HARMONIC DIRECTIONAL VARIATIONAL INEQUALITIES

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In this paper, we prove that the minimum of the sum of two locally Lipschitz continuous harmonic convex functions can be characterized by the harmonic variational inequalities. This motivated us to introduce and study some new classes of harmonic directional variational inequalities Several special cases such as harmonic complementarity problems and related optimization problems are discussed. The auxiliary principle technique is applied to suggest and analyze some inertial iterative schemes for harmonic directional variational inequalities. The convergence criteria of the proposed methods is discussed under some weak conditions. Our method of proof of the convergence criteria is simple as compared with other techniques. Results obtained in this paper continue to hold for various other classes of harmonic variational inequalities and related optimization problems. We have only considered the theoretical aspects of these harmonic directional inequalities. Numerical implementation of these methods required further research efforts.

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1. Introduction

Variational inequalities were introduced by Stamapcchia [42] and Fichera [10] in 1964 independently to study the elliptic equations subject to the obstacles and unilateral problems connected with Signorini criteria. They proved that the optimality conditions of the energy(virtual work, potential) functions on the convex set by the inequality. This inequality is called the variational inequality. Variational inequalities can be viewed as a novel and innovative generalization of the variational principles. Variational inequality describes a broad spectrum of very interesting developments involving a link among various fields of mathematics, physics, economics, regional and engineering sciences. We recall some simple concepts from convex analysis. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{C} \subseteq \mathcal{H}$ be a closed convex set. The inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|.\|$, respectively It is well known that the minimum $u \in \mathcal{C}$ of a differentiable convex functional F on a convex set $\mathcal{C}$ in a normed space can be characterized by an inequality of the type

$$\langle F'(u), v - u \rangle \geq 0, \quad \forall v \in \mathcal{C}, \tag{1}$$

where $F'(u)$ is the derivative of the differentiable convex functional F at $u \in \mathcal{C}$. However, it is worth mentioning that this theory allows many diversified applications. In fact, this theory provides us the most natural, direct, simple, unified and efficient framework for a general treatment of a wide class of unrelated linear and nonlinear problems. For the applications, motivations, generalizations, extensions, numerical methods, dynamical systems, sensitivity analysis, error bounds and related optimization problems, see [4, 9, 11, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 32, 33, 34, 35, 36, 37, 38, 39, 40, 43, 44, 45, 46, 47, 48] and the references therein.

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Noor and Noor [23] proved that the minimum of the differentiable harmonic convex function F on the harmonic convex set $\mathcal{C}_h \subseteq \mathcal{H}$ is equivalent to finding that the minimum $u \in \mathcal{C}_h \subseteq \mathcal{H}$ such that

$$\langle F'(u), \frac{uv}{u-v} \rangle \geq 0, \quad \forall v \in \mathcal{C}_h, \tag{2}$$

which is called the harmonic variational inequality.

In many cases, the problem (2) may not arise as the optimality conditions of the differentiable convex functions. To overcome this drawback, Noor and Noor [23] considered the problem of finding $u \in \mathcal{C}_h$ such that

$$\langle Tu, \frac{uv}{u-v} \rangle \geq 0, \quad \forall \in \mathcal{C}_h. \tag{3}$$

The inequality of the type (3) is known as the harmonic variational inequality, introduced and investigated by Nooe et al. [23]. For the application, formulation, numerical analysis and related optimization problems, see [1, 2, 13, 23, 24, 25, 26, 27, 28, 32] and the references therein.
Motivated and inspired by the ongoing research in this dynamic field, we introduce and study some new classes of harmonic directional variational inequalities involving two arbitrary bifunctions, which is the main aim of this paper. We remark that the projection and resolvent type method cannot be used to suggest the iterative methods for solving such type of problems. To overcome this drawback, we use the auxiliary principle technique, which is mainly due to Glowinski et al[11], to suggest iterative method for solving such type nonlinear variational inequalities. Noor [14, 18, 19] and Noor et al[20, 21, 22, 23, 24, 25, 26, 28, 32, 33, 34, 35, 36] have used this technique to develop some iterative schemes for solving various classes of variational inequalities and equilibrium problems. We point out that this technique does not involve any projection and resolvent of the operator and is flexible. In this paper, we show that the auxiliary principle technique can be applied to suggest and analyze some new classes of inertial iterative methods for solving harmonic directional variational inequalities. The inertial type methods was suggested by Polyak [41] to speed up the convergence of iterative methods. We also prove that the convergence of these new methods requires pseudomonotonicity, which is weaker conation than monotonicity. As special cases, one can obtain several known and new results for harmonic variational inequalities, variational inequalities and related optimization problems. Results obtained in this paper, represent an improvement and refinement of the known results for harmonic variational inequalities and their variant forms. It is perhaps part of the fascination of the subject that so many branches of pure and applied sciences are involved in the variational inequality theory.

