A NOVEL STUDY ON FUZZY CONGRUENCES ON n-ARY SEMIGROUPS

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In this paper, we introduce the concept of fuzzy congruences on n-ary semi-groups and describe quotient n-ary semigroups by fuzzy congruences. Some isomorphism theorems about n-ary semigroups are established. Moreover, we discuss a special kind of n-ary semigroups. We also establish relationships between normal fuzzy ideals and fuzzy congruences. In particular, we prove that there exists a preserving inclusion injective mapping from the set of all normal fuzzy ideals of the special n-ary semigroups to the set of all fuzzy congruences in an n-ary semigroup with one zero. Finally, we obtain that there is a one-to-one correspondence between the set of all invariant fuzzy congruences on an n-ary semigroup and the set of all invariant fuzzy congruences on a quotient n-ary semigroup.

Keywords: n-ary semigroup; quotient n-ary semigroup; congruence; fuzzy congruence; fuzzy ideal; normal fuzzy ideal.
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1. Introduction

The theory of fuzzy sets was first developed by Zadeh [27] and has been applied to many branches in mathematics and other applied areas. Later, the concept of fuzzy subgroups was introduced by Rosenfeld [23]. This work was the first fuzzification of algebraic structures and thus opened a new direction, new exploration, new path of thinking to mathematicians, engineers, computer scientists, and many other researchers. In order to study quotient algebraic structures, we naturally need to consider fuzzy congruence relations of algebraic structures. At present, this work is mainly concentrated on the groups and semigroups. We know that there exists a one-to-one correspondence between the set of all congruences on a group and the set of all normal subgroups of a group. In [20], Kuroki proved that the set of all fuzzy congruences and the set of all fuzzy ideals of a group can be depicted with each other. Similar to the work, there is a lot of work but different emphasis, for example, Dutta and Biswas have discussed the relationships between fuzzy k-ideals and fuzzy congruences in [16]. In 1997, Kim and Bae [18] also proved that the set of all fuzzy

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congruences with respect to usually intersection and union is a modular lattice. The other important results about fuzzy congruences can be found in [21, 24, 25, 26].

The generalization of algebraic structures was in active research for a long time, it was first initiated by Kasner [17] in 1904, but the important study of n-ary semigroups and n-ary groups was done by Dudek. For more details, the reader is referred to [7, 8, 9, 11, 13, 14, 15]. In addition, a new class of mathematical structures called (m, n)-semirings (which generalize the usual semirings) was discussed by [1, 2, 5, 6]. Up till now, the theory of n-ary systems have many applications, for example, in the theory of automata. We know that n-ary semigroups have been applied in the theory of fuzzy sets and rough sets (see [3, 4]). The first fuzzification of an n-ary system was introduced by Dudek [10]. Moreover, as a generalization of Rosenfeld's fuzzy groups, Davvaz and Dudek [4] discussed further fuzzy n-ary groups, and investigated their related properties. The notion of intuitionistic fuzzy sets, as a generalization of the notion of fuzzy sets, was introduced by Dudek [12] in n-ary systems. In particular, n-ary hyper algebras were investigated by many researchers, for examples, see [28].

The purpose of this paper is to introduce fuzzy congruences on an n-ary semigroup, and establish isomorphism theorems about n-ary semigroups in terms of fuzzy congruences. Furthermore, we discuss a special kind of n-ary semigroups with one zero, and give a characterization between normal fuzzy ideals and fuzzy congruences. Moreover, we prove that in an n-ary semigroup with one zero, there exists a preserving inclusion injective mapping from the set of all normal fuzzy ideals of the special n-ary semigroups to the set of all fuzzy congruences.

2. Preliminaries

A non-empty set S together with one n-ary operation $f : S^n \to S$, where $n \geq 2$, is called an n-ary groupoid and is denoted by (S, f). According to the general convention used in the theory of n-ary groupoids, the sequence of elements $x_i, x_{i+1}, \ldots, x_j$ is denoted by x_i^j. In the case $j < i$, it is the empty symbol. If $x_{i+1} = x_{i+2} = \ldots = x_{i+t} = x$, then we write $\overset{(t)}{x}$ instead of x_{i+1}^{i+t}. In this convention,

$$f(x_1, x_2, \ldots, x_n) = f(x_1^n),$$

and

$$f(x_1, \ldots, x_i, \underbrace{x, \ldots, x}_{t}, x_{i+t+1}, \ldots, x_n) = f(x_1^i, \overset{(t)}{x}, x_{i+t+1}^n).$$

An n-ary groupoid (S, f) is called an (i, j)-associative if

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$$

hold for all $x_1, x_2, \ldots, x_{2n-1} \in S$. If this identity holds for all $1 \leq i \leq j \leq n$, then we say that the operation f is associative, and (S, f) is called an n-ary semigroup. Throughout this paper, unless otherwise mentioned, (S, f) will denote an n-ary semigroup.

Definition 2.1. [20] *A fuzzy set μ of R is called a A function α from $S \times S$ to the unit interval [0,1] is called a fuzzy relation on S. Let α and β be two fuzzy relations on S. Then*

the product $\alpha \circ \beta$ of α and β is defined by

$$(\alpha \circ \beta)(x,y) = \bigvee_{z \in S} [\alpha(x,z) \wedge \beta(z,y)]$$

for all $(x,y) \in S \times S$.

Definition 2.2. [18] *A fuzzy relation α of S is called a fuzzy equivalence relation if it satisfies the following conditions:*

(i) $\alpha(x,x) = 1$ for all $x \in S$ (fuzzy reflexive).

