e [N.m^{-1}]-stiffness of the elastic element (spring);
k_{ty}, k_{tz} [N.m^{-1}]-lateral stiffness of the tire, vertical stiffness of the tire;
γ_e [rad]-angle between the spring axis and the normal line to the straight AC;
φ [rad]-angle between the longitudinal swing arm axis and the horizontal line;
$\vec{j}, \vec{k}$ -unit vectors in the y, z directions respectively.

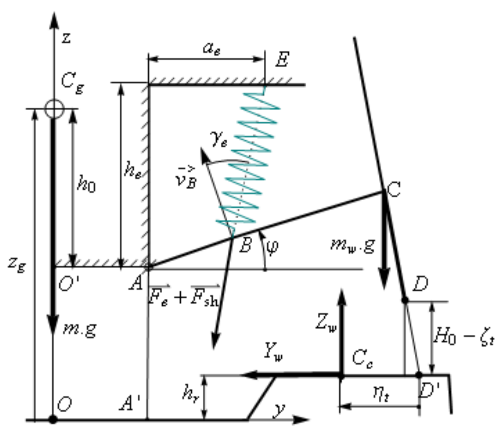


Fig.1. Schematic representation of a swing axle quarter suspension model

The acceleration of the point C is given by relation:

$$\vec{a}_C = -l(\dot{\varphi}^2 \cos\varphi + \ddot{\varphi}\sin\varphi)\vec{j} + (\ddot{z}_g - l\dot{\varphi}^2 \sin\varphi + l\ddot{\varphi}\cos\varphi)\vec{k}. \qquad (1)$$

By applying D'Alembert's principle for the ensemble represented in Fig. 1 and by projecting the obtained vector equation on the vertical direction we get (having relation (1) in view):

$$(m + m_w)\ddot{z}_g + m_w l\ddot{\varphi}\cos\varphi - m_w l\dot{\varphi}^2 \sin\varphi = -(m + m_w)g + Z_w. \qquad (2)$$

From the equilibrium condition for the unsprung part, taking the moment about point A, we obtain:

$$Z_w\left[l\cos\varphi + r_{rm}\sin\varphi + (H_0 - \varsigma_t)\tan\varphi - \eta_t\right] - (Y_w H_0 - \varsigma_t + r_{rm}\cos\varphi - l\sin\varphi) - m_w l(\ddot{z}_g\cos\varphi + l\ddot{\varphi}) - I_{wx}\ddot{\varphi} = (F_e + F_{sh})l_1\cos\gamma_e. \quad (3)$$

The reactions on the wheel acting at tire-road contact patch can be written as some function of the deformations:

$$Y_w = k_{ty}\cdot\eta_t,\quad Z_w = k_{tz}\cdot\varsigma_t. \quad (4)$$

The velocity of point C has two components: the one to the Ox axis direction (corresponding to the translation motion of the automobile), which is assumed to be constant, and the other to the lateral direction. For this reason, because the tire is elastic, the wheel rolling is produced with lateral slip, even if an effective sliding is not generated. In the following, we suppose that point C does not slide with respect to the road. Because the lateral velocity of point C is variable when the heave oscillations are produced, point C_c has a certain running velocity. This velocity has a longitudinal component equal to the automobile speed and a lateral component which is dependent on the variation with respect the time of the lateral tire deformation. Therefore, the tire rolls with variable lateral slip. According to [4], the lateral deformation of tire is given both the rolling with slip and the tire camber angle. In the two cases the so called relaxation length of the tire is the same.

The following relation is considered [4];

$$\dot{\eta}_t + v\frac{k_{ty}}{k}\eta_t = v\alpha + \frac{k_\gamma}{k}v\gamma, \quad (5)$$

where α is the slip angle and γ represents the wheel camber angle (in our case $\gamma=\varphi$). If $D^{'}$ is the point fixed with respect to the wheel rim and it is situated in the lowest part of the rim and on the road surface, then one can write [4]:

$$\alpha = \frac{v_{D'y}}{v}, \quad (6)$$

where $v_{D'y}$ is the projection on the surface plane of the component of the velocity of the point D' on an automobile transversal plane. This has the direction opposed to the lateral tire deformation. One can prove that during the rectilinear movement this component has the expression (the positive direction of the velocity is to right):

$$v_{D'y} = (r_{rm}\cos\varphi + H_0 - \varsigma_t - l\sin\varphi)\dot{\varphi}. \quad (7)$$

By projecting the OO'ACD"O contour on a vertical direction we get:

$$\varsigma_t = H_0 + h_0 - l\sin\varphi + r_{rm}\cos\varphi - z_g + h_r. \quad (8)$$

Taking into account the relations (4), (6), (7) and (8), the Equations (2), (3) and (5) become:

$$(m+m_w)\ddot{z}_g + k_{tz}z_g + m_w l\ddot{\varphi}\cos\varphi - m_w l\sin\varphi\cdot\dot{\varphi}^2 + k_{tz}l\sin\varphi - k_{tz}r_{rm}\cos\varphi = -(m+m_w)g + k_{tz}(H_0+h_0) + k_{tz}h_r, \quad (9)$$

$$m_w l\cos\varphi\cdot\ddot{z}_g+(m_w l^2+I_{wx})\ddot{\varphi}+k_{ty}(z_g-h_0+l\sin\varphi-r_{rm}\cos\varphi-h_r)\eta_t-$$
$$k_{tz}(H_0+h_0-l\sin\varphi+r_{rm}\cos\varphi-z_g+h_r)[l\cos\varphi+r_{rm}\sin\varphi+(H_0-\varsigma_t)\tan\varphi-\eta_t]=$$
$$-(F_e+F_{sh})l_1\cos\gamma_e, \tag{10}$$

$$\dot{\eta}_t-(z_g-h_0-h_r)\dot{\varphi}+\frac{k_{ty}}{k}v\eta_t-\frac{k_\gamma}{k}v\varphi=0. \tag{11}$$