2. **Preliminaries**

Let $\mathcal{H}$ be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|.\|$ respectively. Let $j : \mathcal{H} \longrightarrow R$ be a locally Lipschitz continuous function. First of all, we recall the following concepts and results from nonsmooth analysis, see [5, 6, 8].

Definition 2.1. [5] *Let j be locally Lipschitz continuous at a given point $x \in \mathcal{H}$ and v be any other vector in $\mathcal{H}$. The Clarke's generalized directional derivative of j at x in the direction v, denoted by $j^0(x; v)$, is defined as*

$$f^0(x; v) = \lim_{t \to 0^+} \sup_{h \to 0} \frac{f(x + h + tv) - f(x + h)}{t}.$$

The generalized gradient of j at x, denoted $\partial j(x)$, is defined to be subdifferential of the function $j^0(x;v)$ at 0. That is

$$\partial j(x) = \{w \in \mathcal{H} : \langle w, v\rangle \leq j^0(x;v), \quad \forall v \in \mathcal{H}.\}$$

Lemma 2.1. *Let j be a locally Lipschitz continuous at a given point $x \in \mathcal{H}$ with a constant L. Then*

(i). $\partial j(x)$ *is a none-empty compact subset of* $\mathcal{H}$ *and* $\|\xi\| \leq L$ *for each* $\xi \in \partial j(x)$.

(ii). *For every* $v \in \mathcal{H}$, $j^0(x;v) = max\{\langle \xi, c\rangle : \xi \in \partial j(x)\}$.

(iii). *The function* $v \longrightarrow j^0(x;v)$ *is finite, positively homogeneous, subadditive, convex and continuous.*

(iv). $j^0(x;-v) = (-j)^0(x;v)$.

(v). $j^0(x;v)$ *is upper semicontinuous as a function of* $(x;v)$.

(vi). $\forall x \in \mathcal{H}$, *there exists a constant* $\alpha > 0$ *such that*

$$|j^0(x;v)| \leq \alpha\|v\|, \quad \forall v \in \mathcal{H}.$$

If j is convex on $\mathcal{C}$ and locally Lipschitz continuous at $x \in \mathcal{C}$, then $\partial j(x)$ coincides with the subdifferential $j'(x)$ of j at x in the sense of convex analysis , and $j^0(x;v)$ coincides with the directional derivative $j'(x;v)$ for each $v \in H$, that is, $j^0(x;v) = \langle j'(x), v\rangle$.

For the sake of completeness and to convey the main ideas, we include the relevant details.

Definition 2.2. [5] *The set $\mathcal{C}_h$ is said to be a harmonic convex set, if*

$$\frac{uv}{v + \lambda(u-v)} \in \mathcal{C}_h, \quad \forall u, v \in \mathcal{C}_h, \quad \lambda \in [0,1].$$

For the applications of the harmonic means in circuit theory, risk analysis and related optimization programming, see [3, 5, 12, 27].

Definition 2.3. [5] *The function ϕ on the harmonic convex set $\mathcal{C}_h$ is said to be harmonic convex, if*

$$\phi\Big(\frac{uv}{v + \lambda(u-v)}\Big) \leq (1-\lambda)\phi(u) + \lambda\phi(v), \quad \forall u, v \in \mathcal{C}_h \quad \lambda \in [0,1].$$

The function ϕ is said to be harmonic concave function, if and only if, $-\phi$ is harmonic convex function. For the applications, motivation, integral inequalities and other aspects, see [3, 5, 12, 13, 20, 23, 24, 26, 27, 28, 29, 30, 31].

We prove that the minimum of the locally Lipschitz continuous harmonic on the harmonic convex set $\mathcal{C}_h$ can be characterized by the directional harmonic variational inequality.