(ii) $\alpha(x,y) = \alpha(y,x)$ for all $x,y \in S$ (fuzzy symmetric).

(iii) $\alpha(x,y) \geq \bigvee_{z \in S} [\alpha(x,z) \wedge \alpha(z,y)]$ for all $x,y,z \in S$ (fuzzy transitive).

In particular, the identity relation on S is defined by

$$Id_S(x,y) = \begin{cases} 1 & if\ x = y, \\ 0 & otherwise, \end{cases}$$

for all $(x,y) \in S \times S$.

Definition 2.3. [10] *A fuzzy set μ on an n-ary semigroup (S,f) is called a fuzzy k-ideal if*

$$\mu(f(x_1^n)) \geq \mu(x_k)$$

holds for all $x_1, x_2, \ldots, x_n \in S$. If μ is a fuzzy k-ideal for every $k = 1, 2, \ldots, n$, then it is called a fuzzy ideal. Clearly, if μ satisfies $\mu(f(x_1^n)) \geq \mu(x_1) \vee \mu(x_2) \vee \ldots \vee \mu(x_n)$, then it is a fuzzy ideal of (S,f).

Remark 2.1. *A fuzzy ideal μ of (S,f) is said to be normal if there exists $x \in S$ such that $\mu(x) = 1$.*

3. Quotient n-ary semigroups via fuzzy congruences

In this section, we introduce fuzzy congruences on an n-ary semigroup, and establish isomorphism theorems about n-ary semigroups via fuzzy congruences.

Definition 3.1. *Let α be a fuzzy equivalence relation of (S,f). α is called a fuzzy congruence of (S,f) if it satisfies:*

$$\alpha(f(x_1^n), f(y_1^n)) \geq \alpha(x_1, y_1) \wedge \alpha(x_2, y_2) \wedge \ldots \wedge \alpha(x_n, y_n)$$

for all $1 \leq i \leq n$ and $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in S$.

We denote the set of all fuzzy congruences of (S,f) by $FC(S,f)$.

Example 3.1. *Consider the set $S = \{-i, 0, i\}$ with the 3-ary operation f as the usual multiplication of complex numbers. Then (S,f) is a 3-ary semigroup. The fuzzy relation α on (S,f) defined by*

$$\alpha(x,y) = \begin{cases} 1 & if\ x = y, \\ 0.5 & if\ x \neq y\ and\ both\ x,y\ are\ imaginary\ numbers, \\ 0 & otherwise. \end{cases}$$

is a fuzzy congruence on (S,f).

Let α be a fuzzy relation of (S, f). For each $\lambda \in [0, 1]$, we put $S_\alpha(\lambda) = \{(a, b) | (a, b) \in S \times S, \alpha(a, b) \geq \lambda\}$. This set is called λ-level set of α.

Theorem 3.1. α *is a fuzzy congruence of (S, f) if and only if for each $\lambda \in [0, 1]$, $S_\alpha(\lambda)$ is a congruence of (S, f).*

Proof. It is clear that $S_\alpha(\lambda)$ is an equivalence relation. Since α is a fuzzy congruence of (S, f), let $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n) \in S_\alpha(\lambda)$. Then we have

$$\alpha(f(x_1^n), f(y_1^n)) \geq \alpha(x_1, y_1) \wedge \alpha(x_2, y_2) \wedge \ldots \wedge \alpha(x_n, y_n) \geq \lambda,$$

which implies that $(f(x_1^n), f(y_1^n)) \in S_\alpha(\lambda)$. Hence $S_\alpha(\lambda)$ is a congruence of (S, f).

Conversely, for each $\lambda \in [0, 1]$, since $S_\alpha(\lambda)$ is a congruence of (S, f), for any $x \in S, \alpha(x, x) \geq \lambda$. It implies that $\alpha(x, x) = 1$, so α is a fuzzy reflexive relation. For all $x, y \in S$, if $\alpha(x, y) \neq \alpha(y, x)$, let $\alpha(x, y) = \lambda_1, \alpha(y, x) = \lambda_2$. If $\lambda_1 > \lambda_2$, then $(y, x) \notin S_\alpha(\lambda_1)$, but $(x, y) \in S_\alpha(\lambda_1)$, since $S_\alpha(\lambda_1)$ is a congruence of (S, f), contradiction. So $\alpha(x, y) = \alpha(y, x)$. When $\lambda_1 < \lambda_2$, the proof is similar. Hence α is fuzzy symmetric relation. For any $x, y, z \in S$, let $\alpha(x, z) = t_1, \alpha(z, y) = t_2$. If $t_1 \leq t_2$, then $(x, z), (z, y) \in S_\alpha(t_1)$, so $(x, y) \in S_\alpha(t_1), \alpha(x, y) \geq t_1 = \bigvee_{z \in S} [\alpha(x, z) \wedge \alpha(z, y)]$. When $t_1 \geq t_2$, the proof is similar. This means α is a fuzzy transitive relation. Thus α is a fuzzy equivalence relation of (S, f).

