BI-TANGENT QUATERNION KAEHLER MANIFOLDS

Deniz POYRAZ[1], Bayram ŞAHİN[2]

Complex structures and tangent structures are well known. Almost semi-quaternion manifolds formed by the combination of these two structures have been also studied. In this paper, firstly, the existence of a structure similar to the quaternion Kaehler manifold for an almost semi-quaternion manifold is investigated. For this purpose, the necessary conditions are obtained for the covariant derivative of each cross-section to remain in the vector bundle formed by these cross-sections. After this stage, the interactions of these cross-sections and the curvature tensor field are investigated. In this direction, new relations were found. Finally, the flatness of the manifolds with this type of structure in case of constant curvature is examined.

Keywords: Tangent structure, Quaternion Kaehler structure, Almost bi-tangent quaternion structure, Bi-tangent quaternion Kaehler manifold

MSC2020: 53C15

1. Introduction

In differential geometry, manifold endowed with certain structures have been active field since almost complex manifolds were introduced in 1930's. Nowadays, there are many structures inspiring from complex manifolds and contact manifolds see: [4, 10, 15, 23, 28].

Since Ishihara[16] defined and studied quaternion Kaehler manifolds, manifolds containing more than one linear endomorphism have been studied by many authors. Among these, hypercomplex structures, polysymplectic structures [1, 2], 3-contact structures [17] and Para-quaternionic structures [5] can be mentioned. Various versions of product structures and complex structures were also studied by Cruceanu in [6, 7, 8].

On the other hand, almost tangent structures or almost subtangent structures have interesting properties. Manifolds containing these structures have been studied by many authors [3], [21] and [22]. The almost semiquaternion structures is a degenerate, hypercomplex structure defined by the semiquaternion algebra. It has appeared in F. Tricerri's clasification of the structures of type $F_1F_2 + F_2F_1 = \alpha I$, $F_1^2 = F_2^2 = -I$(I identity, $\alpha \in \mathbb{R}$), on a differentiable manifold, when $\alpha^2 = 4$. In Tricerri's paper [24] this is irregular case. All these considerations hold good for $\alpha \neq 4$. Munteanu [18, 19, 20] examined manifolds with two tangent structures and one complex structure, calling them semi-quaternions, and showed the existence of the Riemannian metric on such manifolds. However, if the covariant derivative of almost complex structures on an almost quaternion manifold remain in the subbundle determined by these almost complex structures, such almost quaternion manifolds are called quaternion Kaehler manifolds and such manifolds are the most studied quaternion manifold types in the literature. However, Munteanu did not examine this situation.

In this paper, we define the bi-tangent quaternion Kaehler manifold with the help of almost semiquaternion structure which we will call almost bi-tangent quaternion structure.

[1], Ege University, Faculty of Science, Department of Mathematics, 35100 Izmir, TURKEY, e-mail: deniz.poyraz@ege.edu.tr

[3] Ege University, Faculty of Science, Department of Mathematics, 35100 Izmir, TURKEY, e-mail: bayram.sahin@ege.edu.tr

We give an example and obtain the covariant derivatives of each cross-section to be in the vector bundle defined by these cross-sections. Then we obtain certain relations for curvature tensor fields with linear endomorphisms. Finally, we show that a manifold endowed with such structure is flat when it is a manifold with constant sectional curvature.

2. Preliminaries

Let $(M(c), g)$ be a complete simply connected Riemannian manifold of constant curvature c. Then the curvature tensor field R is given by

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}$$

for any vector fields X and Y on M.

If M is a $4m$-dimensional manifold with the Riemannian metric g, then M is said to be a quaternion Kaehler manifold [29] if there exists a 3-dimensional vector bundle V of type $(1,1)$ with local basis of almost Hermitian structures J_1, J_2, J_3 satisfying

$$J_1 \circ J_2 = -J_2 \circ J_1 = J_3$$

and

$$\nabla_X J_k = \sum_{l=1}^{3} Q_{kl}(X) J_l, \; k = 1,2,3,$$

for all vector fields X tangent to M, where ∇ is the Levi-Civita connection and Q_{kl} are certain 1-forms locally defined on M such that $Q_{kl} + Q_{lk} = 0$. The second condition can be given by

$$\begin{aligned}
\nabla_X J_1 &= r(X)J_2 - q(X)J_3, \\
\nabla_X J_2 &= -r(X)J_1 + p(X)J_3, \\
\nabla_X J_3 &= q(X)J_1 - p(X)J_2,
\end{aligned}$$

for all vector fields X tangent to M, where p, q and r are certain 1-forms locally defined on M.