Theorem 2.1. *Let ϕ be a locally Lipschitz harmonic convex function on the harmonic convex set $\mathcal{C}_h$. Then $u \in \mathcal{C}_h$ is a minimum of ϕ, if and only if, $u \in \mathcal{C}_h$ satisfies the inequality*

$$\phi'\Big(u; \frac{uv}{u-v}\Big) \geq 0, \quad \forall v \in \mathcal{C}_h. \tag{4}$$

The inequality of type (4) is called the harmonic directional variational inequality.

Proof. Let $u \in \mathcal{C}_h$ is a minimum of a locally Lipschitz continuous harmonic convex function ϕ. Then

$$\phi(u) \leq \phi(v), \quad \forall v \in \mathcal{C}_h. \tag{5}$$

Since $\mathcal{C}_h$ is a harmonic convex set, so $\forall u, v \in \mathcal{C}_h, \quad v_\lambda = \frac{uv}{u+\lambda(u-v)} \in \mathcal{C}_h$. Replacing v by $v_\lambda)$ in (5) and diving by λ and taking limit as $\lambda \to 0$, we have

$$0 \leq \frac{\phi(\frac{uv}{u+\lambda(u-v)}) - \phi(u)}{\lambda} = \phi'(u; \frac{uv}{u-v}).$$

the required result (4). Conversely, let the function ϕ be harmonic convex function on the harmonic convex set $\mathcal{C}_h$. Then

$$\frac{uv}{v+\lambda(u-v)} \leq (1-\lambda)\phi(u) + \lambda\phi(v) = \phi(u) + \lambda(\phi(v) - \phi(u)),$$

which implies that

$$\phi(v) - \phi(u) \quad \geq \quad \lim_{\lambda \to 0} \frac{\phi(\frac{uv}{v+\lambda(u-v)}) - \phi(u)}{\lambda} = \phi'(u; \frac{uv}{u-v}) \geq 0, \quad \text{using} \quad (4).$$

Consequently, it follows that

$$\phi(u) \leq \phi(v), \quad \forall v \in \mathcal{C}_h.$$

This shows that $u \in \mathcal{C}_h$ is the minimum of a locally Lipschitz continuous harmonic convex function. □

The inequality of the type (4) is called the harmonic directional variational inequality. We would like to mention that Theorem 2.1 implies that the locally Lipschitz continuous harmonic optimization programming problem can be studied via the harmonic directional variational inequality (4).
Using the ideas and techniques of Theorem 2.1, we can derive the following result.

Theorem 2.2. *Let ϕ be a locally Lipschitz continuous harmonic convex functions on the harmonic convex set $\mathcal{C}_h$. Then*

$$(i). \qquad \phi(v) - \phi(u) \geq \phi'(u; \frac{uv}{u-v}), \quad \forall u, v \in \mathcal{C}_h.$$

$$(ii). \qquad \phi'(u; \frac{uv}{v-u}) + \phi'(v; \frac{uv}{v-u}), \leq 0, \quad \forall u, v \in \mathcal{C}_h.$$

Motivated by Theorem 2.1 and Theorem 2.2, we introduce some new concepts.

Definition 2.4. *A bifunction $B(.,.)$ is said to be a harmonic monotone bifunction , if and only if,*

$$B(u; \frac{uv}{v-u}) + B(v; \frac{uv}{v-u}), \leq 0 \quad \forall u, v \in \mathrm{H}.$$

Definition 2.5. *A bifunction $B(.,.)$ is said to a harmonic pseudomonotone bifunction with respect to the bifunction $W(.,.)$, if*

$$B(v; \frac{uv}{v-u}) + W(v; \frac{uv}{v-u}) \leq 0, \quad \forall v \in \mathcal{H}.$$

A harmonic monotone bifunction is a harmonic pseudomonotone bifunction, but the converse is not true.
Consider the energy (virtual) functional

$$I[v] = F(v) - \phi(v), \tag{6}$$

where $F(v)$ and $\phi(v)$ are two harmonic convex functions.
We now consider the optimality conditions of the energy function $I[v]$ defined by (6) under suitable conditions.