Since $S_\alpha(\lambda)$ is a congruence of (S, f), for any $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in S$, let $\alpha(x_1, y_1) = a_1, \alpha(x_2, y_2) = a_2, \ldots, \alpha(x_n, y_n) = a_n$, put $a_0 = a_1 \wedge a_2 \ldots \wedge a_n$. Then $\alpha(x_i, y_i) \geq a_0$, where $1 \leq i \leq n$, so $(x_i, y_i) \in S_\alpha(a_0)$, which implies $(f(x_1^n), f(y_1^n)) \in S_\alpha(a_0)$. Hence $\alpha(f(x_1^n), f(y_1^n)) \geq a_0 = \alpha(x_1, y_1) \wedge \alpha(x_2, y_2) \wedge \ldots \wedge \alpha(x_n, y_n)$. This means α is a fuzzy congruence of (S, f). This completes the proof. $\square$

Let α be a fuzzy congruence of an n-ary semigroup (S, f). For any $x, y \in S$, we define a binary relation $\sim$ on S by

$$x \sim y \quad if \quad and \quad only \quad if \quad \alpha(x, y) = 1.$$

Corollary 3.1. $\sim$ *is a congruence of (S, f).*

Let $\alpha_x = \{y \in S | y \sim x\}$. Then α_x is the congruence class containing x and $(S, f)/\alpha = \{\alpha_x | x \in S\}$ is the set of all congruence classes of (S, f) for any $x \in S$.

Remark 3.1. *From the define above, we can obtain that $\alpha_x = \alpha_y$ if and only if $\alpha(x, y) = 1$.*

Theorem 3.2. *If α is a fuzzy congruence of an n-ary semigroup (S, f), then $((S, f)/\alpha, F)$ is an n-ary semigroup under the n-ary operation defined by*

$$F(\alpha_{x_1}, \alpha_{x_1}, \ldots, \alpha_{x_n}) = \alpha_{f(x_1^n)}$$

for all $x_1, x_2, \ldots, x_n \in S$.

Proof. We shall first show that the given operation is well defined. Let $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in S$ be such that $\alpha_{x_1} = \alpha_{y_1}, \alpha_{x_2} = \alpha_{y_2}, \ldots, \alpha_{x_n} = \alpha_{y_n}$. We need to show that

$$F(\alpha_{x_1}, \alpha_{x_1}, \ldots, \alpha_{x_n}) = F(\alpha_{y_1}, \alpha_{y_1}, \ldots, \alpha_{y_n}).$$

In fact, since α is a fuzzy congruence of (S, f), it follows from Remark 3.1 that

$$\alpha(x_1, y_1) = \alpha(x_2, y_2) = \ldots = \alpha(x_n, y_n) = 1.$$

So

$$\alpha(f(x_1^n), f(y_1^n)) \geq \alpha(x_1, y_1) \wedge \alpha(x_2, y_2) \wedge \ldots \wedge \alpha(x_n, y_n) = 1.$$

Thus $\alpha(f(x_1^n), f(y_1^n)) = 1$, which implies $\alpha_{f(x_1^n)} = \alpha_{f(y_1^n)}$. This means $F(\alpha_{x_1}, \alpha_{x_2}, \ldots, \alpha_{x_n}) = F(\alpha_{y_1}, \alpha_{y_1}, \ldots, \alpha_{y_n})$. Hence F is well defined. $(S, f)/\alpha$ is closed under the operation F and F is (i, j)-associative are obvious, we omit their proof. $\qquad \square$

Theorem 3.3. *Let α and β be two fuzzy congruences of an n-ary semigroup (S, f) with $\alpha \subseteq \beta$. Then the fuzzy relation β/α of $((S, f)/\alpha, F)$, defined by $(\beta/\alpha)(\alpha_x, \alpha_y) = \beta(x, y)$ is a fuzzy congruence of $((S, f)/\alpha, F)$ and $((S, f)/\alpha, F)/(\beta/\alpha) \cong (S, f)/\beta$.*

Proof. First we show β/α is well defined. In fact, if $\alpha_x = \alpha_{x'}$ and $\alpha_y = \alpha_{y'}$, then $\alpha(x, x') = \alpha(x, x') = 1$. Since $\alpha \subseteq \beta$, so $\beta(x, x') = \beta(y, y') = 1$.

Again, β is a fuzzy congruence of (S, f), then

$$
\begin{aligned}
\beta(x, y) &\geq \bigvee_{z \in S} [\beta(x, z) \wedge \beta(z, y)] \\
&\geq \beta(x, y') \wedge \beta(y', y) \\
&= \beta(x, y') \\
&\geq \bigvee_{z' \in S} [\beta(x, z') \wedge \beta(z', y')] \\
&\geq \beta(x, x') \wedge \beta(x', y') \\
&= \beta(x', y'),
\end{aligned}
$$

and thus $\beta(x, y) \geq \beta(x', y')$. Similarly, we can prove that $\beta(x', y') \geq \beta(x, y)$. Therefore, $\beta(x, y) = \beta(x', y')$. This means β/α is well defined.

Obviously, β/α is a fuzzy equivalence relation of $((S, f)/\alpha, F)$. Let $\forall x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in S$. Then, since β is a fuzzy congruence of (S, f), we have

$$
\begin{aligned}
&(\beta/\alpha)(F(\alpha_{x_1}, \alpha_{x_2}, \ldots, \alpha_{x_n}), F(\alpha_{y_1}, \alpha_{y_2}, \ldots, \alpha_{y_n})) \\
={}& (\beta/\alpha)(\alpha_{f(x_1^n)}, \alpha_{f(y_1^n)}) \\
={}& \beta(f(x_1^n), f(y_1^n)) \\
\geq{}& \beta(x_1, y_1) \wedge \ldots \wedge \beta(x_n, y_n) \\
={}& (\beta/\alpha)(\alpha_{x_1}, \alpha_{y_1}) \wedge \ldots \wedge (\beta/\alpha)(\alpha_{x_n}, \alpha_{y_n}).
\end{aligned}
$$

Hence β/α is a fuzzy congruence of $((S, f)/\alpha, F)$.

By Theorem 3.2, we know $((S, f)/\alpha, F)/(\beta/\alpha)$ and $(S, f)/\beta$ are n-ary semigroups.