The force developed by the spring and the force developed by the shock absorber (it is supposed that the shock absorber is fitted inside of the spring, the mounting points being the same as those of the spring) are given by relations:

$$F_e=k_e(l_{ef}-l_e),\quad F_{sh}=-c_{sh}\dot{l}_e. \tag{12}$$

Starting from the representation of Fig.1 we obtain the relation

$$l_e=\sqrt{l_1^2+a_e^2+h_e^2-2l_1(a_e\cos\varphi+h_e\sin\varphi}. \tag{13}$$

Differentiating the preceding relation with respect to time, we find the expression

$$\dot{l}_e=\frac{l_1(a_e\sin\varphi-h_e\cos\varphi)}{l_e}\cdot\dot{\varphi}. \tag{14}$$

Also, one establishes the relation

$$\cos\gamma_e=\frac{-a_e\sin\varphi+h_e\cos\varphi}{l_e}. \tag{15}$$

Taking into account the relations (12), (13), (14) and (15), equations (9), (10) and (11) represent a system of the non-linear differential equations, the dependent variables being $z_g(t)$, $\varphi(t)$ and $\eta_t(t)$ (it is supposed that the function $h_r(t)$ is known). If the initial conditions $z_g(0)=z_{g0}$, $\dot{z}_g(0)=\dot{z}_{g0}$, $\varphi(0)=\varphi_0$, $\dot{\varphi}(0)=\dot{\varphi}_0$ and $\eta_t(0)=\eta_{t0}$ are given, then one can determine, as a rule, the function that define the dependent variables.

If we consider η_t, ζ_t and φ as dependent variables, then the motion equations are written as:

$$\ddot{\varsigma}_t+\frac{k_{tz}}{m+m_w}\varsigma_t+[r_{rm}\sin\varphi+(1-\mu)l\cos\varphi]\ddot{\varphi}+$$
$$[r_{rm}\cos\varphi-(1-\mu)l\sin\varphi]\dot{\varphi}^2=\ddot{h}_r+g, \tag{16}$$

$$\ddot{\varsigma}_t \cos\varphi - \left[l + \frac{I_{wx}}{m_w l} - (l\cos\varphi + r_{rm}\sin\varphi) \right]\ddot{\varphi} - \frac{c_{sh} l_1^2}{m_w l} \cdot \frac{(-a_e \sin\varphi + h_e \cos\varphi)^2}{l_e^2}\dot{\varphi}$$

$$+ \frac{k_e l_1}{m_w l}(-a_e \sin\varphi + h_e \cos\varphi) + \frac{k_{tz}}{m_w l}(l\cos\varphi + r_{rm}\sin\varphi + H_0 \tan\varphi - \varsigma_t \tan\varphi - \eta_t)\varsigma_t +$$

$$\frac{k_{ty}}{m_w l}\varsigma_t \eta_t - \frac{k_{ty}}{m_w l} H_0 \eta_t + \frac{k_{ty}}{m_w l}(l \sin\varphi - r_{rm}\cos\varphi)\eta_t = \frac{k_e l_1 l_{ef}}{m_w l}\cos\gamma_e + \ddot{h}_r \cos\varphi, \quad (17)$$

$$\dot{\eta}_t - (r_{rm}\cos\varphi + H_0 - l\sin\varphi - \varsigma_t)\dot{\varphi} + v\frac{k_{ty}}{k}\eta_t - v\frac{k_\gamma}{k}\varphi = 0, \quad (18)$$

where $\mu = m_w/(m_w + m)$.

Thus, we obtain also in this case a system of three non-linear differential equations. Regarding the initial conditions, it is necessary to take into account a certain peculiarity. From relation (8), by differentiating with respect to time, we obtain

$$\dot{\varsigma}_t = -(l\cos\varphi + r_{rm}\sin\varphi)\dot{\varphi} - \dot{z}_g + \dot{h}_r. \quad (19)$$

It is found that $\dot{\varsigma}_t(0)$ *and* $\dot{\varphi}(0)$ cannot be arbitrarily chosen. They must satisfy the relation:

$$\dot{\varsigma}_t(0) = -(l\cos\varphi(0) + r_{rm}\sin\varphi(0))\dot{\varphi}(0) - \dot{z}_g(0) + \dot{h}_r(0). \quad (20)$$

To study comparatively and to facilitate the calculations when the variables do not present important variations, the equations are linearised. Usually, one considers the variations of the dependent variables with respect to the equilibrium static state. So, during steady-state motion on a horizontal even road the following relations are valid (the same notations as in the general case are used, but the index 0 is added):

$$Z_{w0} = (m + m_w)g,\ \varsigma_{t0} = \frac{g(m+m_w)}{k_{tz}},\ \eta_{t0} = \frac{k_\gamma}{k_{ty}}\varphi_0,\ l\cos\varphi_0 + r_{rm}\sin\varphi_0 +$$

$$(H_0 - \varsigma_{t0})\tan\varphi_0 - \eta_{t0} - \frac{k_\gamma \varphi_0}{(m+m_w)g}(H_0 - \varsigma_{t0} + r_{rm}\cos\varphi_0 - l\sin\varphi_0) = \frac{l_1 F_{e0}\cos\gamma_e}{(m+m_w)g}. \quad (21)$$

It is noticed that if the angle φ_0 is imposed, the Eq. (20) comprises l_{ef} as an unknown (one takes into account the first relation (12) and relation (15) in this special case for the given geometrical characteristics). Using a numerical method, l_{ef} can be determined or another geometrical quantity may be chosen if l_{ef} is imposed.