Theorem 2.3. *Let F and $\phi(v)$ be locally Lipschitz continuous harmonic convex functions on the convex set $\mathcal{C}_h$. If $u \in \mathcal{C}_h$ is the minimum of the functional $I[v]$ defined by (6), then*

$$F'(u; \frac{uv}{u-v}) - \phi'(u; \frac{uv}{u-v}) \geq 0, \quad \forall v, u \in \mathcal{C}_h. \tag{7}$$

Proof. Let $u \in \mathcal{C}_h$ be a minimum of the functional $I[v]$. Then

$$I[u] \leq I[v], \quad \forall v \in K.$$

which implies that

$$F(u) - \phi(u) \leq F(v) - \phi(v), \quad \forall v \in \mathcal{C}_h. \tag{8}$$

Since $\mathcal{C}_h$ is a convex set, so, $\forall u, v \in \mathcal{C}_h, \quad \lambda \in [0,1]$, $v_t = \frac{uv}{(1-\lambda)v+\lambda u} \in \mathcal{C}_h$.
Taking $v = v_t$ in (8), we have

$$F(u) - \phi(u) \leq F(v_t) - \phi(v_t), \quad \forall v \in \mathcal{C}_h. \tag{9}$$

This implies that

$$0 \leq F(\frac{uv}{(1-\lambda)v+\lambda u}) - F(u) - \phi(\frac{uv}{(1-\lambda)v+\lambda u}) - \phi(u), \quad \forall v \in \mathcal{C}_h. \tag{10}$$

Dividing the above inequality by λ and taking limit as $\lambda \to 0$, we have

$$\begin{aligned} 0 &\leq \lim_{\lambda \to 0} \frac{F(\frac{uv}{(1-\lambda)v+\lambda u}) - F(u)}{\lambda} - \lim_{\lambda \to 0} \frac{\phi(\frac{uv}{(1-\lambda)v+\lambda u}) - \phi(u)}{\lambda} \\ &= F'(u; \frac{uv}{u-v}) - \phi'(u; \frac{uv}{u-v}), \end{aligned}$$

which is the required (7).
Since F is a locally Lipschitz continuous harmonic convex function, so

$$F(\frac{uv}{v+\lambda(u-v)}) \quad \leq \quad F(u) + \lambda(F(v) - F(u)), \quad \forall u, v \in \mathcal{C}_h$$

from which, we have

$$F(v) - F(u) \quad \geq \quad \lim_{\lambda \to 0} \frac{F(\frac{uv}{(1-\lambda)v+\lambda u}) - F(u)}{\lambda} = F'(u; \frac{uv}{u-v}) \tag{11}$$

In a similar way,

$$\phi(v) - \phi(u) \quad \geq \quad \lim_{\lambda \to 0} \frac{\phi(\frac{uv}{(1-\lambda)v+\lambda u}) - \phi(u)}{\lambda} = \phi'(u; \frac{uv}{u-v}). \tag{12}$$

From (12) and (11), we have

$$F(v) + \phi(v) - (F(u) + \phi(u)) \geq = \langle F'(u; \frac{uv}{u-v}\rangle - \langle \phi'(u; \frac{uv}{u-v}\rangle \geq 0.$$

Consequently, it follows that $u \in \mathcal{C}_h$ such that

$$F(u) - \phi(u) \leq F(v) - \phi(v), \quad \forall v \in \mathcal{C}_h,$$

which shows that $u \in \mathcal{C}_h$ is the minimum of the function $I[v]$ defined by (6). □

Remark 2.1. *The inequality of the type (7) is called the mildly nonlinear harmonic variational inequality.*
Essentially using the technique of Theorem 2.3, one can prove the following result.

Theorem 2.4. *Let F be a differentiable harmonic convex functions and $\phi(v)$ be a locally Lipschitz continuous harmonic convex functions on the convex set $\mathcal{C}_h$. If $u \in \mathcal{C}_h$ is the minimum of the functional $I[v]$ defined by (6), then*

$$\langle F'(u), \frac{uv}{u-v}\rangle - \phi'(u; \frac{uv}{u-v}) \geq 0, \quad \forall v, u \in \mathcal{C}_h, \tag{13}$$

which is called the harmonic hemivariational inequality.
In many applications, the inequalities of the type (7) may not arise as the minimum of the sum of the two locally Lipschitz continous harmonic convex functions. These facts motivated us to consider more general harmonic variational inequality, which contains the inequalities (7) and (13) as a special case.

For given nonlinear continuous bifunction $B(.,.), W(.,.) : H \times H \longrightarrow H$, we consider the problem of finding $u \in \mathcal{C}_h$ such that

$$B(u; \frac{uv}{u-v}) + W(u; \frac{uv}{u-v}) \geq 0, \quad \forall v \in \mathcal{C}_h, \tag{14}$$

which is called the harmonic directional variational inequality.