3. Almost bi-tangent quaternion manifolds

Inspring from quaternion Kaehler manifolds, there are many new manifolds similar to the quaternion Kaehler manifolds such as paraquaternionic manifolds, almost 3-contact structure, bi-product manifolds, etc ([5, 9, 11, 12, 13, 14, 17, 25, 26, 27]).

We recall the definition of semiquaternion manifolds, but we will give a new name for such manifolds because the notion of semiquaternion sounds semi-Riemannian geometry however metric almost semiquaternion manifolds are Riemannian manifolds.

Let M be a differantiable manifold with $dimM = 4m$ and J, T_1, T_2 are $(1,1)$ tensor fields on M. If the following conditions are satisfied

$$J^2 = -I, \qquad T_1^2 = T_2^2 = 0, \qquad rankT_1 = 2m,$$

$$JT_1 = -T_1 J = T_2, \; JT_2 = -T_2 J = -T_1, \; T_1T_2 = T_2T_1 = 0, \tag{1}$$

then M is called almost bi-tangent quaternion (almost semiqternion) [19] structure, shortly SQ-structure on M.

Considering the vertical distribution $\mathcal{V} = KerT_1$ and $\mathcal{H}$ a fixed distribution, called horizontal, complementary to $\mathcal{V}$ in $T(M)$ $(i.e. T_x(M) = \mathcal{H}_x \otimes \mathcal{V}_x, \; \forall(x) \in M)$ which is preserved by J and denoting by h and v the corresponding projectors, from [20] and [28] it is known the existence of a $(1,1)$ tensor field T_1^*, $(T_1^*)^2 = 0$, called generalized inverse of T_1, uniquelly defined (for chosen distribution $\mathcal{H}$) by the conditions:

$$T_1^* T_1 = h, \qquad vT_1^* = 0, \qquad T_1^* h = 0.$$

The triad $SQ^* = (J, T_1^*, T_2^* = J \cdot T_1^*)$ defines also on M a bi-tangent quaternion structure, called adjoint to SQ.

If g is a Rimannian metric on M, then we can define uniquely $\mathcal{H}$ as the distribution orthogonal to $\mathcal{V}$ by g, i.e.:

$$g(vX, hY) = 0, \qquad X, Y \in \chi(M). \tag{2}$$

Definition 3.1. [19] *Let M be an almost bi-tangent quaternion manifold. Then we define a Riemannian metric on M by*

$$g(JX, JY) = g(X, Y), \tag{3}$$

$$X, Y \in \chi(M)$$

$$g(T_1 X, T_1 Y) = g(hX, hY), \tag{4}$$

where h is the projector on $\mathcal{H}$, orthogonal to $\mathcal{V}$, with respect to g. In this case, (M, J, T_1, T_2, g) is called metric almost bi-tangent quaternion manifold.

Proposition 3.1. [19] *On every paracompact manifold M, endowed with a QS-structure, there exist a metric almost bi-tangent quaternion structure.*

Proof. Let be the SQ-structure given by (1) . Choosing a Riemannian metric f on M, let $\mathcal{H}$ be the orthocomplement of $\mathcal{V}$ with respect to the metric f. Let be the Riemannian metric:

$$\tilde{g}(X, Y) = f(X, Y) + f(T_1 X, T_1 Y) + f(T_1^* X, T_1^* Y).$$

Then the metric

$$g(X, Y) = \tilde{g}(X, Y) + \tilde{g}(JX, JY)$$

determines on M a metric almost bi-tangent quaternion structure. □

Example 3.1. *Let $\mathbb{R}^{4m}$, $(m \geq 1)$ be a Euclidean space. Then the canonical structures J, T_1, T_2 of $\mathbb{R}^{4m}$ and the Riemannian metric g are given by*

$$J = \begin{pmatrix} 0 & 0 & -I & 0 \\ 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \end{pmatrix}, T_1 = \begin{pmatrix} 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \end{pmatrix}, T_2 = \begin{pmatrix} 0 & 0 & 0 & -I \\ 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{aligned} & g\big((x_1, y_1, z_1, w_1, .., x_m, y_m, z_m, w_m), (x_1', y_1', z_1', w_1', .., x_m', y_m', z_m', w_m')\big) \\ & = x_1 x_1' + y_1 y_1' + z_1 z_1' + w_1 w_1' + ... + x_m x_m' + y_m y_m' + z_m z_m' + w_m w_m' \end{aligned}$$

for all $X = (x, y, z, w), Y = (x', y', z', w') \in \mathbb{R}^{4m}$, where I is the $m \times m$ identity matrix. Now we can define

$$\begin{aligned} \mathcal{V} = KerT_1 &= \{(x, y, z, w) : T_1(x, y, z, w) = 0\} \\ &= \{(x, 0, z, 0) : x, z \in \mathbb{R}\}, \\ \mathcal{H} &= \{(0, y, 0, w) : y, w \in \mathbb{R}\}. \end{aligned}$$

It is easy to see that Definition 3.1 is satisfied.