Define a mapping

$$h : ((S, f)/\alpha, F)/(\beta/\alpha) \to (S, f)/\beta,$$

by $h((\beta/\alpha)_{\alpha_x}) = \beta_x$ for all $x \in S$. If $(\beta/\alpha)_{\alpha_x} = (\beta/\alpha)_{\alpha_y}$, then $(\beta/\alpha)(\alpha_x, \alpha_y) = \beta(x, y) = 1$, so $\beta_x = \beta_y$. Hence h is well-defined.

Let F^* and F' be two n-ary operations of $((S,f)/\alpha, F)/(\beta/\alpha)$ and $(S,f)/\beta$, respectively. Then we have

$$
\begin{aligned}
& h(F^*((\beta/\alpha)_{\alpha_{x_1}}, (\beta/\alpha)_{\alpha_{x_2}}, \ldots, (\beta/\alpha)_{\alpha_{x_n}})) \\
=\ & h((\beta/\alpha)_{F(\alpha_{x_1}, \alpha_{x_2}, \ldots, \alpha_{x_n})}) \\
=\ & h((\beta/\alpha)_{\alpha_{f(x_1^n)}}) \\
=\ & \beta_{f(x_1^n)} \\
=\ & F'(\beta_{x_1}, \beta_{x_2}, \ldots, \beta_{x_n}) \\
=\ & F'((\beta/\alpha)_{\alpha_{x_1}}, (\beta/\alpha)_{\alpha_{x_2}}, \ldots, (\beta/\alpha)_{\alpha_{x_n}}).
\end{aligned}
$$

This means h is a homomorphism. If $\beta_x = \beta_y$, then $\beta(x,y) = (\beta/\alpha)(\alpha_x, \alpha_y) = 1$. It thus follows that $(\beta/\alpha)_{\alpha_x} = (\beta/\alpha)_{\alpha_y}$, and h is injective.

Furthermore, for any $\beta_x \in (S,f)/\beta$, there exists $\alpha = \beta$ such that $h((\beta/\alpha)_{\beta_x}) = h((\beta/\alpha)_{\alpha_x}) = \beta_x$. Hence h is surjective. This completes the proof. $\square$

Let (S,f) and (H,g) be two n-ary semigroups, φ a homomorphism of S to H. If α is a fuzzy relation on (S,f) and α' is a fuzzy relation on (H,g), then the inverse image $\varphi^{-1}(\alpha')$ of α' is the fuzzy relation on (S,f) defined by

$$
\varphi^{-1}(\alpha')(x,y) = \alpha'(\varphi(x), \varphi(y))
$$

for all $x, y \in S$.

The image $\varphi(\alpha)$ of α is the fuzzy relation on (H,g) defined by

$$
\varphi(\alpha)(x,y) = \begin{cases} \bigvee_{(x_i, y_i) \in \varphi^{-1}(x,y)} \alpha(x_i, y_i) & \text{if } \varphi^{-1}(x,y) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}
$$

for all $x, y \in H, x_i, y_i \in S$ and $1 \leq i \leq n$.

The following basic assertions hold:

(i) For any $\alpha \in FC(S,f), \alpha \subseteq \varphi^{-1}(\varphi(\alpha))$. If φ is injective, then $\alpha = \varphi^{-1}(\varphi(\alpha))$.

(ii) For any $\beta \in FC(H,g), \varphi(\varphi^{-1}(\beta)) \subseteq \beta$. If φ is surjective, then $\varphi(\varphi^{-1}(\beta)) = \beta$.

Proposition 3.1. *Let (S,f) and (H,g) be two n-ary semigroups, φ a homomorphism from (S,f) into (H,g). If α' is a fuzzy congruence on (H,g), then $\varphi^{-1}(\alpha')$ is a fuzzy congruence on (S,f).*

Proof. Let $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in S$, then we have

$$
\begin{aligned}
& \varphi^{-1}(\alpha')(f(x_1^n), f(y_1^n)) \\
=\ & \alpha'(\varphi(f(x_1^n)), \varphi(f(y_1^n))) \\
=\ & \alpha'(g(\varphi(y_1), \varphi(y_2), \ldots, \varphi(y_n)), g(\varphi(y_1), \varphi(y_2), \ldots, \varphi(y_n))) \\
\geq\ & \alpha'(\varphi(x_1, y_1)) \wedge \alpha'(\varphi(x_2, y_2)) \wedge \ldots \wedge \alpha'(\varphi(x_n, y_n)) \\
=\ & \varphi^{-1}(\alpha')(x_1, y_1) \wedge \varphi^{-1}(\alpha')(x_2, y_2) \wedge \ldots \wedge \varphi^{-1}(\alpha')(x_n, y_n).
\end{aligned}
$$

Hence $\varphi^{-1}(\alpha')$ is a fuzzy congruence on (S,f). $\square$

Proposition 3.2. *Let (S,f) and (H,g) be two n-ary semigroups, φ an epimorphism from (S,f) into (H,g). If α is a fuzzy congruence on (S,f), then $\varphi(\alpha)$ is a fuzzy congruence on (H,g).*

Proof. Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in H$. Since φ is an epimorphism, there exist $x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots, y_n \in S$ such that $\varphi(x_1) = a_1, \varphi(x_2) = a_2, \ldots, \varphi(x_n) = a_n, \varphi(y_1) = b_1, \varphi(y_2) = b_2, \ldots, \varphi(y_n) = b_n$. So

$$\varphi(f(x_1^n)) = g(\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_n)) = g(a_1^n),$$

and

$$\varphi(f(y_1^n)) = g(\varphi(y_1), \varphi(y_2), \ldots, \varphi(y_n)) = g(b_1^n).$$