We can write: $\varsigma_t = \varsigma_{t0} + \Delta\varsigma_t,\ \varphi = \varphi_0 + \Delta\varphi,\ \eta_t = \eta_{t0} + \Delta\eta_t$,

where $\Delta\zeta_t$, $\Delta\varphi_t$ and $\Delta\eta_t$ are the variations of ζ_t, φ and η_t respectively with respect to the values at steady-state motion. If it is supposed that these variations are small, then from the Taylor's series expansion of the functions ζ_t, φ and η_t the first two terms of them are kept. After that, in the motion equations the products and the powers of these variations are neglected. Taking into account also the static equilibrium conditions we obtain the following linear differential system with constant coefficients:

$$M \cdot \ddot{X} + N \cdot \dot{X} + P \cdot X = [1, \cos\varphi_0, 0]^T \cdot \ddot{h}_r \tag{22}$$

where $X = [\Delta\varsigma_t, \Delta\varphi, \Delta\eta_t]^T$ is the column vector of the new dependent variables and M, N and P are square matrices 3x3: $M=(m_{ij})$, $N=(n_{ij})$, $P=(p_{ij})$, $E_x=(e_{ij})$, i=1, 2, 3, j=1, 2, 3. The coefficients of these matrices are given by expressions:

$m_{11} = 1, m_{12} = r_{rm}\sin\varphi_0 + (1-\mu)l\cos\varphi_0, m_{13} = 0, m_{21} = \cos\varphi_0,$

$m_{22} = (l\cos\varphi_0 + r_{rm}\sin\varphi_0)\cos\varphi_0 - [l + I_{wx}/(m_w l)], m_{23} = 0, m_{31} = m_{32} = m_{33} = 0,$

$n_{11} = n_{12} = n_{13} = 0, n_{21} = n_{23} = 0, n_{22} = -\dfrac{c_{sh} l_1^2}{m_w l} \cdot \dfrac{(-a_e\sin\varphi_0 + h_e\cos\varphi_0)^2}{l_{e0}^2}, n_{31} = 0,$

$n_{32} = -(r_{rm}\cos\varphi_0 + H_0 - l\sin\varphi_0 - \varsigma_{t0}), n_{33} = 1,\ p_{11} = \dfrac{k_{tz}}{m + m_w}, p_{12} = p_{13} = 0,$

$p_{21} = \dfrac{k_{tz}}{m_w l}(l\cos\varphi_0 + H_0\tan\varphi_0 + r_{rm}\sin\varphi_0 - 2\varsigma_{t0}\tan\varphi_0 - \eta_{t0}) + \dfrac{k_{ty}}{m_w l}\cdot\eta_{t0},$

$p_{22} = \dfrac{k_e l_1}{m_w l}[-a_e\cos\varphi_0 - h_e\sin\varphi_0 - l_{ef}(\dfrac{l_1(h_e\cos\varphi_0 - a_e\sin\varphi_0)^2}{l_{e0}^3} - a_e\cos\varphi_0$

$-h_e\sin\varphi_0)] + \dfrac{1}{m_w l}(k_{ty} r_{rm}\eta_{t0}\cos\varphi_0 + \dfrac{H_0 k_{tz}\varsigma_{t0}}{\cos^2\varphi_0} + k_{tz} r_{rm}\cos\varphi_0\varsigma_{t0} + \dfrac{k_{tz}\varsigma_{t0}^2}{\cos^2\varphi_0}) +$

$\dfrac{1}{m_w}(k_{ty}\eta_{t0}\cos\varphi_0 - k_{tz}\varsigma_{t0}),\ p_{23} = -\dfrac{k_{tz}}{m_w l}\varsigma_{t0} + k_{ty}\dfrac{\varsigma_{t0} + l\sin\varphi_0 - r_{rm}\cos\varphi_0 - H_0}{m_w l},$

$p_{31} = 0, p_{32} = -v\dfrac{k_\gamma}{k}, p_{33} = v\dfrac{k_{ty}}{k}.$

An interesting case is that when the automobile does not move and it is tested in connection with the suspension behavior. For this purpose, the ride simulator with hydraulic actuators or test stand for determination of the elastic characteristic of the suspension is used [2, 5]. In general, in similar cases the non-rolling wheels are supported on plates with balls so that the tires do not take over the lateral forces. However, there are cases when the suspension testing is used without to take similar precautions: during passing of the wheel on an irregularity or when the oscillations are excited by pulling down the body or raising the vehicle (some diagnosis testing stands are not equipped with the mentioned devices).