4. **Bi-tangent quaternion Kaehler manifolds**

In this section we introduce bi-tangent quaternion Kaehler manifolds and investigate curvature relations. We also show that a bi-tangent quaternion Kaehler manifold M with $dimM = 4m$ is flat when M is of constant curvature.

Definition 4.1. *Let M be a almost bi-tangent quaternion manifold with a 3-dimensional vector bundle V consisting of $\{J, T_1, T_2\}$. If Φ is a cross-section (local or global) of the bundle V, then $\nabla_X \Phi$ also a cross-section of V, X being an arbitrary vector field in M. In this case, M will be called bi-tangent quaternion Kaehler manifold.*

We now obtain the meaning of the above definition.

Theorem 4.1. *Let M be a bi-tangent quaternion Kaehler manifold, then we have*

$$\begin{aligned} \nabla_X J &= p(X)T_1 + q(X)T_2, && (5)\\ \nabla_X T_1 &= r(X)T_1 + s(X)T_2, && (6)\\ \nabla_X T_2 &= -s(X)T_1 + r(X)T_2, && (7) \end{aligned}$$

for $X \in \chi(M)$, where $p(X)$, $q(X)$, $r(X)$ and $s(X)$ differentiable functions on M.

Proof. From Definition 4.1, we can write

$$\begin{aligned} \nabla_X J &= a_{11}J + a_{12}T_1 + a_{13}T_2 && (8)\\ \nabla_X T_1 &= a_{21}J + a_{22}T_1 + a_{23}T_2 && (9)\\ \nabla_X T_2 &= a_{31}J + a_{32}T_1 + a_{33}T_2. && (10) \end{aligned}$$

for $X \in \chi(M)$. From (1) we have

$$\begin{aligned} \nabla_X T_1 &= \nabla_X T_2 J\\ &= (\nabla_X T_2)J + T_2(\nabla_X J). \end{aligned}$$

Using (8) and (10) we obtain

$$\begin{aligned} \nabla_X T_1 &= (a_{31}J + a_{32}T_1 + a_{33}T_2)J + T_2(a_{11}J + a_{12}T_1 + a_{13}T_2)\\ &= a_{31}JJ + a_{32}T_1J + a_{33}T_2J + a_{11}T_2J + a_{12}T_2T_1 + a_{13}T_2T_2\\ &= -a_{31} + (a_{33} + a_{11})T_1 - a_{32}T_2. \end{aligned}$$

Similarly from (1) we have

$$\begin{aligned} \nabla_X T_2 &= \nabla_X JT_1\\ &= (\nabla_X J)T_1 + J(\nabla_X T_1) \end{aligned}$$

and using (8) and (9) we get

$$\begin{aligned} \nabla_X T_2 &= (a_{11}J + a_{12}T_1 + a_{13}T_2)T_1 + J(a_{21}J + a_{22}T_1 + a_{23}T_2)\\ &= a_{11}JT_1 + a_{12}T_1T_1 + a_{13}T_2T_1 + a_{21}JJ + a_{22}JT_1 + a_{23}JT_2\\ &= -a_{21} - a_{23}T_1 + (a_{11} + a_{22})T_2. \end{aligned}$$

Therefore we have $a_{21} = a_{31} = a_{11} = 0$, $a_{23} = -a_{32}$ and $a_{22} = a_{33}$. □

There is such a relationship between the above equations.

Lemma 4.1. *Let M be a bi-tangent quaternion Kaehler manifold. Then the following relation is satisfied,*

$$\nabla_X J = \left(\frac{p(X)r(X) + q(X)s(X)}{s(X)^2 + r(x)^2}\right)\nabla_X T_1 + \left(\frac{q(X)r(X) - p(X)s(X)}{s(X)^2 + r(x)^2}\right)\nabla_X T_2,$$

for any vector field X on M.

Proof. Multiplying (6) with $s(X)$ and multiplying (7) with $r(X)$ gives T_2. Also multiplying (6) with $r(X)$ and multiplying (7) with $-s(X)$ gives T_1. If we substitute T_1 and T_2 in (5), then we obtain the relation. □

Now we will examine the relationship between the curvature tensors.

Lemma 4.2. *Let M be a bi-tangent quaternion Kaehler manifold. Then the following curvature relations are satisfying*

$$(R(X,Y)J)Z = C(X,Y)T_1Z + B(X,Y)T_2Z, \tag{11}$$

$$(R(X,Y)T_1)Z = A(X,Y)T_1Z + D(X,Y)T_2Z, \tag{12}$$

$$(R(X,Y)T_2)Z = -D(X,Y)T_1Z + A(X,Y)T_2Z, \tag{13}$$

for $X, Y, Z \in \chi(M)$, where A, B, C and D are differentiable functions on M.

