A PROPERTY OF LOGARITHMICALLY ABSOLUTELY MONOTONIC FUNCTIONS AND THE LOGARITHMICALLY COMPLETE MONOTONICITY OF A POWER-EXPONENTIAL FUNCTION

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In this work, the notion of a "logarithmically absolutely monotonic function" is introduced, the inclusion that a logarithmically absolutely monotonic function is also absolutely monotonic is revealed, the logarithmically complete monotonicity and the logarithmically absolute monotonicity of the function $(1+\alpha/x)^{x+\beta}$ are proved, where α and β are given real parameters. A new proof for the inclusion that a logarithmically completely monotonic function is also completely monotonic is given, and an open problem is posed.

Keywords: absolutely monotonic function, completely monotonic function, Faá di Bruno's formula, logarithmically absolutely monotonic function, logarithmically completely monotonic function, open problem, property

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1. Introduction

Recall [31, 33, 46, 48] that a function f is said to be completely monotonic on an interval I if f has derivative of all orders on I such that

$$(-1)^k f^{(k)}(x) \geq 0 \tag{1}$$

for $x \in I$ and $k \geq 0$. For our own convenience, the set of the completely monotonic functions on I is denoted by $\mathcal{C}[I]$.

Recall also [31, 33, 45, 46, 48] that a function f is said to be absolutely monotonic on an interval I if it has derivatives of all orders and

$$f^{(k-1)}(t) \geq 0 \tag{2}$$

for $t \in I$ and $k \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers. The set of the absolutely monotonic functions on I is denoted by $\mathcal{A}[I]$.

Recall again [6, 35, 38, 40] that a positive function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies

$$(-1)^k [\ln f(x)]^{(k)} \geq 0 \tag{3}$$

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for $k \in \mathbb{N}$ on I. Similar to above, the set of the logarithmically completely monotonic functions on I is denoted by $\mathcal{C}_{\mathcal{L}}[I]$.

The famous Bernstein-Widder's Theorem [48, p. 161] states that $f \in \mathcal{C}[(0,\infty)]$ if and only if there exists a bounded and nondecreasing function $\mu(t)$ such that

$$f(x) = \int_0^\infty e^{-xt} \mathrm{d}\mu(t) \tag{4}$$

converges for $0 < x < \infty$, and that $f(x) \in \mathcal{A}[(0,\infty)]$ if and only if there exists a bounded and nondecreasing function $\sigma(t)$ such that

$$f(x) = \int_0^\infty e^{xt} \mathrm{d}\sigma(t) \tag{5}$$

converges for $0 \leq x < \infty$.

In [7, 35, 38, 40, 41, 46] and many other references, the inclusions $\mathcal{C}_{\mathcal{L}}[I] \subset \mathcal{C}[I]$ and $\mathcal{S} \subset \mathcal{C}_{\mathcal{L}}[(0,\infty)]$ were revealed implicitly or explicitly, where $\mathcal{S}$ denotes the class of Stieltjes transforms.

The class $\mathcal{C}_{\mathcal{L}}[(0,\infty)]$ is characterized in [7, Theorem 1.1] explicitly and in [21, Theorem 4.4] implicitly by

$$f \in \mathcal{C}_{\mathcal{L}}[(0,\infty)] \Longleftrightarrow f^\alpha \in \mathcal{C} \text{ for all } \alpha > 0 \Longleftrightarrow \sqrt[n]{f} \in \mathcal{C} \text{ for all } n \in \mathbb{N}.$$

In other words, the functions in $\mathcal{C}_{\mathcal{L}}[(0,\infty)]$ are those completely monotonic functions for which the representing measure μ in (4) is infinitely divisible in the convolution sense: For each $n \in \mathbb{N}$ there exists a positive measure ν on $[0,\infty)$ with n-th convolution power equal to μ.