When the automobile does not move, Eq.(5) is not valid (even for small speeds its validity ceases because the angle α gets large values and certain approximations cannot be made [4]). Besides, the point C_c remains fixed with respect to the road; it has not a certain lateral running velocity. At a certain velocity, on the road without irregularities the oscillatory system of the automobile arrives at steady-state operation and the lateral tire deformation is η_{t0} as it has been noticed before. When the automobile stops there is a certain lateral tire deformation. In the following, we assume that this is even η_{t0} and the point C_c becomes C_{c0}. Its position is specified by the distance A'C_0, which is denoted by Δy_{Cc0} (A' is the projection of the point A on the road; it is fixed with respect to road). In initial position, the point D' becomes D_0' and we can write the following relationships:

$$\begin{aligned} y_{D'0} &= y_A + l\cos\varphi_0 + r_{rm}\sin\varphi_0 + (H_0 - \varsigma_{t0})\tan\varphi_0, \\ y_{D'} &= y_A + l\cos\varphi + r_{rm}\sin\varphi + (H_0 - \varsigma_t)\tan\varphi, \\ \Delta y_{Cc0} &= l\cos\varphi_0 + r_{rm}\sin\varphi_0 + (H_0 - \varsigma_{t0})\tan\varphi_0 - \eta_{t0}, \end{aligned} \tag{23}$$

where y_A is the abscissa of point A (y_A=const.).

The lateral tire deformation is $\Delta y = y_{D'} - (y_A + \Delta y_{Cc0})$

which becomes

$$\Delta y = l(\cos\varphi - \cos\varphi_0) + r_{rm}(\sin\varphi - \sin\varphi_0) + (H_0 - \varsigma_t)\tan\varphi - (H_0 - \varsigma_{t0})\tan\varphi_0 + \eta_{t0}. \tag{24}$$

If we consider that a certain external force F_{ex}(t) acting upon sprung mass corresponding to a quarter of the automobile at a point situated on the Oz axis, then the motion equation on the direction of this axis is similar to Eq. (9) in which h_r=0:

$$\ddot{z}_g + \mu.l.\cos\varphi\cdot\ddot{\varphi} - \mu.l.\sin\varphi\cdot\dot{\varphi}^2 = -g + \frac{k_{tz}\cdot\varsigma_t}{m+m_w} + \frac{F_{ex}(t)}{m+m_w}. \tag{25}$$

The equation similar to Eq. (10) is written:

$$\begin{aligned} &(I_{wx} + m_w l^2)\ddot{\varphi} + k_{ty}.\Delta y.(r_{rm}\cos\varphi - l\sin\varphi + H_0 - \varsigma_t) \\ &- k_{tz}.\varsigma_t.\Delta y_{Cc0} - l_1(F_e + F_{sh})\cos\gamma_e = 0, \end{aligned} \tag{26}$$

where ζ_t is given by (8) for h_r=0.

For static condition, for a given external force, taking into account that Z_w=$k_{tz}\zeta_t$, Eq. (26) is written as

$$\begin{aligned} &k_{ty}.[l(\cos\varphi - \cos\varphi_0) + r_{rm}(\sin\varphi - \sin\varphi_0) + (H_0 - Z_w/k_{tz})\tan\varphi - (H_0 - \varsigma_{t0})\tan\varphi_0 \\ &+ \eta_{t0}].(r_{rm}\cos\varphi - l\sin\varphi + H_0 - Z_w/k_{tz}) - Z_w.\Delta y_{Cc0} + l_1 F_e\cos\gamma_e = 0. \end{aligned} \tag{27}$$

From this equation we can directly obtain Z_w as a function of φ. In this way the elastic characteristic of the suspension can be constructed at circumstances specified here.

3. Simulation results

Due to the complexity of the models it is necessary to use numerical data. For this purpose the following values have been chosen : l=0.520 m, l_1=0.400 m, a_e=0.400 m, h_e=0.200 m, h_0=0.2175 m, φ_0=1°, m=400 kg, m_w=15.0 kg, I_{wx}=2.0 kg.m^2. The considered tire is 180/65 R 15, so that r_{rm}=0.1905 m and H_0=0.117 m. For the one degree-of-freedom oscillatory model, one considers the undamped natural frequency 1.10 Hz, to obtain the suspension stiffness k_s=19107 N/m and the spring stiffness k_e=32291 N/m which is calculated by the known relation

$$k_e = k_s (l/l_1)^2 . \tag{28}$$

Assuming a usual value of the suspension damping ratio ζ=0.2354, one obtains the value of the damping coefficient of the shock absorber c_{sh}=2200 N.s/m which has been calculated by the relations

$$c_s = 2\varsigma\sqrt{m.k_s}\,,\; c_{sh} = c_s (l/l_1)^2 . \tag{29}$$

Also, the following values have been chosen : k=25000 N/rad, k_{tz}=157000 N/m, k_{ty}=0.5k_{tz} and k_γ=4000 N/rad.

In order to simulate numerical the preceding models we have conceived various programs in *Mathematica* software.

First, we discuss the results corresponding to the case when the automobile does not move and the wheels are supported directly on the road (not on plates with balls). We define the relative displacement of the sprung mass

$$\Delta z_g = z_{g0} - z_g \,, \tag{30}$$

where z_{g0} is the ordinate of the gravity center of the sprung mass corresponding to the automobile static position.