Proof. From (5) we obtain

$$\begin{aligned}
\nabla_X\nabla_Y JZ &= \nabla_X((\nabla_Y J)Z + J\nabla_Y Z)\\
&= \nabla_X(p(Y)T_1Z + q(Y)T_2Z + J\nabla_Y Z)\\
&= \nabla_X p(Y)T_1Z + \nabla_X q(Y)T_2Z + \nabla_X J\nabla_Y Z\\
&= X(p(Y))T_1Z + p(Y)\nabla_X T_1Z + X(q(Y))T_2Z\\
&\quad + q(Y)\nabla_X T_2Z + (\nabla_X J)(\nabla_Y Z) + J\nabla_X\nabla_Y Z.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\nabla_X\nabla_Y JZ &= X(p(Y))T_1Z + p(Y)[(\nabla_X T_1)Z + T_1\nabla_X Z]\\
&\quad + X(q(Y))T_2Z + q(Y)[(\nabla_X T_2)Z + T_2\nabla_X Z]\\
&\quad + p(X)T_1\nabla_Y Z + q(X)T_2\nabla_Y Z + J\nabla_X\nabla_Y Z.
\end{aligned}$$

From (6) and (7) we derive

$$\begin{aligned}
\nabla_X\nabla_Y JZ &= X(p(Y))T_1Z + p(Y)[r(X)T_1Z + s(X)T_2Z + T_1\nabla_X Z]\\
&\quad + X(q(Y))T_2Z + q(Y)[-s(X)T_1Z + r(X)T_2Z + T_2\nabla_X Z]\\
&\quad + p(X)T_1\nabla_Y Z + q(X)T_2\nabla_Y Z + J\nabla_X\nabla_Y Z.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
\nabla_X\nabla_Y JZ &= X(p(Y))T_1Z + p(Y)r(X)T_1Z + p(Y)s(X)T_2Z + p(Y)T_1\nabla_X Z\\
&\quad + X(q(Y))T_2Z - q(Y)s(X)T_1Z + q(Y)r(X)T_2Z\\
&\quad + q(Y)T_2\nabla_X Z + p(X)T_1\nabla_Y Z + q(X)T_2\nabla_Y Z + J\nabla_X\nabla_Y Z.
\end{aligned}$$

Similarly we get

$$\begin{aligned}
\nabla_Y\nabla_X JZ &= \nabla_Y((\nabla_X J)Z + J\nabla_X Z)\\
&= Y(p(X))T_1Z + p(X)r(Y)T_1Z + p(X)s(Y)T_2Z + p(X)T_1\nabla_Y Z\\
&\quad + Y(q(X))T_2Z - q(X)s(Y)T_1Z + q(X)r(Y)T_2Z\\
&\quad + q(X)T_2\nabla_Y Z + p(Y)T_1\nabla_X Z + q(Y)T_2\nabla_X Z + J\nabla_Y\nabla_X Z.
\end{aligned}$$

From (5) we have

$$\begin{aligned}\nabla_{[X,Y]}JZ &= (\nabla_{[X,Y]}J)Z + J\nabla_{[X,Y]}Z\\ &= p([X,Y])T_1Z + q([X,Y])T_2Z + J\nabla_{[X,Y]}Z.\end{aligned}$$

Summing up we conclude that

$$\begin{aligned}(R(X,Y)J)Z &= (X(p(Y))T_1Z - Y(p(X))T_1Z - p([X,Y])T_1Z) + (p(Y)r(X)T_1Z\\ &\quad -p(X)r(Y)T_1Z) + (p(Y)s(X)T_2Z - p(X)s(Y)T_2Z)\\ &\quad +(X(q(Y))T_2Z - Y(q(X))T_2Z - q([X,Y])T_2Z)\\ &\quad -(q(Y)s(X)T_1Z - q(X)s(Y)T_1Z) + (q(Y)r(X)T_2Z\\ &\quad -q(X)r(Y)T_2Z) + (J\nabla_X\nabla_YZ - J\nabla_Y\nabla_XZ - J\nabla_{[X,Y]}Z).\end{aligned}$$

Thus we find

$$\begin{aligned}(R(X,Y)J)Z &= 2dp(X,Y)T_1Z + (p\wedge r)(Y,X)T_1Z + (p\wedge s)(Y,X)T_2Z\\ &\quad 2dq(X,Y)T_2Z - (q\wedge s)(Y,X)T_1Z + (q\wedge r)(Y,X)T_2Z\\ &\quad +JR(X,Y)Z - JR(X,Y)Z\\ &= (2dp + (r\wedge p) + (q\wedge s))(X,Y)T_1Z + (2dq + (s\wedge p)\\ &\quad +(r\wedge q))(X,Y)T_2Z.\end{aligned}$$