To the best of our knowledge, the terminology "logarithmically completely monotonic function" and some properties of it appeared firstly without explicit definition in [6], re-coined independently with explicit definition by the first author in [38], the preprints of [20, 35]. Since then, a further deep investigation on the logarithmically completely monotonic functions was explicitly carried out in [7] and a citation of the logarithmically completely monotonic functions appeared in [16].

It is said in [7] that "In various papers complete monotonicity for special functions has been established by proving the stronger statement that the function is a Stieltjes transform". It is also said in [8] that "In concrete cases it is often easier to establish that a function is a Stieltjes transform than to verify complete monotonicity". Because the logarithmically completely monotonic functions must be completely monotonic, in order to show some functions, especially the power-exponential functions or the exponential functions, are completely monotonic, maybe it is sufficient and much simpler to prove their logarithmically complete monotonicity or to show that they are Stieltjes transforms, if possible. These techniques have been used in [1, 2, 3, 5, 9, 10, 11, 12, 13, 14, 16, 22, 28, 29, 30, 31, 34, 37, 40, 41, 43, 44, 46, 50] and many other articles. It can be imagined that, if there would not any inclusion relationship between the sets of completely monotonic functions, logarithmically completely monotonic functions and Stieltjes transforms, it would be very complex, difficult, even impossible, to verify some power-exponential functions to be completely monotonic.

It is worthwhile to point out that, in most related papers before, although the logarithmically completely monotonicities of some functions had been established essentially, they were stated using the notion "completely monotonic function" instead

of "logarithmically completely monotonic function". Because a completely monotonic function may be not logarithmically completely monotonic, in our opinion, it should be better that many results or conclusions on completely monotonic functions are rewritten or restated using the term "logarithmically completely monotonic function".

The main results of this paper are as follows.

Similar to the definition of the logarithmically completely monotonic function, we would like to coin a notion "logarithmically absolutely monotonic functions".

Definition 1. A positive function f is said to be logarithmically absolutely monotonic on an interval I if it has derivatives of all orders and $[\ln f(t)]^{(k)} \geq 0$ for $t \in I$ and $k \in \mathbb{N}$.

For our own convenience, the set of the logarithmically absolutely monotonic functions on an interval I is denoted by $\mathcal{A}_{\mathcal{L}}[I]$.

Similar to the inclusion $\mathcal{C}_{\mathcal{L}}[I] \subset \mathcal{C}[I]$ mentioned above, the logarithmically absolutely monotonic functions have the following nontrivial property.

Theorem 1. *A logarithmically absolutely monotonic function on an interval I is also absolutely monotonic on I, but not conversely. Equivalently, $\mathcal{A}_{\mathcal{L}}[I] \subset \mathcal{A}[I]$ and $\mathcal{A}[I] \setminus \mathcal{A}_{\mathcal{L}}[I] \neq \emptyset$.*

This theorem hints us that, in order to show some functions, especially the power-exponential functions or the exponential functions, are absolutely monotonic, maybe it is much simpler or easier to prove the stronger statement that they are logarithmically absolutely monotonic.

Let

$$F_{\alpha,\beta}(x) = \left(1 + \frac{\alpha}{x}\right)^{x+\beta} \tag{6}$$

for $\alpha \neq 0$ and either $x > \max\{0, -\alpha\}$ or $x < \min\{0, -\alpha\}$. In [17, 18, 19, 23, 25, 32, 36, 39, 42, 49] (see also related content in [26, 27]), the sufficient and necessary conditions such that the function $F_{\alpha,\beta}(x)$, its simplified forms, its variants and their corresponding sequences are monotonic are obtained.

In [43, Theorem 1.2] and [50], it was proved that $F_{\alpha,\beta}(x) \in \mathcal{C}_{\mathcal{L}}[(0, \infty)]$ for $\alpha > 0$ and $\beta \in \mathbb{R}$ if and only if $2\beta \geq \alpha > 0$. From $\mathcal{C}_{\mathcal{L}}[I] \subset \mathcal{C}[I]$ it is deduced that the function $F_{\alpha,\beta}(x) - e^{\alpha} \in \mathcal{C}[(0, \infty)]$ if and only if $0 < \alpha \leq 2\beta$, which is a conclusion obtained in [4].