It is found that the suspension elastic characteristic is non-linear although the elastic element has the linear characteristic. The suspension stiffness is

$$k_s = \frac{dZ_w}{d(\Delta z_g)} = \frac{dZ_w / d\varphi}{d(\Delta z_g)/d\varphi}. \tag{31}$$

We have directly used the expression behind the second equality of the relationship (31) by employing the differentiating function in *Mathematica*. The result is plotted in Fig. 2. It is noticed that the suspension stiffness variation is large, which proves the pronounced non-linearity of the suspension elastic characteristic. Also, in initial static position (Δz_g=0), for example, the suspension stiffness has the value of 34000 N/m, which differs much from the usual value of the equivalent stiffness. This value is of 19170.7 N/m, as it has been shown. Therefore, an increment of stiffness of 78% takes place. This difference is explained by the effect of the lateral tire deformation and the contribution of the lateral tire force. This fact is illustrated in Fig. 3, where one can see that the magnitudes of the lateral tire forces are very large for Δz_g=±0.10 m.

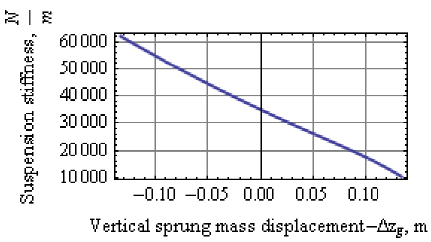


Fig.2. Suspension stiffness as a function of relative displacement of sprung mass

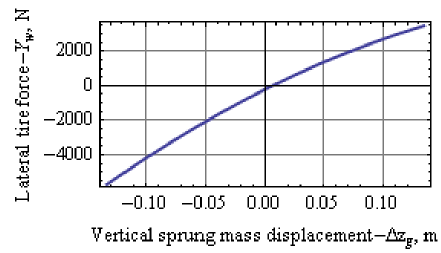


Fig.3. Lateral force as a function of sprung mass displacement during testing of a standing automobile

As a result of the large differences between the two suspension stiffness, the free vibration characteristics of the automobile will be different. The free vibrations may be generated by two methods: (i) the application of a large force on the sprung mass for a short time and (ii) pulling down the body and quickly letting it or raising the body and quickly letting it down. In case (i) we define a function with the mentioned peculiarity (for instance, a half sinusoidal) and the motion is studied by the differential system (25) and (26). In case (ii), F_{ex}=0, but it is necessary to specify the initial conditions. If Δz_g is imposed, then $\Delta\varphi$ can be determined, these quantities being considered as the initial values for the differential system (25) and (26). Also, for the initial conditions we take $\Delta \dot{z}_g(0+) = 0,\ \Delta\dot{\varphi}(0+) = 0$.

To make some comparisons it is necessary to consider also the free vibrations of the usual equivalent vibration wheel with two degree-of-freedom (two masses) with the above mentioned essential characteristics. The system of equations which describes the vibration of this model is [6]:

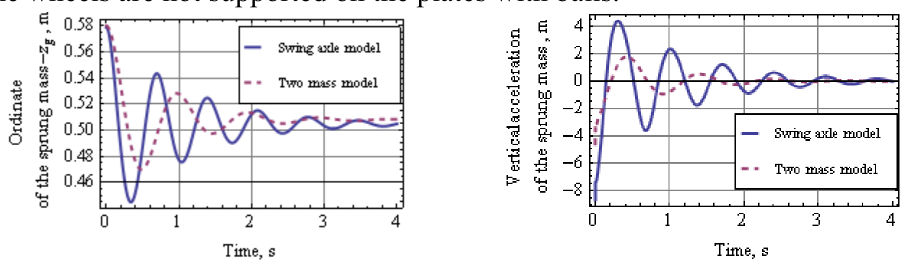
$$\begin{bmatrix} m & 0 \\ 0 & m_r \end{bmatrix} \cdot \begin{bmatrix} \Delta\ddot{z}_g \\ \Delta\ddot{\zeta}_t \end{bmatrix} + \begin{bmatrix} c_s & -c_s \\ -c_s & c_t + c_s \end{bmatrix} \cdot \begin{bmatrix} \Delta\dot{z}_g \\ \Delta\dot{\varphi} \end{bmatrix} + \begin{bmatrix} k_s & -k_s \\ -k_s & k_{tz} + k_s \end{bmatrix} \cdot \begin{bmatrix} \Delta z_g \\ \Delta\varphi \end{bmatrix} = \begin{bmatrix} 0 \\ c_t \end{bmatrix} \cdot \dot{h}_r + \begin{bmatrix} 0 \\ k_{tz} \end{bmatrix} h_r, \tag{32}$$

where c_t represents the tire damping coefficient.

Obviously, for free vibrations we consider $h_r \equiv dh_r/dt \equiv 0$. In the following it is assumed that c_t=0. Fig. 4 shows the simulation results of the free vibrations for the two models (case (ii)) and variables z_g and d^2z_g/dt^2. Obviously, the adequate initial values have been chosen for $\Delta z_g(0)$ and $\Delta\zeta_t(0)$ taking into account that

$$\Delta\varsigma_t(0) = \frac{k_s}{k_t + k_s} \cdot \Delta z_g(0). \tag{33}$$

The differences between the two models are clearly made evident on the mentioned figure. To make some comparisons, the damped natural frequency and damping ratios have been determined. By means of the computer program, we get the pseudo-periods and after that we determine the damping ratios by the relations similar to those used in the model with one degree of freedom [6]. Thus, for the two degrees of freedom, the value of the pseudo-period is T=0.9731 s to which corresponds the natural frequency of 1.0276 Hz and ζ≈const.=0.199415 (ζ does not dependent on the order number of the amplitude in the case of the viscous damping). For the model with swing arms, the pseudo-periods are not constant; the mean value is 0.6927 s to which the natural frequency of 1.444 Hz corresponds. The mean value of the damping ratio is ζ =0.151. Therefore, the natural frequency of this model is high and damping ratio is reduced. As a result, even in the case of the simple testing we can not correctly assess the suspension if the wheels are not supported on the plates with balls.