We now discuss some new and known classes of harmonic directional variational inequalities and related optimization problems.

Special Cases

(I). if $B(u; \frac{uv}{u-v}) = \langle Tu, \frac{uv}{u-v}\rangle\rangle$ and $W(u; \frac{uv}{u-v}) = \langle A(u); \frac{uv}{u-v}\rangle$, then the problem (14) reduces to finding $u \in \mathcal{C}_h$ such that

$$\langle Tu, \frac{uv}{u-v}\rangle + \langle A(u); \frac{uv}{u-v}\rangle \geq 0, \quad \forall v \in \mathcal{C}_h, \tag{15}$$

which is called the nonlinear harmonic hemivariational inequality.

(II). If $\langle A(u); \frac{uv}{u-v}\rangle = \phi'(u; \frac{uv}{u-v})$, where $\phi'(u)$ denotes derivative of the harmonic convex function $\phi(u)$ in the direction $\frac{uv}{u-v}$, then problem (14) reduces to finding $u \in \mathcal{C}_h$, such that

$$\langle Tu; \frac{uv}{u-v}\rangle + \phi'(u; \frac{uv}{u-v}) \quad 0, \quad \forall v \in \mathcal{C}_h, \tag{16}$$

which is also called the harmonic directional variational inequality.
(III). For $\langle A(u), \frac{uv}{v-u}\rangle = J^0(u; \frac{uv}{u-v})$, the problem (14) reduces to finding $u \in \mathcal{C}$ such that

$$B(u; \frac{uv}{u-v}) + J^0(u; \frac{uv}{u-v}) \geq 0, \quad \forall v \in \mathcal{C}_h, \tag{17}$$

which is known as harmonic directional hemivariational inequality. One can show discuss the applications of hemivariational inequalities in superpotential analysis of elasticity and structural analysis following the techniques discussed in [38, 39, 40].
(IV). If $\phi(.)$ is a differentiable harmonic convex function, then
$W(u, \frac{uv}{u-v}) = \phi'(u; \frac{uv}{u-v}) = \langle \phi'(u), \frac{uv}{u-v}\rangle$, and consequently problem (14) is equivalent to finding $u \in \mathcal{C}_h$ such that

$$B(u; \frac{uv}{u-v}) + \langle \phi'(u), \frac{uv}{u-v}\rangle \geq 0, \quad \forall v \in \mathcal{C}_h, \tag{18}$$

which is called the harmonic directional variational inequality.
(V). If $\langle A(u); \frac{uv}{u-v}) = -\langle Au, \frac{uv}{u-v}\rangle$ then the problem (15) reduces to finding $u \in \mathcal{C}_h$ such that

$$\langle Tu, \frac{uv}{u-v}\rangle - \langle A(u), \frac{uv}{u-v}\rangle \geq 0, \quad \forall v \in \mathcal{C}_h, \tag{19}$$

which is called the mildly nonlinear harmonic variational inequality involving the difference of two monotone operators

(VI). If $\mathcal{C}_h^\star = \{u \in \mathcal{H} : \langle u, \frac{uv}{u-v}\rangle \geq 0, \quad \forall v \in \mathcal{C}_h\}$ is a polar harmonic convex cone of the harmonic convex $\mathcal{C}_h$, then problem (15) is equivalent to fining $u \in \mathcal{H}$, such that

$$\frac{uv}{u-v} \in \mathcal{C}_h, \quad Tu + A(u) \in \mathcal{C}_h^\star, \quad \langle Tu + A(u), \frac{uv}{u-v}\rangle = 0, \tag{20}$$

is called the general harmonic complementarity problem. For the applications, numerical methods and other aspects of complementarity problems, see [7, 15, 16, 17, 18, 22, 32, 37] and the references therein.

(VII). If $\mathcal{C}_h = \mathcal{H}$, then problem (15 is equivalent to fining $u \in \mathcal{H}$, such that

$$\langle Tu + A(u), \frac{uv}{u-v}\rangle = 0, \quad \forall v \in \mathcal{H}, \tag{21}$$

which is called the weak formulation of the mildly nonlinear harmonic boundary value problem.