Now it is natural to pose a problem: How about the logarithmically complete or absolute monotonicity of the function $F_{\alpha,\beta}(x)$ for all real numbers $\alpha \neq 0$ and β on the interval $(-\infty, \min\{0, -\alpha\})$ or $(\max\{0, -\alpha\}, \infty)$? The following Theorem 2 answers this problem.

Theorem 2. *Let $\alpha \neq 0$.*
 (1) *For $\alpha < 0$,*
     (a) *$F_{\alpha,\beta}(x) \in \mathcal{C}_{\mathcal{L}}[(-\alpha, \infty)]$ if and only if $\beta \leq \alpha$,*
     (b) *and $\dfrac{1}{F_{\alpha,\beta}(x)} \in \mathcal{C}_{\mathcal{L}}[(-\alpha, \infty)]$ if and only if $2\beta \geq \alpha$.*
 (2) *For $\alpha > 0$,*
     (a) *$F_{\alpha,\beta}(x) \in \mathcal{C}_{\mathcal{L}}[(0, \infty)]$ if and only if $2\beta \geq \alpha$,*

    (b) *and* $\dfrac{1}{F_{\alpha,\beta}(x)} \in \mathcal{C}_{\mathcal{L}}[(0,\infty)]$ *if and only if* $\beta \leq 0$.

(3) *For* $\alpha < 0$,

    (a) $F_{\alpha,\beta}(x) \in \mathcal{A}_{\mathcal{L}}[(-\infty,0)]$ *if and only if* $\beta \geq 0$,

    (b) *and* $1F_{\alpha,\beta}(x) \in \mathcal{A}_{\mathcal{L}}[(-\infty,0)]$ *if and only if* $2\beta \leq \alpha$.

(4) *For* $\alpha > 0$,

    (a) $F_{\alpha,\beta}(x) \in \mathcal{A}_{\mathcal{L}}[(-\infty,-\alpha)]$ *if and only if* $2\beta \leq \alpha$,

    (b) *and* $\dfrac{1}{F_{\alpha,\beta}(x)} \in \mathcal{A}_{\mathcal{L}}[(-\infty,-\alpha)]$ *if and only if* $\beta \geq \alpha$.

    As an immediate consequence of combining Theorem 2 with the inclusion $\mathcal{C}_{\mathcal{L}}[I] \subset \mathcal{C}[I]$, the following complete monotonicity relating to the function $F_{\alpha,\beta}(x)$, which extends the corresponding results in [4, 43, 50], can be obtained easily.

Theorem 3. *Let* $\alpha \neq 0$.

(1) *For* $\alpha > 0$,

    (a) $F_{\alpha,\beta}(x) - e^{\alpha} \in \mathcal{C}[(0,\infty)]$ *if and only if* $\alpha \leq 2\beta$,

    (b) *and* $\dfrac{1}{F_{\alpha,\beta}(x)} - e^{-\alpha} \in \mathcal{C}[(0,\infty)]$ *if and only if* $\beta \leq 0$.

(2) *For* $\alpha < 0$,

    (a) $F_{\alpha,\beta}(x) - e^{\alpha} \in \mathcal{C}[(-\alpha,\infty)]$ *if and only if* $\beta \leq \alpha$,

    (b) *and* $\dfrac{1}{F_{\alpha,\beta}(x)} - e^{-\alpha} \in \mathcal{C}[(-\alpha,\infty)]$ *if and only if* $2\beta \geq \alpha$.

    In [7] and [35, 38], two different proofs for the inclusion $\mathcal{C}_{\mathcal{L}}[I] \subset \mathcal{C}[I]$ were given. Now we would like to present a new proof for this inclusion by using Faá di Bruno's formula [15, 24, 47].