a) b)

Fig.4. Free vibrations during testing of a standing automobile

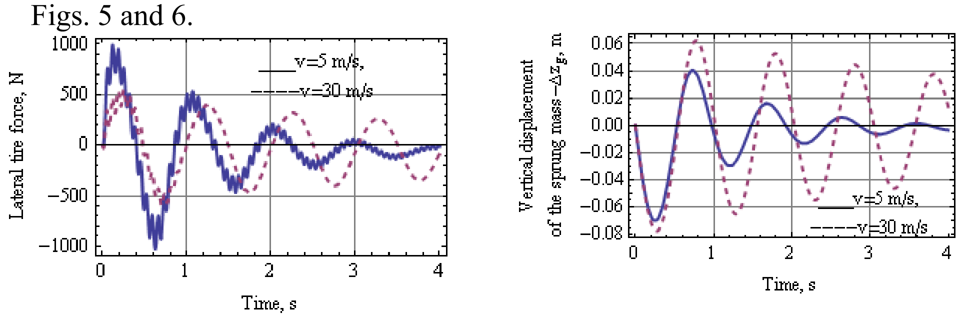
The micro profile of the road is defined by the function:

$$h_r = h_r(x). \tag{33}$$

Considering v=const., we get

$$h_r = h_r(vt). \tag{34}$$

From the analysis of the equations of motion it is found that the automobile velocity arises directly in Eq. (11) and Eq. (17) respectively, which shows that the free vibrations are dependent on the automobile velocity in contrast with the case of the usual models. To make evident this fact, the free vibrations have been studied in the case when the shock absorber is not fitted, namely c_{sh}=0. The results regarding Δz_g(t) and Y_w(t) for the velocities of 5 m/s and 30 m/s are plotted in Figs. 5 and 6.



a) b)

Fig.5. Free vibrations without shock absorber: a) vertical displacement of the sprung mass as a function of time; b) tire lateral force as a function of time

It is found that for small velocity there is an important damping effect even if shock absorbers do not exist. Thus, for v=5 m/s the damping ratio represents 0.149. For velocity of 30 m/s, the damping ratio decreases to 0.032. The damping effect is given by the tire rolling with slip which is in connection with losses expressed indirectly by the cornering stiffness. For small velocities, the slip angle is large ($\alpha=\arctan(v_{D'y}/v)$ and therefore the damping effect increases. In exchange, for large velocities the slip angle is small and the damping decreases. In Fig. 5b, the graph of the high frequency component is also plotted.

When the automobile travels over an undulating surface, forced vibrations are also produced. Usually, to study the behavior of the oscillator system of an automobile the following functions are considered:

$$h_r(vt) = \frac{h_{r\max}}{2}(1-\cos(\frac{2\pi v}{\lambda}\cdot t)), \quad h_r(vt) = h_{r\max}\sin(\frac{2\pi v}{\lambda}\cdot t), \tag{35}$$

where λ is wave length of the irregularity. The second relation (35) leads to the same depth of the road, namely $2h_{rmax}$. For $t\le 0$ it is assumed that h_r=0. In the case of the first function (35) with the initial conditions $z_g(0)=z_{g0}$, $dz_g/dt|_{t=0}=0$, $\varphi(0)=\varphi_0$, $d\varphi/dt|_{t=0}=0$, $\eta_t(0)=\eta_{t0}$ the obtained results are plotted in Figs. 6, 7. The following data have been chosen: h_{rmax}=0.16 m, λ=5.0 m and two velocities v=5.5 m/s and v=18.0 m/s. The second velocity has been chosen so that the normal tire reaction attains the limit of the contact loss with the road of the tire.

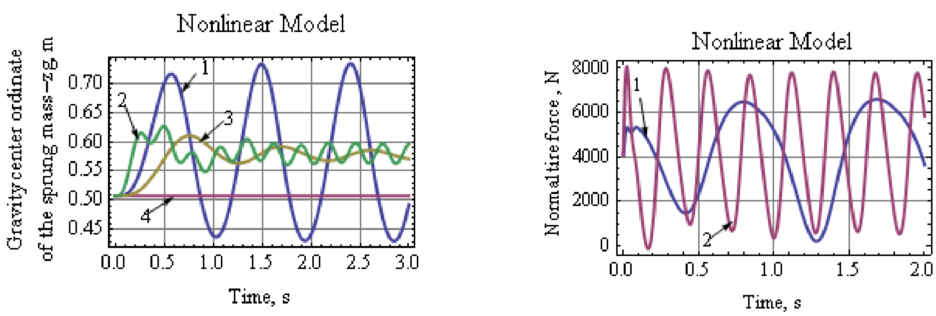


Fig.6. Forced vibrations of swing axle system: 1. v=5.5 m/s; 2. v=18.0 m/s; 3. mean gravity center ordinate for v=18.0 m/s; 4. gravity center ordinate at static equilibrium position