(VIII). For $Au = A|u|$, the problem (21) reduces to finding $u \in \mathcal{H}$ such that

$$\langle Tu + A|u|, \frac{uv}{u-v}\rangle = 0, \quad \forall v \in H, \tag{22}$$

which is called the system of absolute value harmonic equations.

(IX). If $\langle A(u), \frac{uv}{u-v}\rangle = 0$, then problem (14) reduces to finding $u \in \mathcal{C}_h$ such that

$$B(u; \frac{uv}{v-u}) \geq 0, \quad \forall v \in \mathcal{C}_h, \tag{23}$$

which is called the harmonic directional variational inequality.

Remark 2.2. *For different and suitable choice of the bifunctions, operators and the spaces, one can obtained several new and known classes of the harmonic variational inequalities and related optimization problems. For the recent applications, motivation, numerical methods, sensitivity analysis and local uniqueness of solutions of harmonic variational inequalities and related optimization problems, see* [1, 2, 5, 13, 19, 22, 23, 24, 26, 28, 32] *and the references therein. This show that the problem (14) is quite general and unified one. Due to the structure and nonlinearity involved, one has to consider its own. It is an open problem to develop unified numerical implementation numerical methods for solving the harmonic variational inequalities.*

3. **Iterative Methods and Convergence Analysis**

There are several techniques such as projection, resolvent, descent methods for solving the variational inequalities and their variant forms. None of these techniques can not be applied for suggesting the iterative methods for solving the harmonic variational inequalities. The inertial type iterative methods were suggested by Polyak [41] to speed the convergence analysis of the iterative methods. Alvarez [4] analyzed the weak convergence of the relaxed and inertial hybrid projection-proximal point algorithm for maximal monotone operators in Hilbert space. For the applications of the inertial type methods for solving variational inequalities, variational inclusions and their variant forms, see [1, 2, 4, 18, 20, 21, 24, 25, 26, 27, 32, 33] and the references therein. To overcome these drawbacks, we apply the auxiliary principle technique, which is mainly due to Glowinski et al [11] as developed in [1, 2, 14, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 32, 33], to suggest and analyze some inertial iterative methods for solving harmonic directional variational inequalities (14).

For a given $u \in \mathcal{C}_h$ satisfying (14), consider the problem of finding $w \in \mathcal{C}_h$ such that

$$\rho B(w+\eta(u-w)); \frac{vw}{w-v}) \quad + \quad \langle w-u, v-w\rangle$$
$$+ \quad W(w+\eta(u-w); \frac{vw}{v-w}) \geq 0, \forall v \in \mathcal{C}_h, \qquad (24)$$

where $\rho > 0, \eta \in [0,1]$ are constants.
Inequality of type (24) is called the auxiliary harmonic directional variational inequality.
If $w = u$, then w is a solution of (14). This simple observation enables us to suggest the following iterative method for solving (14).

Algorithm 3.1. *For a given $u_0 \in \mathcal{C}_h$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho B(u_{n+1}+\eta(u_n-u_{n+1})); \frac{vu_{n+1}}{v-u_{n+1}}) + \langle u_{n+1}-u_n, v-u_{n+1}\rangle$$
$$\geq -\rho W(u_{n+1}+\eta(u_n-u_{n+1})); \frac{vu_{n+1}}{v-u_{n+1}}\rangle, \quad \forall v \in \mathcal{C}_h. \qquad (25)$$

Algorithm 3.1 is called the hybrid proximal point algorithm for solving harmonic hemivariational inequalities(14).

Special Cases

We now consider some cases of Algorithm 3.1.

(I). For $\eta = 0$, Algorithm 3.1 reduces to:

Algorithm 3.2. *For a given $u_0 \in \mathcal{C}_h$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho B(u_{n+1}; \frac{vu_{n+1}}{v-u_{n+1}}) \quad + \quad \langle u_{n+1}-u_n, v-u_{n+1}\rangle$$
$$\geq \quad -\rho W(u_{n+1}; \frac{vu_{n+1}}{v-u_{n+1}}\rangle, \forall v \in \mathcal{C}_h. \qquad (26)$$

(II). If $\eta = 1$, then Algorithm 3.1 reduces to:

Algorithm 3.3. *For a given $u_0 \in \mathcal{C}_h$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho B(u_n; \frac{u_n u_{n+1}}{v-u_{n+1}}) + \langle u_{n+1}-u_n, v-u_{n+1}\rangle \geq -\rho W(u_n; \frac{vu_{n+1}}{v-u_{n+1}}\rangle, \quad \forall v \in \mathcal{C}_h.$$