Similarly from (6) we find

$$\begin{aligned}\nabla_X\nabla_YT_1Z &= \nabla_X((\nabla_YT_1)Z + T_1\nabla_YZ)\\ &= \nabla_X(r(Y)T_1Z + s(Y)T_2Z + T_1\nabla_YZ)\\ &= \nabla_Xr(Y)T_1Z + \nabla_Xs(Y)T_2Z + \nabla_XT_1\nabla_YZ\\ &= X(r(Y))T_1Z + r(Y)\nabla_XT_1Z + X(s(Y))T_2Z\\ &\quad +s(Y)\nabla_XT_2Z + (\nabla_XT_1)(\nabla_YZ) + T_1\nabla_X\nabla_YZ.\end{aligned}$$

Hence we have

$$\begin{aligned}\nabla_X\nabla_YT_1Z &= X(r(Y))T_1Z + r(Y)[(\nabla_XT_1)Z + T_1\nabla_XZ]\\ &\quad +X(s(Y))T_2Z + s(Y)[(\nabla_XT_2)Z + T_2\nabla_XZ]\\ &\quad +r(X)T_1\nabla_YZ + s(X)T_2\nabla_YZ + T_1\nabla_X\nabla_YZ.\end{aligned}$$

From (6) and (7) we derive

$$\begin{aligned}\nabla_X\nabla_YT_1Z &= X(r(Y))T_1Z + r(Y)[r(X)T_1Z + s(X)T_2Z + T_1\nabla_XZ]\\ &\quad +X(s(Y))T_2Z + s(Y)[-s(X)T_1Z + r(X)T_2Z + T_2\nabla_XZ]\\ &\quad +r(X)T_1\nabla_YZ + s(X)T_2\nabla_YZ + T_1\nabla_X\nabla_YZ.\end{aligned}$$

Hence we obtain

$$\begin{aligned}\nabla_X\nabla_YT_1Z &= X(r(Y))T_1Z + r(Y)r(X)T_1Z + r(Y)s(X)T_2Z + r(Y)T_1\nabla_XZ\\ &\quad +X(s(Y))T_2Z - s(Y)s(X)T_1Z + s(Y)r(X)T_2Z\\ &\quad +s(Y)T_2\nabla_XZ + r(X)T_1\nabla_YZ + s(X)T_2\nabla_YZ + T_1\nabla_X\nabla_YZ.\end{aligned}$$

Similarly we get

$$\begin{aligned}\nabla_Y\nabla_XT_1Z &= \nabla_Y((\nabla_XT_1)Z + T_1\nabla_XZ)\\ &= Y(r(X))T_1Z + r(X)r(Y)T_1Z + r(X)s(Y)T_2Z + r(X)T_1\nabla_YZ\\ &\quad +Y(s(X))T_2Z - s(X)s(Y)T_1Z + s(X)r(Y)T_2Z\\ &\quad +s(X)T_2\nabla_YZ + r(Y)T_1\nabla_XZ + s(Y)T_2\nabla_XZ + T_1\nabla_Y\nabla_XZ\end{aligned}$$

and from (6) we have

$$\begin{aligned}
\nabla_{[X,Y]}T_1Z &= (\nabla_{[X,Y]}T_1)Z + T_1\nabla_{[X,Y]}Z \\
&= r([X,Y])T_1Z + s([X,Y])T_2Z + T_1\nabla_{[X,Y]}Z.
\end{aligned}$$

Summing up we conclude that

$$\begin{aligned}
(R(X,Y)T_1)Z &= (X(r(Y))T_1Z - Y(r(X))T_1Z - r([X,Y])T_1Z) \\
&\quad +(X(s(Y))T_2Z - Y(s(X))T_2Z - s([X,Y])T_2Z) \\
&= 2dr(X,Y)T_1Z + 2ds(X,Y)T_2Z.
\end{aligned}$$

Similarly from (7) we find

$$\begin{aligned}
\nabla_X\nabla_YT_2Z &= \nabla_X((\nabla_YT_2)Z + T_2\nabla_YZ) \\
&= \nabla_X(-s(Y)T_1Z + r(Y)T_2Z + T_2\nabla_YZ) \\
&= -\nabla_Xs(Y)T_1Z + \nabla_Xr(Y)T_2Z + \nabla_XT_2\nabla_YZ \\
&= -X(s(Y))T_1Z - s(Y)\nabla_XT_1Z + X(r(Y))T_2Z \\
&\quad +r(Y)\nabla_XT_2Z + (\nabla_XT_2)(\nabla_YZ) + T_2\nabla_X\nabla_YZ.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
\nabla_X\nabla_YT_2Z &= -X(s(Y))T_1Z - s(Y)[(\nabla_XT_1)Z + T_1\nabla_XZ] \\
&\quad +X(r(Y))T_2Z + r(Y)[(\nabla_XT_2)Z + T_2\nabla_XZ] \\
&\quad -s(X)T_1\nabla_YZ + r(X)T_2\nabla_YZ + T_2\nabla_X\nabla_YZ.
\end{aligned}$$