Theorem 4. *A logarithmically completely monotonic function on an interval* I *is also completely monotonic on* I, *but not conversely. Equivalently,* $\mathcal{C}_{\mathcal{L}}[I] \subset \mathcal{C}[I]$ *and* $\mathcal{C}[I] \setminus \mathcal{C}_{\mathcal{L}}[I] \neq \emptyset$.

    Now we are in a position to pose an open problem: How to characterize the logarithmically completely monotonic functions and the logarithmically absolutely monotonic functions, as we characterize the completely monotonic functions and the absolutely monotonic functions by Bernstein-Widder's Theorem?

2. Proofs of theorems

Proof of Theorem 1. The Faá di Bruno's formula [15, 24, 47] gives an explicit formula for the n-th derivative of the composition $g(h(t))$: If $g(t)$ and $h(t)$ are functions for which all the necessary derivatives are defined, then

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}[g(h(x))] = \sum_{\substack{1 \leqslant i \leqslant n, i_k \geqslant 0 \\ \sum_{k=1}^{n} i_k = i \\ \sum_{k=1}^{n} k i_k = n}} \frac{n!}{\prod_{k=1}^{n} i_k!} g^{(i)}(h(x)) \prod_{k=1}^{n} \left[\frac{h^{(k)}(x)}{k!}\right]^{i_k}. \tag{7}$$

Applying (7) to $g(x) = e^x$ and $h(x) = \ln f(x)$ leads to

$$f^{(n)}(x) = \left[e^{\ln f(x)}\right]^{(n)} = n! f(x) \sum_{\substack{1 \leqslant i \leqslant n, i_k \geqslant 0 \\ \sum_{k=1}^{n} i_k = i \\ \sum_{k=1}^{n} k i_k = n}} \prod_{k=1}^{n} \frac{\left\{[\ln f(x)]^{(k)}\right\}^{i_k}}{[i_k!(k!)^{i_k}]} \tag{8}$$

for $n \in \mathbb{N}$. If $f(x) \in \mathcal{A}_{\mathcal{L}}[I]$, then $[\ln f(x)]^{(k)} \geq 0$ for $k \in \mathbb{N}$, and then $f^{(n)}(x) \geq 0$ for $n \in \mathbb{N}$, that means $f(x) \in \mathcal{A}[I]$.

Conversely, it is clear that $0 \in \mathcal{A}[I]$, but $0 \notin \mathcal{A}_{\mathcal{L}}[I]$. Therefore $\mathcal{A}[I] \setminus \mathcal{A}_{\mathcal{L}}[I] \neq \emptyset$. The proof of Theorem 1 is complete. $\qquad\square$

Proof of Theorem 2. Direct computations yield

$$\ln F_{\alpha,\beta}(x) = (x + \beta) \ln \left(1 + \frac{\alpha}{x}\right), \tag{9}$$

$$[\ln F_{\alpha,\beta}(x)]' = \ln \left(1 + \frac{\alpha}{x}\right) - \frac{\alpha(x + \beta)}{x(x + \alpha)}, \tag{10}$$

$$[\ln F_{\alpha,\beta}(x)]'' = \frac{\alpha[(2\beta - \alpha)x + \alpha\beta]}{x^2(x + \alpha)^2}, \tag{11}$$

and

$$\lim_{x \to \pm\infty} [\ln F_{\alpha,\beta}(x)]'' = 0, \quad \lim_{x \to \pm\infty} [\ln F_{\alpha,\beta}(x)]' = 0. \tag{12}$$

For $\alpha < 0$ and $x > -\alpha$, in virtue of formula

$$di\frac{1}{x^r} = \frac{1}{\Gamma(r)} \int_0^\infty t^{r-1} e^{-xt} dt \tag{13}$$

for $x > 0$ and $r > 0$, equation (11) can be rewritten as

$$\begin{aligned} [\ln F_{\alpha,\beta}(x)]'' &= \frac{1}{x + \alpha} - \frac{1}{x} - \frac{\beta - \alpha}{(x + \alpha)^2} + \frac{\beta}{x^2} \\ &\triangleq \int_0^\infty [\beta - q_\alpha(t)] t\left(e^{\alpha t} - 1\right) e^{-(x+\alpha)t} dt, \end{aligned} \tag{14}$$