(III). If $\eta = \frac{1}{2}$, then Algorithm 3.1 collapses to:

Algorithm 3.4. *For a given $u_0 \in \mathcal{C}_h$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho B(\frac{u_{n+1}+u_n}{2}); \frac{vu_{n+1}}{v-u_{n+1}}) \quad + \quad \langle u_{n+1}-u_n, v-u_{n+1}$$
$$+ \quad \rho W(\frac{u_{n+1}+u_n}{2}, \frac{vu_{n+1}}{v-u_{n+1}}) \geq 0, \quad \forall v \in \mathcal{C}_h,$$

which is called the mid-point proximal method for solving the problem (14).

If $W(.;.) = 0$, then Algorithm 3.1 reduces to:

Algorithm 3.5. *For a given $u_0 \in \mathcal{C}_h$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho B(u_{n+1}+\eta(u_n-u_{n+1})); \frac{vu_{n+1}}{v-u_{n+1}}) + \langle u_{n+1}-u_n, v-u_{n+1}\rangle \geq 0, \quad \forall v \in \mathcal{C}_h.$$

for solving harmonic directional variational inequality.

Lemma 3.1. $\forall u, v \in \mathcal{H}$,

$$2\langle u, v\rangle = \|u+v\|^2 - \|u\|^2 - \|v\|^2. \quad (27)$$

We now consider the convergence criteria of Algorithm 3.2. The analysis is in the spirit of Noor et al.[27, 32]. We include the proof for the sake of completeness and to convey an idea of the technique involved.

Theorem 3.1. *Let $u \in \mathcal{C}_h$ be a solution of (14) and let u_{n+1} be the approximate solution obtained from Algorithm 3.2. Let the bifunction $B(.,.)$ be pseudomonotone with respect to the bifunction$W(.,.)$. Then*

$$\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2 - \|u_{n+1} - u_n\|^2. \quad (28)$$

Proof. Let $u \in \mathcal{C}_h$ be a solution of (14). Then

$$-B(v; \frac{uv}{v-u}) - W(v; \frac{uv}{v-u}) \geq 0, \quad \forall v \in \mathcal{C}_h. \quad (29)$$

since the bifunction $B(.,.)$ pseudomonotone with respect to the bifunction$W(.,.)$.
Now taking $v = u_{n+1}$ in (29), we have

$$-B(u_{n+1}; \frac{uu_{n+1}}{u_{n+1}-u}) - W(u_{n+1}; \frac{uu_{n+1}}{u_{n+1}-u}) \geq 0. \quad (30)$$

Taking $v = u$ in (26), we get

$$\langle \rho B(u_{n+1}; \frac{uu_{n+1}}{u-u_{n+1}}) + \langle u_{n+1} - u_n, u - u_{n+1}\rangle + W(u_{n+1}; \frac{uu_{n+1}}{u-u_{n+1}}),$$

which can be written as

$$\langle u_{n+1} - u_n, u - u_{n+1}\rangle \geq -\rho B(u_{n+1}; \frac{uu_{n+1}}{u-u_{n+1}}) - \rho W(u_{n+1}; \frac{uu_{n+1}}{u-u_{n+1}}) \geq 0, \quad (31)$$

where we have used (30).
Setting $u = u - u_{n+1}$ and $v = u_{n+1} - u_n$ in (27), we obtain

$$2\langle u_{n+1} - u_n, u - u_{n+1}\rangle = \|u - u_n\|^2 - \|u - u_{n+1}\|^2 - \|u_{n+1} - u_n\|^2. \quad (32)$$

Combining (31) and (32), we have

$$\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2 - \|u_{n+1} - u_n\|^2,$$

the required result (28). □

Theorem 3.2. *Let H be a finite dimensional space and all the assumptions of Theorem 3.1 hold. Then the sequence $\{u_n\}_0^\infty$ given by Algorithm 3.2 converges to a solution u of (14).*