From (6) and (7) we derive

$$\begin{aligned}
\nabla_X\nabla_YT_2Z &= -X(s(Y))T_1Z - s(Y)[r(X)T_1Z + s(X)T_2Z + T_1\nabla_XZ] \\
&\quad +X(r(Y))T_2Z + r(Y)[-s(X)T_1Z + r(X)T_2Z + T_2\nabla_XZ] \\
&\quad -s(X)T_1\nabla_YZ + r(X)T_2\nabla_YZ + T_2\nabla_X\nabla_YZ.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\nabla_X\nabla_YT_2Z &= -X(s(Y))T_1Z - s(Y)r(X)T_1Z - s(Y)s(X)T_2Z - s(Y)T_1\nabla_XZ \\
&\quad +X(r(Y))T_2Z - r(Y)s(X)T_1Z + r(Y)r(X)T_2Z \\
&\quad +r(Y)T_2\nabla_XZ - s(X)T_1\nabla_YZ + r(X)T_2\nabla_YZ + T_2\nabla_X\nabla_YZ.
\end{aligned}$$

Similarly we get

$$\begin{aligned}
\nabla_Y\nabla_XT_2Z &= \nabla_Y((\nabla_XT_2)Z + T_2\nabla_XZ) \\
&= -Y(s(X))T_1Z - s(X)r(Y)T_1Z - s(X)s(Y)T_2Z - s(X)T_1\nabla_YZ \\
&\quad +Y(r(X))T_2Z - r(X)s(Y)T_1Z + r(X)r(Y)T_2Z \\
&\quad +r(X)T_2\nabla_YZ - s(Y)T_1\nabla_XZ + r(Y)T_2\nabla_XZ + T_2\nabla_Y\nabla_XZ.
\end{aligned}$$

From (7) we find

$$\begin{aligned}
\nabla_{[X,Y]}T_2Z &= (\nabla_{[X,Y]}T_2)Z + T_2\nabla_{[X,Y]}Z \\
&= -s([X,Y])T_1Z + r([X,Y])T_2Z + T_2\nabla_{[X,Y]}Z.
\end{aligned}$$

Summing up we conclude that

$$\begin{aligned}
(R(X,Y)T_2)Z &= (-X(s(Y))T_1Z + Y(s(X))T_1Z + s([X,Y])T_1Z) \\
&\quad (X(r(Y))T_2Z - Y(r(X))T_2Z - r([X,Y])T_2Z) \\
&= -2ds(X,Y)T_1Z + 2dr(X,Y)T_2Z.
\end{aligned}$$

□

There is a relationship between the equations we found above as follows:

Lemma 4.3. *Let M be a bi-tangent quaternion Kaehler manifold. Then the following relation is satisfied,*

$$\begin{aligned}(R(X,Y)J)Z &= \frac{C(X,Y)A(X,Y)+B(X,Y)D(X,Y)}{A(X,Y)^2+D(X,Y)^2}(R(X,Y)T_1)Z \\ &+\frac{B(X,Y)A(X,Y)-C(X,Y)D(X,Y)}{A(X,Y)^2+D(X,Y)^2}(R(X,Y)T_2)Z.\end{aligned}$$

Proof. Multiplying (12) with $D(X,Y)$ and multiplying (13) with $A(X,Y)$ gives T_2Z. Also multiplying (12) with $A(X,Y)$ and multiplying (13) with $-D(X,Y)$ gives T_1Z. If we substitute T_1Z and T_2Z in (11), then we obtain the relation. □

Now we will show that a bi-tangent quaternion Kaehler manifold M with $dimM = 4m$ is flat when M is of constant curvature.