where

$$q_\alpha(t) = \frac{e^{\alpha t} - \alpha t - 1}{t\left(e^{\alpha t} - 1\right)} = \frac{\alpha(e^u - u - 1)}{u(e^u - 1)} \triangleq \alpha q(u) \tag{15}$$

for $t > 0$ and $u = \alpha t < 0$. Since $q(u)$ is decreasing on $(-\infty, 0)$ with

$$\lim_{u \to 0^-} q(u) = \frac{1}{2} \quad \text{and} \quad \lim_{u \to -\infty} q(u) = 1,$$

then

(1) when $\beta \leq \alpha$ the function $(-1)^i [\ln F_{\alpha,\beta}(x)]^{(i+2)} \geq 0$, and
(2) when $2\beta \geq \alpha$ the function $(-1)^i [\ln F_{\alpha,\beta}(x)]^{(i+2)} \leq 0$ on $(-\alpha, \infty)$ for $i \geq 0$.

Since $[\ln F_{\alpha,\beta}(x)]'$ increases for $\beta \leq \alpha$ and decreases for $2\beta \geq \alpha$, considering one of the limits in (12) shows that $[\ln F_{\alpha,\beta}(x)]' \leq 0$ for $\beta \leq \alpha$ and $[\ln F_{\alpha,\beta}(x)]' \geq 0$ for $2\beta \geq \alpha$. In conclusion, $(-1)^k [\ln F_{\alpha,\beta}(x)]^{(k)} \geq 0$ for $\beta \leq \alpha$ and $(-1)^k [\ln F_{\alpha,\beta}(x)]^{(k)} \leq 0$ for $2\beta \geq \alpha$ and $k \in \mathbb{N}$. This means that $F_{\alpha,\beta}(x) \in \mathcal{C}_{\mathcal{L}}[(-\alpha, \infty)]$ for $\beta \leq \alpha < 0$ and $\frac{1}{F_{\alpha,\beta}(x)} \in \mathcal{C}_{\mathcal{L}}[(-\alpha, \infty)]$ for both $2\beta \geq \alpha$ and $\alpha < 0$.

Conversely, if $F_{\alpha,\beta}(x) \in \mathcal{C}_{\mathcal{L}}[(-\alpha,\infty)]$ for $\alpha < 0$, then $[\ln F_{\alpha,\beta}(x)]' \leq 0$ which can be rearranged as

$$\beta \leq x\left[\left(1 + \frac{x}{\alpha}\right)\ln\left(1 + \frac{\alpha}{x}\right) - 1\right] \triangleq \theta_\alpha(x) \tag{16}$$

and $\lim\limits_{x \to (-\alpha)^+} \theta_\alpha(x) = \alpha$, thus $\beta \leq \alpha$. If $\dfrac{1}{F_{\alpha,\beta}(x)} \in \mathcal{C}_{\mathcal{L}}[(-\alpha,\infty)]$ for $\alpha < 0$, then $[\ln F_{\alpha,\beta}(x)]' \geq 0$ which can be rearranged as $\beta \geq \theta_\alpha(x) \to \dfrac{\alpha}{2}$ as $x \to \infty$, hence $2\beta \geq \alpha$ holds.

If $\alpha > 0$, the formulas (14) and (15) are valid for $x > 0$ and $u > 0$. Since $q(u)$ is decreasing on $(0,\infty)$ with

$$\lim_{u \to 0^+} q(u) = \frac{1}{2} \quad \text{and} \quad \lim_{u \to \infty} q(u) = 0,$$

by the same argument as above, it follows easily that $F_{\alpha,\beta}(x) \in \mathcal{C}_{\mathcal{L}}[(0,\infty)]$ for $2\beta \geq \alpha$ and $\dfrac{1}{F_{\alpha,\beta}(x)} \in \mathcal{C}_{\mathcal{L}}[(0,\infty)]$ for $\beta \leq 0$.