Hence $\{(x_i, y_i) \in S \times S (i = 1, 2, \ldots, n) | (x_i, y_i) \in \varphi^{-1}(g(a_1^n), g(b_1^n))\} \supseteq \{(f(x_1^n), f(y_1^n)) \in S \times S | (x_1, y_1) \in \varphi^{-1}(a_1, b_1), (x_2, y_2) \in \varphi^{-1}(a_2, b_2), \ldots, (x_n, y_n) \in \varphi^{-1}(a_n, b_n)\}$. Then we have

$$
\begin{aligned}
&\varphi(\alpha)(g(a_1^n), g(b_1^n)) \\
=\ & \bigvee_{(x_i, y_i) \in \varphi^{-1}(g(a_1^n), g(b_1^n))} \alpha(x_i, y_i) \\
\geq\ & \bigvee_{(x_1, y_1) \in \varphi^{-1}(a_1, b_1), \ldots, (x_n, y_n) \in \varphi^{-1}(a_n, b_n)} \alpha(f(x_1^n), f(y_1^n)) \\
\geq\ & \bigvee_{(x_1, y_1) \in \varphi^{-1}(a_1, b_1), \ldots, (x_n, y_n) \in \varphi^{-1}(a_n, b_n)} \alpha(x_1, y_1) \wedge \ldots \wedge \alpha(x_n, y_n) \\
=\ & \bigvee_{(x_1, y_1) \in \varphi^{-1}(a_1, b_1)} \alpha(x_1, y_1) \wedge \ldots \wedge \bigvee_{(x_n, y_n) \in \varphi^{-1}(a_n, b_n)} \alpha(x_n, y_n) \\
=\ & \varphi(\alpha)(a_1, b_1) \wedge \ldots \wedge \varphi(\alpha)(a_n, b_n).
\end{aligned}
$$

Hence $\varphi(\alpha)$ is a fuzzy congruence on (H, g). This completes the proof. $\square$

Theorem 3.4. *Let (S, f) and (H, g) be two n-ary semigroups, φ an isomorphism of (S, f) into (H, g). If α is a fuzzy congruence on (S, f), then $(S, f)/\alpha \cong (H, g)/\varphi(\alpha)$.*

Proof. By Theorem 3.2 and Proposition 3.2, $(S, f)/\alpha$ and $(H, g)/\varphi(\alpha)$ are n-ary semigroups. Define a mapping $\theta : (S, f)/\alpha \to (H, g)/\varphi(\alpha)$ by $\theta(\alpha_x) = \varphi(\alpha)_{\varphi(x)}$ for all $x \in S$. We first show that θ is well defined. In fact, let $x, x' \in S$. If $\alpha_x = \alpha_{x'}$, then $\alpha(x, x') = 1$, and we have

$$\varphi(\alpha)(\varphi(x), \varphi(x')) = \bigvee_{(x, x') \in \varphi^{-1}(\varphi(x), \varphi(x'))} \alpha(x, x') = 1,$$

which implies that $\varphi(\alpha)_{\varphi(x)} = \varphi(\alpha)_{\varphi(x')}$. Hence θ is well defined.

Moreover, θ is also a homomorphism. In fact, let $x_1, x_2, \ldots, x_n \in S$, T and T' be two n-ary operations of n-ary semigroup $(S, f)/\alpha$ and $(H, g)/\varphi(\alpha)$, respectively. Then we have

$$
\begin{aligned}
\theta(T(\alpha_{x_1}, \alpha_{x_2}, \ldots, \alpha_{x_n})) &= \theta(\alpha_{f(x_1^n)}) \\
&= \varphi(\alpha)_{\varphi(f(x_1^n))} \\
&= \varphi(\alpha)_{g(\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_n))} \\
&= T'(\varphi(\alpha)_{\varphi(x_1)}, \varphi(\alpha)_{\varphi(x_2)}, \ldots, \varphi(\alpha)_{\varphi(x_n)}).
\end{aligned}
$$

This means θ is a homomorphism.

Since φ is surjective, for any $\varphi(\alpha)_y \in (H, g)/\varphi(\alpha), y \in H$, there exists $x \in S$ such that $\varphi(x) = y$. So $\theta(\alpha_x) = \varphi(\alpha)_{\varphi(x)} = \varphi(\alpha)_y$, which implies that θ is surjective. Again, φ is injective, so $\alpha = \varphi^{-1}(\varphi(\alpha))$. For any $x, x' \in S$, if $\varphi(\alpha)_{\varphi(x)} = \varphi(\alpha)_{\varphi(x')}$, then we have $\alpha(x, x') = \varphi^{-1}(\varphi(\alpha))(x, x') = \varphi(\alpha)(\varphi(x), \varphi(x')) = 1$, which implies that $\alpha_x = \alpha'_{x'}$. This means θ is injective. Thus θ is a isomorphism and $(S, f)/\alpha \cong (H, g)/\varphi(\alpha)$. $\square$

Theorem 3.5. *Let (S, f) and (H, g) be two n-ary semigroups, φ an epimorphism of (S, f) into (H, g). If α' is a fuzzy congruence on (H, g), then $(S, f)/\varphi^{-1}(\alpha') \cong (H, g)/\alpha$.*

Proof. The proof is similar to that of Theorem 3.4, and we omit it.   □

4. **The relationships between fuzzy ideals and fuzzy congruences**

In this section, we give the concept of an n-ary semigroup with one zero and investigate the characterizations of normal fuzzy ideals and fuzzy congruences on a special kind of n-ary semigroup. Moreover, some conclusions are given based on the concept of an invariant fuzzy congruence about another fuzzy congruence on an n-ary semigroup.

