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APPLYING LEM TO DÜFFING'S OSCILLATOR

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Metoda echivalenței lineare (LEM) a fost creată de Toma în scopul determinării şi studiului numeric şi calitativ al soluțiilor sistemelor dinamice nelineare într-un cadru clasic linear. LEM se aplică aici ecuației Düfing. Reprezentarea LEM normală generală posedă avantajul de a evidenția dependența de parametri a soluțiilor, fiind eficientă cu precădere pentru studiul pe termen lung al acestora. Stabilim în această lucrare reprezentarea LEM normală pentru un oscilator Düffing de tip Ueda. Soluțiile LEM sunt apoi testate numeric folosind metoda Runge-Kutta.

The linear equivalence method (LEM) was previously introduced by Toma to for getting and studying - both numerically and qualitatively - the solutions of nonlinear dynamical systems. LEM is applied here to Düffing equation. The normal LEM representations emphasize the dependence on parameters and are particularly fitted for the study of long term behaviour of the solution. We established it in the case of a damped Düffing oscillator of Ueda type. The LEM solutions are then numerically tested by using the Runge-Kutta method.

Keywords: Düffing oscillator, linear equivalence method, normal LEM representation.

1. Introduction

The Düffing oscillator is mathematically modelled as [1]

$$\ddot{x} + \delta\dot{x} + \beta x + \alpha x^3 = A\cos\omega t\,, \tag{1}$$

with a positive damping constant δ. For positive values of β, this can be physically interpreted as a forced oscillator with a spring of non-linear restoring force; for positive α, one has a hardening, while for negative α – a softening spring. For $\beta < 0$, it can be regarded as describing the dynamics of a point mass in a double well potential [2[5].

Let us also note that Düffing model is an algebraically simple equation involving time-dependent acceleration (jerks) that have chaotic solutions, as previously shown by Ueda [6].

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In what follows, we take $\alpha > 0, \beta \geq 0, \delta \geq 0$.

Gottlieb pointed out [7] that the simplest ODE in a single variable exhibiting chaos is of third order, following Poincaré-Bendixson theorem; Düffing equation with $\alpha = \omega = 1$ may be written as a fourth order homogeneous polynomial equation depending on an unique parameter, δ [8]. The fourth derivative is, in fact, the time derivative of the jerk; it is also called spasm, jounce or sprite, because of its behaviour [8]. Sprott deduced numerically polynomial jerks allowing chaotic solutions [8] and simple first order polynomial ODS with three equations and three unknown functions allowing solutions with chaotic behaviour; among them, one can recognize Lorenz's system and Rössler type ODSs, both with the corresponding chaotic attractors [9].

In this paper, we treat Düffing equation by using LEM – the linear equivalence method – previously introduced by Toma (see [10] and the first LEM monograph [11]) for the qualitative and numerical study of the solutions of non-linear ODEs depending on parameters. The method was successfully applied to various nonlinear dynamical systems modelling various physical and mechanical phenomena (e.g. [12-[*18*]); some of them were also included in [19].

More precisely, we use here the normal LEM representations [19] [22], establishing the parametric LEM solutions emphasizing third order effects; a numerical comparison with the Runge-Kutta method is then provided. For wide ranges of the involved parameters it is shown that the LEM formulae can be applied on large time intervals, thus emphasizing qualitatively the long term behaviour of the solution.

2. The normal LEM representations

While LEM can be applied to more general ODSs, as the involved model studied here is polynomial and with constant coefficients, we will restrict to this case. Consider therefore the polynomial ODS

$$\mathcal{F}\mathbf{y} \equiv \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} - \mathbf{P}(\mathbf{y}) = \mathbf{0}, \quad \mathbf{P} = \left[P_j(\mathbf{y})\right]_{j=\overline{1,n}}, \quad P_j(\mathbf{y}) \equiv \sum_{|\eta| \leq p_j} a_{j\eta}\mathbf{y}^{\eta}, \qquad (2)$$
$$a_{j\mu} \in \Re,\ j = \overline{1,n}, |\mu| \leq p_j, \quad j = \overline{1,n}.$$

As it was mentioned – firstly in [10] – the LEM mapping is

$$v(t,\xi) = \mathrm{e}^{\langle \xi, \mathbf{y} \rangle},\ \boldsymbol{\xi} = (\xi_1, \xi_2, ..., \xi_n) \in \Re^n, \qquad (3)$$

where $\boldsymbol{\xi}$ are newly introduced parameters. This mapping associates to the nonlinear ODS two linear equivalents [10], [11]:

- a linear PDE, always of first order with respect to t

$$\mathfrak{L}\, v(t,\xi) \equiv \frac{\partial v}{\partial t} - \langle \xi, \mathbf{P}(D) \rangle v = 0\,, \tag{4}$$

and

- a linear, while infinite, first order ODS, that may be also written in matrix form

$$\begin{gathered} \mathbb{S}\mathbf{V} \equiv \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} - \mathbf{AV} = \mathbf{0}, \\ \mathbf{V} = \left(\mathbf{V}_j\right)_{j \in \mathfrak{N}}, \quad \mathbf{V}_j = \left(v_\gamma\right)_{\|\gamma\| = j}. \end{gathered} \tag{5}$$

The second LEM equivalent, the system (5), is obtained from the first one, by searching the unknown function v in the class of analytic in $\boldsymbol{\xi}$ functions

$$v(t,\xi) = 1 + \sum_{|\gamma|=1}^{\infty} v_\gamma(t) \frac{\xi^j}{j!}\,. \tag{6}$$

The LEM matrix **A** is row and column-finite, as the differential operator is polynomial. It has a cell-diagonal structure.