Theorem 4.2. *Let M be a real 4m-dimensional bi-tangent quaternion Kaehler manifold. If M is of constant curvature, then M is flat.*

Proof. If M is of constant curvature c, then

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\} \tag{14}$$

for any vector fields X, Y and Z on M. From (12) and (13) we can write

$$R(X,Y)T_1Z = T_1R(X,Y)Z + A(X,Y)T_1Z + D(X,Y)T_2Z, \tag{15}$$

$$R(X,Y)T_2Z = T_2R(X,Y)Z - D(X,Y)T_1Z + A(X,Y)T_2Z. \tag{16}$$

Choosing $Z = T_1Z$ in (14) and using (15) we have

$$\begin{aligned}c\{g(Y,T_1Z)X - g(X,T_1Z)Y\} &= c\{g(Y,Z)T_1X - g(X,Z)T_1Y\} \\ &+A(X,Y)T_1Z + D(X,Y)T_2Z.\end{aligned} \tag{17}$$

Now substituting Z by T_2Z in (14) and using (16) we have

$$\begin{aligned}c\{g(Y,T_2Z)X - g(X,T_2Z)Y\} &= c\{g(Y,Z)T_2X - g(X,Z)T_2Y\} \\ &-D(X,Y)T_1Z + A(X,Y)T_2Z.\end{aligned} \tag{18}$$

If we take $Z = -JY$ in (17) then we have

$$\begin{aligned}c\{g(Y,T_2Y)X - g(X,T_2Y)Y\} &= c\{g(Y,-JY)T_1X + g(X,JY)T_1Y\} \\ &+A(X,Y)T_2Y - D(X,Y)T_1Y.\end{aligned}$$

Taking inner product of both sides the above equation with T_2Y

$$\begin{aligned}&c\{g(Y,T_2Y)g(X,T_2Y) - cg(X,T_2Y)g(Y,T_2Y)\} \\ &= cg(X,JY)g(T_1Y,T_2Y) + A(X,Y)g(T_2Y,T_2Y) - D(X,Y)g(T_1Y,T_2Y).\end{aligned}$$

Hence we find

$$A(X,Y) = 0. \tag{19}$$

Similarly taking inner product of both sides the same equation with JY

$$\begin{aligned}&c\{g(Y,T_2Y)g(X,JY) - cg(X,T_2Y)g(Y,JY)\} \\ &= cg(X,JY)g(T_1Y,JY) + A(X,Y)g(T_2Y,JY) - D(X,Y)g(T_1Y,JY).\end{aligned}$$

Hence we find

$$D(X,Y) = 2cg(X,JY). \tag{20}$$

If we substitute (19) and (20) in (18) then we have

$$c\{g(Y,T_2Z)X - g(X,T_2Z)Y\} = c\{g(Y,Z)T_2X - g(X,Z)T_2Y\} - 2cg(X,JY)T_1Z.$$

Taking inner product of both sides of above equation with any vector field W on M

$$c\{g(Y,T_2Z)g(X,W) - g(X,T_2Z)g(Y,W)\}$$
$$= c\{g(Y,Z)g(T_2X,W) - g(X,Z)g(T_2Y,W)\} - 2cg(X,JY)g(T_1Z,W).$$

Let $\{e_1, e_2, , , , e_{4m}\}$ be a orthonormal frame of M. Taking $X = W = e_i$

$$\sum_{i=1}^{4m} \big(c\{g(Y,T_2Z)g(e_i,e_i) - g(e_i,T_2Z)g(Y,e_i)\}\big)$$
$$= \sum_{i=1}^{4m} \big(c\{g(Y,Z)g(T_2e_i,e_i) - g(e_i,Z)g(T_2Y,e_i)\} - 2cg(e_i,JY)g(T_1Z,e_i)\big).$$

Then we get

$$4mcg(Y,T_2Z) - cg(T_2Z,Y) = cg(Y,Z)g(T_2e_i,e_i) - cg(T_2Y,Z) + 2cg(Y,T_2Z)$$

and therefore we have

$$(4m-3)cg(Y,T_2Z) = cg(Y,Z)g(T_2e_i,e_i) - cg(Z,T_2Y).$$

Choosing $Z = JY$

$$(4m-3)cg(Y,T_1Y) = cg(Y,JY)g(T_2e_i,e_i) - cg(JY,T_2Y).$$

Then we find

$$(4m-2)cg(Y,T_1Y) = 0.$$

This completes the proof. □

Remark 4.1. *We note that* $g(Y,T_1Y) \neq 0$. *Indeed, from our Example* 3.1 *it is easy to see that* $T_1Y = (y,0,w,0)$ *for non zero* $Y = (x,y,z,w) \in \mathbb{R}^4$ *and* $g(T_1Y,Y) = xy + zw$. *In a similar way, one can see that* $g(Y,T_2Y) \neq 0$.

REFERENCES

[1] *A. Awane*, *k*-Symplectic structures. J. Math. Phys. **33**, (1982), 4046–4052.

[2] *A. M. Blaga*, Riemannian metrics on a polysymplectic manifold. An. Univ. Oradea Fasc. Mat. **16**, (2009) 241–246.

[3] *A. M. Blaga*, Harmonic subtangent structures. J. Math. Art. ID 603078, (2014), 5 pp.