Definition 4.1. *An n-ary semigroup (S, f) has zero element 0 if it satisfies:*

$$f(x_1^{i-1}, 0, x_{i+1}^n) = 0$$

for all $x_1, x_2, \ldots, x_n \in S$ and $1 \leqslant i \leqslant n$.

Example 4.1. [6] *Let $\mathbb{N}$ be the set of all natural numbers and f an n-ary operation $(n \geqslant 2)$ defined on $\mathbb{N}$ by the formula $f(x_1^n) = x_1 \cdot x_2 \ldots x_n$. Then it is not difficult to see that (S, f) is an n-ary semigroup with zero element 0.*

Example 4.2. *Let $S = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$, where the 3-ary operation f is the usual matrix multiplication. Then (S, f) is a 3-ary semigroup with zero element. Clearly, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is the zero element of (S, f).*

Theorem 4.1. *Let α be a fuzzy congruence on an n-ary semigroup (S, f) with zero element 0 and μ_α be the fuzzy set of S defined by*

$$\mu_\alpha(x) = \alpha(x, 0),$$

for all $x \in S$. Then μ_α is a normal fuzzy ideal of (S, f).

Proof. It is straighforward.   □

Theorem 4.2. *Let μ be a fuzzy ideal of (S, f). Then the fuzzy relation α_μ on (S, f) defined by*

$$\alpha_\mu(x, y) = (\mu(x) \wedge \mu(y)) \vee Id_S(x, y),$$

for all $x, y \in S$ is a fuzzy congruence on (S, f).

Proof. It is clear that α_μ is fuzzy reflexive and fuzzy symmetric. Now we show that α_μ is fuzzy transitive. In fact, if $x = y$, then $(\alpha_\mu \circ \alpha_\mu)(x, y) = \bigvee_{z \in S} \alpha_\mu(x, z) = 1 = \alpha_\mu(x, y)$. Let

$x \neq y$. Then we have

$$
\begin{aligned}
& (\alpha_\mu \circ \alpha_\mu)(x, y) \\
= {} & (\bigvee_{z \in S - \{x,y\}} (\alpha_\mu(x,z) \wedge \alpha_\mu(z,y))) \vee (\alpha_\mu(x,x) \wedge \alpha_\mu(x,y)) \vee (\alpha_\mu(x,y) \wedge \alpha_\mu(y,y)) \\
= {} & \alpha_\mu(x,y) \vee (\bigvee_{z \in S - \{x,y\}} \mu(x) \wedge \mu(z) \wedge \mu(y)) \\
\leq {} & \alpha_\mu(x,y) \vee (\bigvee_{z \in S - \{x,y\}} \mu(x) \wedge \mu(y)) \\
= {} & \alpha_\mu(x,y) \vee ((\mu(x) \wedge \mu(y)) \vee Id_S(x,y)) \\
= {} & \alpha_\mu(x,y).
\end{aligned}
$$

Thus α_μ is fuzzy transitive. Hence α_μ is a fuzzy equivalence relation on (S, f).

For all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in S$, if $f(x_1^n) = f(y_1^n)$, then $\alpha_\mu(f(x_1^n), f(y_1^n))$ $= 1 \geq \alpha_\mu(x_1, y_1) \wedge \alpha_\mu(x_2, y_2) \wedge \ldots \wedge \alpha_\mu(x_n, y_n)$, and if $f(x_1^n) \neq f(y_1^n)$ then at most $n-1$ equations hold as follows:

$$
x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n.
$$

Since μ is a fuzzy ideal of (S, f), when there exist i $(0 \leq i \leq n-1)$ equations hold in above, we have

$$
\begin{aligned}
& \alpha_\mu(f(x_1^n), f(y_1^n)) \\
= {} & (\mu(f(x_1^n)) \wedge \mu(f(y_1^n))) \vee Id_S(f(x_1^n), f(y_1^n)) \\
= {} & \mu(f(x_1^n)) \wedge \mu(f(y_1^n)) \\
\geq {} & (\mu(x_1) \vee \mu(x_2) \vee \ldots \vee \mu(x_n)) \vee (\mu(y_1) \vee \mu(y_2) \vee \ldots \vee \mu(y_n)) \\
\geq {} & (\mu(x_1) \wedge \mu(y_1)) \vee (\mu(x_2) \wedge \mu(y_2)) \vee \ldots \vee (\mu(x_n) \wedge \mu(y_n)) \\
= {} & \mu(x_1) \vee \mu(x_2) \vee \ldots \vee \mu(x_i) \vee (\mu(x_{i+1}) \wedge \mu(y_{i+1})) \vee \ldots \vee (\mu(x_n) \wedge \mu(y_n)) \\
\geq {} & (\mu(x_{i+1}) \wedge \mu(y_{i+1})) \vee \ldots \vee (\mu(x_n) \wedge \mu(y_n)) \\
\geq {} & (\mu(x_{i+1}) \wedge \mu(y_{i+1})) \wedge \ldots \wedge (\mu(x_n) \wedge \mu(y_n)) \\
= {} & \underbrace{1 \wedge \ldots \wedge 1}_{i} \wedge (\mu(x_{i+1}) \wedge \mu(y_{i+1})) \wedge \ldots \wedge (\mu(x_n) \wedge \mu(y_n)) \\
= {} & \alpha_\mu(x_1, y_1) \wedge \ldots \wedge \alpha_\mu(x_i, y_i) \wedge \alpha_\mu(x_{i+1}, y_{i+1}) \wedge \ldots \wedge \alpha_\mu(x_n, y_n).
\end{aligned}
$$