Proof. Let $u \in K$ be a solution of (14). From (28), it follows that the sequence $\{||u - u_n||\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Furthermore, we have

$$\sum_{n=0}^{\infty} \|u_{n+1} - u_n\|^2 \leq \|u_0 - u\|^2,$$

which implies that

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \quad (33)$$

Let $\hat{u}$ be the limit point of $\{u_n\}_0^\infty$; whose subsequence $\{u_{n_j}\}_1^\infty$ of $\{u_n\}_0^\infty$ converges to $\hat{u} \in H$. Replacing w_n by u_{n_j} in (26), taking the limit $n_j \longrightarrow \infty$ and using (33), we have

$$B(\hat{u}; \frac{\hat{u}v}{v-\hat{u}}) + W(\hat{u}; \frac{\hat{u}v}{v-\hat{u}}) \geq 0, \quad \forall v \in \mathcal{C}_h,$$

which implies that $\hat{u}$ solves the harmonic directional variational inequality (14) and

$$\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2.$$

Thus, it follows from the above inequality that $\{u_n\}_1^\infty$ has exactly one limit point $\hat{u}$ and

$$\lim_{n\to\infty} (u_n) = \hat{u}.$$

the required result. □

We again consider the auxiliary principle technique to suggest some hybrid inertial proximal point methods for solving the problem (14).

For a given $u \in \mathcal{C}_h$ satisfying (14), consider the problem of finding $w \in \mathcal{C}_h$ such that

$$\begin{aligned}\rho B(w + \eta(u-w)), \frac{uw}{u-w}) \quad &+ \quad \langle M(w) - M(u) + \alpha(u-u), v-w\rangle \\ &+W((w+\eta(w-u))), \frac{uw}{u-w}) \geq 0, \quad \forall v \in \mathcal{C}_h, \quad (34)\end{aligned}$$

where $\rho > 0, \alpha, \eta, \in [0,1]$ are constants and M is an arbitrary operator.

Clearly, for $w = u, \quad w$ is a solution of (14). This fact motivated us to to suggest the following inertial iterative method for solving (14).

Algorithm 3.6. *For given $u_0, u_1 \in \mathcal{C}_h$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}&\langle \rho B(u_{n+1} + \eta(u_n - u_{n+1})); \frac{vu_{n+1}}{v-u_{n+1}}) + \langle M(u_{n+1}) - M(u_n) + \alpha(u_n - u_{n-1}, v - u_{n+1}\rangle \\ &\geq -W((u_{n+1} + \eta(u_n - u_{n+1})); \frac{vu_{n+1}}{v-u_{n+1}}), \quad \forall v \in \mathcal{C}_h.\end{aligned}$$

which is known as the inertial iterative method.

Note that, for $\alpha = 0 \quad M = I$, the identity operator, Algorithm 3.6 is exactly the Algorithm 3.1. Using essentially the technique of Theorem 3.1, Alshejari et al. [1, 2] and Noor et al.[27], one can study the convergence analysis of Algorithm 3.6.

If $\eta = \frac{1}{2}$, the Algorithm 3.6 reduces to:

Algorithm 3.7. *For given $u_0, u_1 \in \mathcal{C}_h$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}&\langle \rho B(\frac{u_{n+1}+u_n}{2}); \frac{vu_{n+1}}{v-u_{n+1}}) + \langle M(u_{n+1}) - M(u_n) + \alpha(u_n - u_{n-1}, v - u_{n+1}\rangle \\ &\geq -W(\frac{u_{n+1}+u_n}{2}); \frac{vu_{n+1}}{v-u_{n+1}}), \quad \forall v \in \mathcal{C}_h.\end{aligned}$$

which is known as the inertial mid-point iterative method.

Remark 3.1. *For different and appropriate values of the parameters η, α, the bifunctions $B(.,.), W(.,.)$, Operator M and spaces, one can obtain a wide class of inertial type iterative methods for solving the harmonic variational inequalities and related optimization problems.*

Conclusion: Some new classes of harmonic directional variational inequalities are introduced and discussed. Several important problems such as harmonic complementarity problems, system of harmonic absolute value problems and related problems can be obtained as special cases. We have applied the auxiliary principle technique to suggest several inertial type methods for solving harmonic directional variational inequalities with suitable modifications. This technique does not involve the projection and the resolvent of the operator. The convergence analysis of these new methods are analyzed under weaker conditions. We have only considered the theoretical aspects of the hybrid inertial iterative methods. It is an interesting problem to develop implementable numerical methods for solving the harmonic variational inequalities. One can explore the applications of the harmonic variational inequalities in different branches of pure and applied sciences.

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