[4] *D. E. Blair*, Contact manifolds in Riemannian Geometry, Lectures Notes in Math. **509**, Springer, 1976.

[5] *N. Blažić*, Paraquaternionic projective space and pseudo-Riemannian geometry. Publications de l'Institut Mathématique. Nouvelle Série, **60**, (1996), 101-107.

[6] *V. Cruceanu*, On almost biproduct complex manifolds. An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) **52**(1), (2006), 5–24.

[7] *V. Cruceanu*, Almost product bicomplex structures on manifolds. An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) **51**(1), (2003), 99–118.

[8] *V. Cruceanu*, Almost hyperproduct structures on manifolds. An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) **48**(2), (2002), 337–354.

[9] *E. García-Río, Y. Matsushita, R. Vázquez-Lorenzo*, Paraquaternionic Kähler manifolds. The Rocky Mountain Journal of Mathematics, (2001), 237-260.

[10] *Y. Gündüzalp*, Anti-invariant semi-Riemannian submersions from almost para-Hermitian manifolds. J. Funct. Spaces Appl., (2013).

[11] *S. Ianuş, G. E. Vilcu*, Some constructions of almost para-hyperhermitian structures on manifolds and tangent bundles. International Journal of Geometric Methods in Modern Physics, **5**(06), (2008), 893-903.

[12] *S. Ianuş, M. Visinescu, G. E. Vîlcu*, Conformal Killing-Yano tensors on manifolds with mixed 3-structures. SIGMA. Symmetry, Integrability and Geometry: Methods and Applications, **5**, 022, (2009).

[13] *S. Ianuş, S. Marchiafava, G. E. Vîlcu*, Paraquaternionic CR-submanifolds of paraquaternionic Kähler manifolds and semi-Riemannian submersions. Open Mathematics, **8**(4), (2010), 735-753.

[14] *S. Ianuş, G. E. Vilcu*, Semi-Riemannian hypersurfaces in manifolds with metric mixed 3-structures. Acta Mathematica Hungarica, **127**(1-2), (2010), 154-177.

[15] *S. Ianuş, A. M. Ionescu, R. Mocanu, R., G. E. Vilcu*, Riemannian submersions from almost contact metric manifolds. Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg, **81**, (2011), 101-114, Springer-Verlag.

[16] *S. Ishihara*, Quaternion Kaehlerian manifolds. Journal of Differential Geometry, **9**(4), (1974), 483-500.

[17] *Y. Y. Kuo*, On almost contact 3-structure. Tohoku Mathematical Journal, Second Series, **22**(3), (1970), 325-332.

[18] *G. Munteanu* Integrability conditions for almost semiquaternion structures. Riv. Mat. Univ. Parma, **13**, (1987), 153-160.

[19] *G. Munteanu*, Metric almost semiquaternion structures. Bull. Math. Soc. Sci. Math. R. S. Roumanie (N.S.), **32**(4), (1988), 319-325.

[20] *G. Munteanu*, Almost semiquaternion structures: existence and connection, An. Stiint. Univ, Al. I. Cuza, Iaşi Sect. I a Mat. **34**(2), (1988), 167-176.

[21] *J. L. Rosendo, and P. M. Gadea*, Almost tangent structures of order k on spheres, Analele Stiintifice ale Universitatii Al I Cuza din Iasi, **23**(2), (1977), 281–286.

[22] *F. Şahin*, Symplectic groupoids and generalized almost subtangent manifolds. Turkish J. Math. **39**(2), (2015), 156–167.

[23] *S. Sasaki*, On differentiable manifolds with certain structures which are closely related to almost contact structure I. Tohoku Math. J., II. Ser. **12**, (1960), 459–476.

[24] *F. Tricerri*, Sulle varieta dotate di due strutture quasi complesse linearmente indipendenti. Riv. Mat. Univ. Parma, **3**(3), (1975), 349-358.

[25] *G. E. Vîlcu*, Mixed paraquaternionic 3-submersions. Indagationes Mathematicae, **24**(2), (2013), 474-488.

[26] *G. E. Vîlcu*, Canonical foliations on paraquaternionic Cauchy–Riemann submanifolds. Journal of Mathematical Analysis and Applications, **399**(2), (2013), 551-558.

[27] *G. E. Vîlcu* On generic submanifolds of manifolds endowed with metric mixed 3-structures. Communications in Contemporary Mathematics, **18**(06), 1550081, 2016.

[28] *K. Yano, E. T. Davies*, Differential geometry on almost tangent manifolds. Annali di Matematica Pura ed Applicata, 103,(1975), 131-160.

[29] *K. Yano, M. Kon*, Structures on Manifolds, World Scientific, 1984.