Therefore, α_μ is a fuzzy congruence on (S, f). This completes the proof. $\qquad \square$

Proposition 4.1. *If μ is a normal fuzzy ideal of (S, f) in the above theorem, then $\alpha_\mu(x, 0) = \mu(x)$ for all $x \in S$. Moreover, α_μ is the smallest fuzzy congruence.*

Proof. Since (S, f) is an n-ary semigroup with zero element 0 and μ is a normal fuzzy ideal of (S, f), we have $\mu(0) = 1$ and $\mu(0) \geq \mu(x)$ for all $x \in S$. If $x = 0$, then $\alpha_\mu(x, 0) = \alpha_\mu(0, 0) = 1 = \mu(0) = \mu(x)$, and if $x \neq 0$, then $\alpha_\mu(x, 0) = (\mu(x) \wedge \mu(0)) \vee Id_S(x, 0) = \mu(x)$.

Let β be a fuzzy congruence on (S, f) such that $\beta(x, 0) = \mu(x)$. Then $\beta(x, y) \geq (\beta \circ \beta)(x, y) = \bigvee_{x \in S} (\beta(x, z) \wedge \beta(z, y)) \geq \beta(x, 0) \wedge \beta(0, y) = \mu(x) \wedge \mu(y)$. If $x = y, \beta(x, y) = \beta(x, x) = 1 = \alpha_\mu(x, x) = \alpha_\mu(x, y)$, and $x \neq y, \beta(x, y) \geq \mu(x) \wedge \mu(y) = (\mu(x) \wedge \mu(y)) \vee Id_S(x, y) = \alpha_\mu(x, y)$. So $\beta(x, y) \geq \alpha_\mu(x, y)$ for all $(x, y) \in S \times S$. This completes the proof. $\qquad \square$

Theorem 4.3. *Let (S, f) be an n-ary semigroup with zero element 0. Then there exists a preserving inclusion injective mapping from the set of all normal fuzzy ideals of (S, f) to the set of all fuzzy congruences on (S, f).*

Proof. Let $FC(S, f)$ be the set of all fuzzy congruences on (S, f) and $NFI(S, f)$ the set of all normal fuzzy ideals of (S, f). We define a mapping $\delta : FC(S, f) \rightarrow NFI(S, f)$ and a mapping $\eta : NFI(S, f) \rightarrow FC(S, f)$ by $\delta(\alpha) = \mu_\alpha$ and $\eta(\mu) = \alpha_\mu$, respectively, for $\alpha \in FC(S, f)$ and $\mu \in NFI(S, f)$ (μ_α and α_μ are defined above). Since $(\delta \circ \eta)(\mu) = \delta(\alpha_\mu) = \mu_{\alpha_\mu}$ and $\mu_{\alpha_\mu}(x) = \alpha_\mu(x, 0) = \mu(x)$ for all $x \in S$. Hence $(\delta \circ \eta)(\mu) = \mu = id_{NFI(S,f)}(\mu)$ for all $\mu \in NFI(S, f)$. So $\delta \circ \eta = id_{NFI(S,f)}$, which implies that η is injective. Let $\mu_1, \mu_2 \in NFI(S, f)$ be such that $\mu_1 \subseteq \mu_2$. Then

$$
\begin{aligned}
\alpha_{\mu_1}(x, y) &= (\mu_1(x) \wedge \mu_1(y)) \vee Id_S(x, y) \\
&\leq (\mu_2(x) \wedge \mu_2(y)) \vee Id_S(x, y) \\
&= \alpha_{\mu_2}(x, y)
\end{aligned}
$$

for all $(x, y) \in S \times S$. Thus $\alpha_{\mu_1} \subseteq \alpha_{\mu_2}$. This completes the proof. $\qquad\square$

Definition 4.2. *Let α be a fuzzy congruence on (S, f). A fuzzy congruence β on (S, f) is said to be α-invariant if $\alpha(x, y) = \alpha(a, b)$ implies that $\beta(x, y) = \beta(a, b)$ for all $(x, y), (a, b) \in S \times S$.*

Example 4.3. *Let (S, f) be the n-ary semigroup in Example 4.1. The fuzzy congruence α on (S, f) defined by*

$$
\alpha(x, y) = \begin{cases} 1 & \text{if } x = y, \\ 0.5 & \text{if } x \neq y \text{ and both } x, y \text{ are even or both } x, y \text{ are odd,} \\ 0 & \text{otherwise.} \end{cases}
$$

Define a fuzzy set μ of (S, f) by

$$
\mu(x) = \begin{cases} s & \text{if } x \text{ is an even number,} \\ t & \text{otherwise.} \end{cases}
$$

where $0 \leq t < s \leq 1$. Then μ is a fuzzy ideal of (S, f). Define β is a fuzzy relation of (S, f) by

$$
\beta(x, y) = (\mu(x) \wedge \mu(y)) \vee Id_S(x, y).
$$

By Theorem 4.2, then β is a fuzzy congruence on (S, f). It is easy to see that β is said to be α-invariant on (S, f).