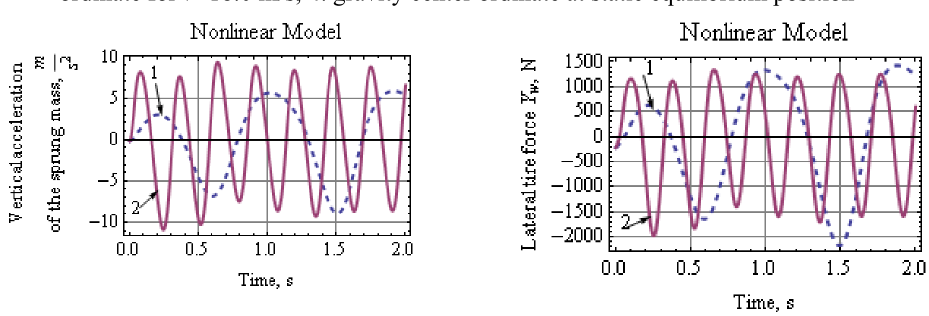


Fig.7. Forced vibrations of swing axle system: 1. v=5.5 m/s; 2. v=18.0 m/s

For a velocity of 18.0 m/s, the variations of the quantities plotted in these figures may be enough assimilate to some sinusoidal variations. But, for a velocity of 5.5 m/s, excepting z_g, the others quantities present variations which are not sinusoidal. So, the non-linearity of the system is emphasized. In connection with this it is necessary to observe that the maximum acceleration of the sprung mass is smaller than the absolute value of the minimum acceleration (for v=5.5 m/s there are the following values: $\ddot{z}_{g\,\max} = 5.80$ m.s^{-2}, $\ddot{z}_{g\,\min} = -8.83$ m.s^{-2} and for v=18.0 m/s: $\ddot{z}_{g\,\max} = 8.75$ m.s^{-2}, $\ddot{z}_{g\,\min} = -9.75$ m.s^{-2}). In this example the values of the angle of the swing arm does not go beyond 11.5 degrees. Consequently, the following approximations can be made: sin$\varphi \approx \varphi$ and cos$\varphi \approx$1.0. It is found that for the two velocities of the sprung mass displacement variations are clearly different, which is not case for the others quantities.

Although the two functions (35) describe the same depth of the road irregularity, the micro profiles are not identical. Therefore, if the second function (35) is used, the normal tire reaction ceases for h_{rmax}=0.085 m. The analysis of the obtained results, which is not presented here, shows that the conclusions which can be draw are similar to those of the preceding excitation.

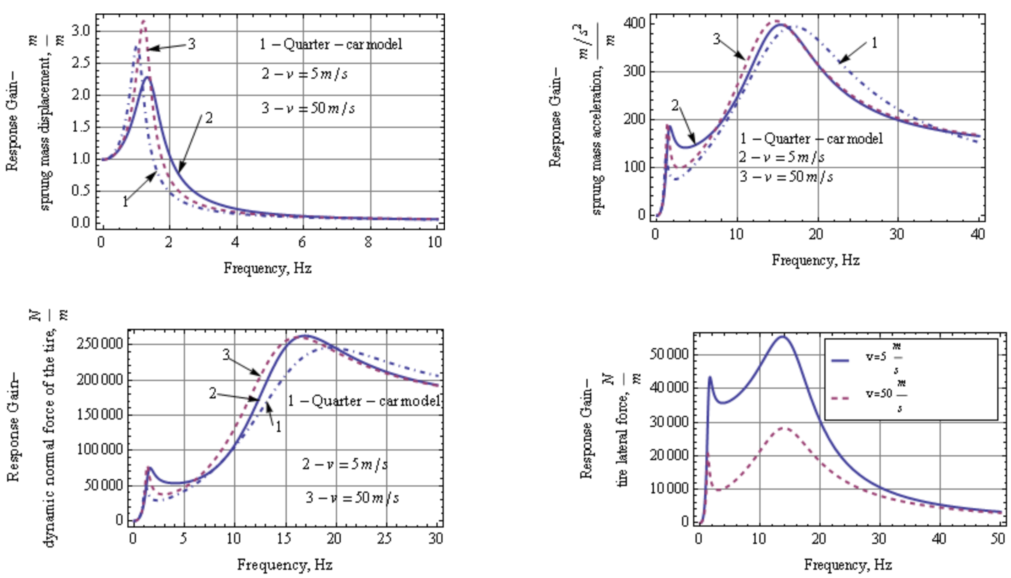


Fig.8. Frequency response functions of sprung mass displacement, sprung mass acceleration, dynamic normal force of the tire and tire lateral force

In usual conditions, the heights of the road undulations considered before are uncommon. For this reason, the linearised equations (21) can be employed. In this case, it is very advantageous to use the frequency characteristics of the system. They are obtained from (21) by using Laplace transformation, considering that the initial values are zero. By means of a computer program written in *Mathematica®* we get the expressions of the transfer functions which are enough complicated and they are not presented here. After that, substituting $\omega_e\sqrt{-1}$ for the Laplace variable s we get the frequency response expressed by gain factor (ω_e is circular frequency of the excitation produced by the road). The frequency response is dependent on the automobile velocity since the coefficients p_{32} and p_{33} are dependent on the velocity (see Fig. 8).