Conversely, if $F_{\alpha,\beta}(x) \in \mathcal{C}_{\mathcal{L}}[(0,\infty)]$ for $\alpha > 0$, then $[\ln F_{\alpha,\beta}(x)]' \leq 0$ which can be rewritten as $\beta \geq \theta_\alpha(x) \to \dfrac{\alpha}{2}$ as x tends to ∞; if $\dfrac{1}{F_{\alpha,\beta}(x)} \in \mathcal{C}_{\mathcal{L}}[(0,\infty)]$ for $\alpha > 0$, then $[\ln F_{\alpha,\beta}(x)]' \geq 0$ which can be rewritten as $\beta \leq \theta_\alpha(x) \to 0$ as $x \to 0$.

For $\alpha < 0$ and $x < 0$, it is easy to obtain

$$\begin{aligned}
[\ln F_{\alpha,\beta}(x)]'' &= -\frac{1}{-(x+\alpha)} + \frac{1}{-x} - \frac{\beta - \alpha}{[-(x+\alpha)]^2} + \frac{\beta}{(-x)^2} \\
&\triangleq \int_0^\infty [\beta + p_\alpha(t)]t\left(1 - e^{\alpha t}\right)e^{xt}\mathrm{d}t,
\end{aligned} \tag{17}$$

where

$$p_\alpha(t) = \frac{1 + (\alpha t - 1)e^{\alpha t}}{t(1 - e^{\alpha t})} = \frac{\alpha[1 + (u-1)e^u]}{u(1 - e^u)} \triangleq \alpha p(u) \tag{18}$$

for $t > 0$ and $u = \alpha t < 0$ and $p(u)$ is decreasing on $(-\infty, 0)$ with

$$\lim_{u \to -\infty} p(u) = 0 \quad \text{and} \quad \lim_{u \to 0^-} p(u) = -\frac{1}{2}.$$

Accordingly, for $i \geq 0$ and on $(-\infty, 0)$, if $\beta - \dfrac{\alpha}{2} \leq 0$ then $[\ln F_{\alpha,\beta}(x)]^{(i+2)} \leq 0$, if $\beta \geq 0$ then $[\ln F_{\alpha,\beta}(x)]^{(i+2)} \geq 0$. By virtue of (12), it is deduced immediately that $[\ln F_{\alpha,\beta}(x)]^{(k)} \leq 0$ for $2\beta \leq \alpha$ and $[\ln F_{\alpha,\beta}(x)]^{(k)} \geq 0$ for $\beta \geq 0$ and $k \in \mathbb{N}$ on $(-\infty, 0)$.

Conversely, if $F_{\alpha,\beta}(x)$ is logarithmically absolutely monotonic on $(-\infty, 0)$, then $[\ln F_{\alpha,\beta}(x)]' \geq 0$ which can be rewritten as $\beta \geq \theta_\alpha(x)$ for $x \in (-\infty, 0)$. From $\lim\limits_{x \to 0^-} \theta_\alpha(x) = 0$, it follows that $\beta \geq 0$; if $\dfrac{1}{F_{\alpha,\beta}(x)}$ is logarithmically absolutely monotonic on $(-\infty, 0)$, then $[\ln F_{\alpha,\beta}(x)]' \leq 0$ which can be rearranged as $\beta \leq \theta_\alpha(x)$ for $x \in (-\infty, 0)$. From $\lim\limits_{x \to -\infty} \theta_\alpha(x) = \dfrac{\alpha}{2}$, it concludes that $2\beta \leq \alpha$.