Theorem 4.4. *Let μ be a fuzzy ideal of (S, f) and α_μ the fuzzy congruence on (S, f) induced by μ. Then there exists a one-to-one correspondence between the set $FC_{\alpha_\mu}(S, f)$ of all α_μ-invariant fuzzy congruences on (S, f) and the set $FC_{\alpha_\mu/\alpha_\mu}((S, f)/\alpha_\mu, F)$ of all α_μ/α_μ-invariant fuzzy congruences on $((S, f)/\alpha_\mu, F)$.*

Proof. Let β be an α_μ-invariant fuzzy congruence on (S, f) and $(x, y), (a, b) \in S \times S$. If $\alpha_\mu(x, y) = \alpha_\mu(a, b)$, then $\beta(x, y) = \beta(a, b)$. By Theorem 3.5, β/α_μ and α_μ/α_μ are fuzzy congruences on $((S, f)/\alpha_\mu, F)$, and when $(\alpha_\mu/\alpha_\mu)((\alpha_\mu)_x, (\alpha_\mu)_y) = (\alpha_\mu/\alpha_\mu)((\alpha_\mu)_a, (\alpha_\mu)_b)$ we

have $(\beta/\alpha_\mu)((\alpha_\mu)_x, (\alpha_\mu)_y) = (\beta/\alpha_\mu)((\alpha_\mu)_a, (\alpha_\mu)_b)$. This means β/α_μ is an α_μ/α_μ-invariant fuzzy congruence on $((S, f)/\alpha_\mu, F)$.

Now we define a map $\psi : FC_{\alpha_\mu}(S, f) \to FC_{\alpha_\mu/\alpha_\mu}((S, f)/\alpha_\mu, F)$ by $\psi(\beta) = \beta/\alpha_\mu$. Obviously, ψ is well-defined. Let β_1, β_2 be α_μ-invariant fuzzy congruences on (S, f) such that $\beta_1 \neq \beta_2$. Then there exists $(x, y) \in S \times S$ such that $\beta_1(x, y) \neq \beta_2(x, y)$. So $(\beta_1/\alpha_\mu)((\alpha_\mu)_x, (\alpha_\mu)_y) = \beta_1(x, y) \neq \beta_2(x, y) = (\beta_2/\alpha_\mu)((\alpha_\mu)_x, (\alpha_\mu)_y)$. Therefore ψ is injective.

Let β' be an α_μ/α_μ-invariant fuzzy congruence on $((S, f)/\alpha_\mu, F)$. We define a fuzzy relation β^* on (S, f) as follows:

$$\beta^*(x, y) = \beta'((\alpha_\mu)_x, (\alpha_\mu)_y).$$

It is easy to verify that β^* is a fuzzy equivalence relation.

Again, for any $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in S$, we have

$$
\begin{aligned}
&\beta^*(f(x_1^n), f(y_1^n)) \\
=\ &\beta'((\alpha_\mu)_{f(x_1^n)}, (\alpha_\mu)_{f(y_1^n)})) \\
=\ &\beta'(F((\alpha_\mu)_{x_1}, (\alpha_\mu)_{x_2}, \ldots, (\alpha_\mu)_{x_n}), F((\alpha_\mu)_{y_1}, (\alpha_\mu)_{y_2}, \ldots, (\alpha_\mu)_{y_n})) \\
\geq\ &\beta'((\alpha_\mu)_{x_1}, (\alpha_\mu)_{y_1}) \wedge \beta'((\alpha_\mu)_{x_2}, (\alpha_\mu)_{y_2}) \wedge \ldots \wedge \beta'((\alpha_\mu)_{x_n}, (\alpha_\mu)_{y_n}) \\
=\ &\beta^*(x_1, y_1) \wedge \beta^*(x_2, y_2) \wedge \ldots \wedge \beta^*(x_n, y_n).
\end{aligned}
$$

Therefore, β^* is a fuzzy congruence on (S, f).

Finally, $\alpha_\mu(x, y) = \alpha_\mu(a, b)$ implies that $(\alpha_\mu/\alpha_\mu)((\alpha_\mu)_x, (\alpha_\mu)_y) = (\alpha_\mu/\alpha_\mu)((\alpha_\mu)_a, (\alpha_\mu)_b)$. This implies that $\beta'((\alpha_\mu)_x, (\alpha_\mu)_y) = \beta'((\alpha_\mu)_a, (\alpha_\mu)_b)$, so we have $\beta^*(x, y) = \beta^*(a, b)$. Hence β^* is α_μ-invariant, and we have $(\beta^*/\alpha_\mu)((\alpha_\mu)_x, (\alpha_\mu)_y) = \beta^*(x, y) = \beta'((\alpha_\mu)_x, (\alpha_\mu)_y)$ for all $((\alpha_\mu)_x, (\alpha_\mu)_y) \in S/\alpha_\mu \times S/\alpha_\mu$. Thus, for any $\beta' \in FC_{\alpha_\mu/\alpha_\mu}((S, f)/\alpha_\mu, F)$, there always exists $\beta^* \in FC_{\alpha_\mu}(S, f)$ such that $\psi(\beta^*) = \beta^*/\alpha_\mu = \beta'$. Hence ψ is surjective. This completes the proof. $\square$

5. Conclusion

It is well known that congruences (fuzzy congruences) always play an important role in the study of algebraic structures. In this paper we introduced the concept of fuzzy congruences in n-ary semigroups, and investigated its related properties. Furthermore, we discussed the quotient n-ary semigroups in terms of fuzzy congruences, and established isomorphism theorems about n-ary semigroups. In particular, we proved that there exists a preserving inclusion injective mapping from the set of all normal fuzzy ideals of the special n-ary semigroups to the set of all fuzzy congruences.

In the future study of n-ary semigroups, we can apply fuzzy congruences of other n-ary algebras, such as, n-ary groups, and (m, n)-ary semirings, and so on. We hope this theory can be served as a foundation of some applied fields, such as decision making, data analysis, and forecasting.

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