To emphasize more clearly the effects of the tire lateral deformations it is advantageous to compare the root-mean-square values of the various quantities corresponding to the swing axle suspension model and the quarter-car model. For this purpose it is necessary to take into account the characteristics of the road, which are defined by the expressions for the power spectral density of roughness roads. Between them, there is the one considered by ISO [7]:

$$S_d(f_s) = \begin{cases} S_d(f_{s0})\left(\dfrac{f_s}{f_{s0}}\right)^{-n_1} & for \quad f_s \le f_{s0} = \dfrac{1}{2\pi}\dfrac{cycle}{m}, \\ S_d(f_{s0})\left(\dfrac{f_s}{f_{s0}}\right)^{-n_2} & for \quad f_s > f_{s0}, \end{cases} \tag{36}$$

where f_{s0} [cycle/m] is the spatial frequency and the following exponent values are considered: n_1=2.0 and n_2=1.5. According to ISO, the class of a road is given by the value of $S_d(f_{s0})$ [7].

To turn from the expression with the spatial frequency to the expression with the so-called frequency f the following relations are used:

$$f = f_s \cdot v, \quad S_{df}(f) = S_d(f/v)/v. \tag{37}$$

The root-mean-square (rms) value of the sprung mass acceleration is given by the relation [6, 7]

$$\sigma_{\ddot{z}_g} = 4\pi^2\sqrt{\int_0^{\infty} f^4 \left|H_{\Delta z_g}(f)\right|^2 S_{df}(f)df}, \tag{38}$$

where $H_{\Delta zg}$ represents the transfer function for Δz_g. Also, for the root-mean-square value of the dynamic tire normal reaction we can write

$$\sigma_{\Delta Zw} = 2\pi\sqrt{\int_0^{\infty} \left|H_{\Delta Zw}(f)\right|^2 S_{df}(f)df}, \tag{39}$$

where $H_{\Delta Zw}$ is the transfer function for the increase of the tire normal reaction.

In the case of the quarter-car model, the suitable transfer functions are employed. If $(\sigma_{\ddot{z}_g})_0$ and $(\sigma_{\Delta Zw})_0$ are the values corresponding to this model, the following expression are employed in order to make comparisons:

$$\delta_1 = \frac{\sigma_{\ddot{z}_g} - (\sigma_{\ddot{z}_g})_0}{(\sigma_{\ddot{z}_g})_0}.100\%, \quad \delta_2 = \frac{\sigma_{\Delta Zw} - (\sigma_{\Delta Zw})_0}{(\sigma_{\Delta Zw})_0}.100\%. \qquad (40)$$

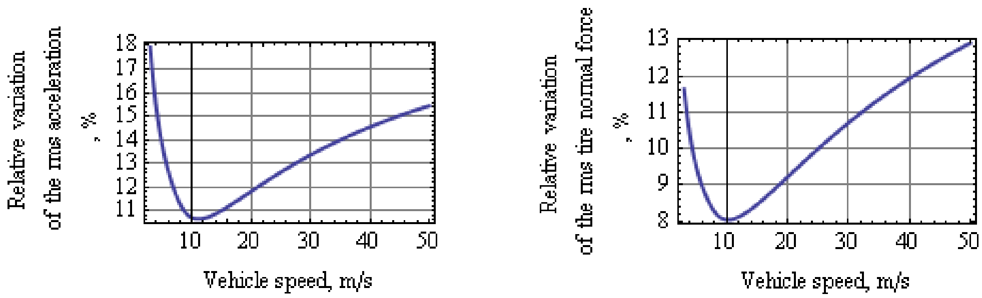


Fig.9. Variations of δ_1 and δ_2 with vehicle velocity

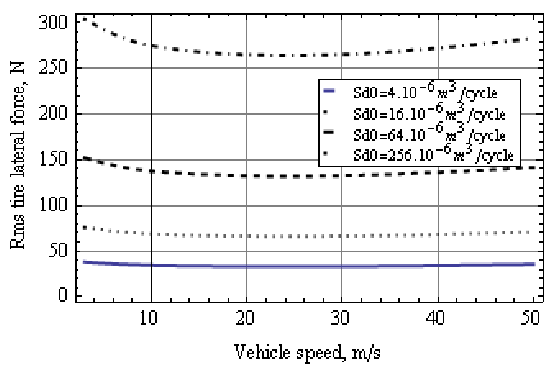


Fig.10. Root-mean-square of lateral force as a function of vehicle speed

It is easy to see that δ_1 and δ_2 do not depend on the road class defined by $S_d(f_{s0})$. Results relating to quantities δ_1 and δ_2 are shown in Figure 9.

From Figure 9 it is found that the effect of the lateral deformations is to increase the root-mean-square values of the vertical acceleration and the wheel normal reaction. This increase is dependent of the vehicle velocity. For $v \approx 10$ m.s^{-1} δ_1 and δ_2 have the minimum values of 10.5% and 8% respectively. The relative increases δ_1 and δ_2 attain the values of 18% and 13% respectively.

Regarding the root-mean-square value of the lateral force we use a relation similar to (39) which contains the transfer function of the lateral force.

From Fig. 10 one can see that for a given class, the rms of the lateral forces is independent of the vehicle velocity. Obviously, when the degree of roughness increases, the root-mean-square of the lateral forces increases also.

4. Conclusions

The non-linear differential equations established in this paper for the swing axle suspension allow the study of heave vibrations of an automobile when it travels over surface with large undulations.

During traveling over the surface of common roads, it is advantageous to use the linearised equations which are deduced from the non-linear ones.

In the case of the swing axle suspension, the tire lateral deformations produced by the track width variation diminish the ride quality and the road holding of an automobile at both small and large velocity.

The usual ride quarter-car model is not enough suitable for the swing axle suspension.

During the testing of the suspension and the experimental determination the suspension elastic characteristic, significant errors are produced if the wheels are not supported on the plates with balls.

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