For $\alpha > 0$ and $x < -\alpha$, the formulas (17) and (18) hold for $x \in (-\infty, -\alpha)$ and $u > 0$. The function $p(u)$ is negative and decreasing on $(0, \infty)$ with

$$\lim_{u \to 0^+} p(u) = -\frac{1}{2} \quad \text{and} \quad \lim_{u \to \infty} p(u) = -1.$$

Consequently, if $\beta - \frac{1}{2}\alpha \leq 0$ then $[\ln F_{\alpha,\beta}(x)]^{(i+2)} \geq 0$ for $i \geq 0$ on $(-\infty, -\alpha)$, if $\beta - \alpha \geq 0$ then $[\ln F_{\alpha,\beta}(x)]^{(i+2)} \leq 0$ for $i \geq 0$ on $(-\infty, -\alpha)$. In virtue of (12), it is readily concluded that $[\ln F_{\alpha,\beta}(x)]^{(k)} \geq 0$ for $2\beta \leq \alpha$ and $[\ln F_{\alpha,\beta}(x)]^{(k)} \leq 0$ for $\beta \geq \alpha$ and $k \in \mathbb{N}$ on $(-\infty, -\alpha)$.

Conversely, if $F_{\alpha,\beta}(x)$ is logarithmically absolutely monotonic on $(-\infty, -\alpha)$, then $[\ln F_{\alpha,\beta}(x)]' \geq 0$ which can be rewritten as $\beta \leq \theta_\alpha(x)$ for $x \in (-\infty, -\alpha)$. From the fact that $\lim_{x \to -\infty} \theta_\alpha(x) = \frac{\alpha}{2}$, it follows that $2\beta \leq \alpha$; if $\frac{1}{F_{\alpha,\beta}(x)}$ is logarithmically absolutely monotonic on $(-\infty, -\alpha)$, then $[\ln F_{\alpha,\beta}(x)]' \leq 0$ which can be rearranged as $\beta \geq \theta_\alpha(x)$ for $x \in (-\infty, -\alpha)$. From the fact that $\lim_{x \to (-\alpha)^-} \theta_\alpha(x) = \alpha$, it concludes that $\beta \geq \alpha$. The proof of Theorem 2 is complete. $\qquad \square$

Proof of Theorem 3. This follows from taking limits by L'Hôspital rule, considering the inclusion $\mathcal{C}_\mathcal{L}[I] \subset \mathcal{C}[I]$ and using Theorem 2. $\qquad \square$

Proof of Theorem 4. If $f(x) \in \mathcal{C}_\mathcal{L}[I]$, then $(-1)^k[\ln f(x)]^{(k)} \geq 0$ for $k \in \mathbb{N}$ and

$$\begin{aligned}
\prod_{k=1}^{n} \big\{[\ln f(x)]^{(k)}\big\}^{i_k} &= \prod_{k=1}^{n} (-1)^{ki_k} \big\{(-1)^k[\ln f(x)]^{(k)}\big\}^{i_k} \\
&= \prod_{k=1}^{n} (-1)^{ki_k} \prod_{k=1}^{n} \big\{(-1)^k[\ln f(x)]^{(k)}\big\}^{i_k} \\
&= (-1)^{\sum_{k=1}^{n} ki_k} \prod_{k=1}^{n} \big\{(-1)^k[\ln f(x)]^{(k)}\big\}^{i_k}
\end{aligned} \tag{19}$$

for $n \in \mathbb{N}$. Substituting (19) into (8) yields

$$(-1)^n f^{(n)}(x) = n! f(x) \sum_{\substack{1 \leqslant i \leqslant n, i_k \geqslant 0 \\ \sum_{k=1}^{n} i_k = i \\ \sum_{k=1}^{n} ki_k = n}} \prod_{k=1}^{n} \frac{\big\{(-1)^k[\ln f(x)]^{(k)}\big\}^{i_k}}{[i_k!(k!)^{i_k}]} \geq 0 \tag{20}$$

for $n \in \mathbb{N}$. This means that $f(x) \in \mathcal{C}[I]$.

Conversely, it is clear that $0 \in \mathcal{C}[I]$ for any interval I, but $0 \notin \mathcal{C}_\mathcal{L}[I]$. Therefore $\mathcal{C}[I] \setminus \mathcal{C}_\mathcal{L}[I] \neq \emptyset$. The proof of Theorem 4 is complete. $\qquad \square$

Acknowledgements

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