The involved cells $\mathbf{A}_{k,k+s}$ are generated by those $f_{j\mu}$ with $|\mu| = s+1$ only; for instance, $\mathbf{A}_{11}$ is the linear part of the operator. This special form of **A** allows the calculus by block partitioning.

Let us associate to (2) the initial conditions

$$\mathbf{y}(t_0) = \mathbf{y}_0, \quad t_0 \in \mathrm{I}\,. \tag{7}$$

By LEM, they are transferred to

$$v(t_0, \xi) = \mathrm{e}^{\langle \xi, \mathbf{y}_0 \rangle}, \quad \xi \in \mathfrak{R}^n\,, \tag{8}$$

a condition that must be associated to (4), and

$$\mathbf{V}(t_0) = \left(\mathbf{y}_0^\gamma\right)_{|\gamma| \in \mathfrak{N}}\,, \tag{9}$$

indicating an initial condition for the second LEM equivalent (5).

The linear equivalents are consistent on Exp-type spaces [11][19].

The following result holds true

Theorem 1. [10][11][19] *The solution of the nonlinear initial problem* ((1), (7)

i) coincides with the first n components of the infinite vector

$$\mathbf{V}(t) = \mathrm{e}^{\mathbf{A}(t-t_0)} \mathbf{V}(t_0), \tag{10}$$

where the exponential matrix, defined as for finite matrices, can be computed by block partitioning, each step involving finite sums;

ii) coincides with the series

$$y_j(t) = y_{j0} + \sum_{l=1}^{\infty} \sum_{|\gamma|=l} u_{j\gamma}(t) y_0^{\gamma}, \quad j = \overline{1,n}, \tag{11}$$

where $u_{j\gamma}(t)$ *satisfy the finite linear ODSs*

$$\frac{\mathrm{d}\mathbf{U}_k^j}{\mathrm{d}t} = \mathbf{A}_{1k}^{\mathrm{T}} \mathbf{U}_k^j + \mathbf{A}_{2k}^{\mathrm{T}} \mathbf{U}_k^j + \ldots + \mathbf{A}_{kk}^{\mathrm{T}} \mathbf{U}_k^j, \quad k = \overline{1,l}, \quad \mathbf{U}_s^j(t) = \left[u_{j\gamma}(t)\right]_{|\gamma|=s}, \tag{12}$$

and the Cauchy conditions

$$\mathbf{U}_1^j(t_0) = \mathbf{e}_j^n \equiv \left[\delta_i^j\right]_{i=\overline{1,n}}, \quad \mathbf{U}_s^j(t_0) = \mathbf{0},\ s = \overline{2,l}, \tag{13}$$

T *standing for transpose matrix and* δ_i^j *– for the Kronecker delta.*

The representation (11) was called *normal* by analogy with the linear case [11]. The eigenvalues of the diagonal cells $\mathbf{A}_{kk}$ are always known [11][19]. It was used in many applications requiring the qualitative behavior of the solution and in stability problems, in general (see e.g. [[10-[23], where it was used along with other LEM representations).

3. Normal LEM solutions for Düffing oscillator

We establish here the normal LEM solutions for Düffing oscillator in two different cases: a) free undamped and b) forced damped.

a) FREE UNDAMPED OSCILLATOR

Let us note that in this case, the Düffing equation becomes

$$\ddot{x} + \beta x + \alpha x^3 = 0, \tag{14}$$

and coincides with the intrinsic equation, found as a mathematical hard core of several physical and mechanical phenomena, completely distinct, both mathematically and physically: the Bernoulli-Euler bar deflection, non-linear rigid pendulum's oscillations, the deflection of the non-linear two bar frame, modelled by Teodorescu [25], and Troesch's plasma model. The intrinsic equation was found and studied in [19][25]

$$z_j^{IV} z_j' - z_j''' z_j'' = (-1)^j z_j'' z_j'^3, \quad j = 1,2\ ; \tag{15}$$

its coefficients do not depend on the physical data, thus it was called *intrinsic* [19][23][25]. Consequently, solving the intrinsic equation led to the solutions of each of the above problems. We apply LEM to get the normal LEM representations for z_j, allowing both quantitative and qualitative interpretations.

Equation (15) may be integrated once, to give

$$z'''_j = -\beta z'_j + \frac{(-1)^j}{2} z'^3_j, \tag{16}$$

with β constant. Putting $z'_j = x_j$, this equation becomes

$$x''_j + \beta x_j + \frac{(-1)^{j+1}}{2} x_j^3 = 0, \tag{17}$$

therefore a particular free undamped Düffing's oscillator.

According to [25], we get, with the arbitrary Cauchy data $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$,

1. for $\beta = a^2$, the following formulae

$$\begin{aligned} u(x) &\cong \beta \cos ax + \frac{\gamma}{a} \sin ax + \\ &+ \frac{(-1)^{j+1}}{2^6} \left[x_0^3 \varphi_1(ax) + x_0^2 \dot{x}_0 \varphi_2(ax) + x_0 \dot{x}_0^2 \varphi_3(ax) + \dot{x}_0^3 \varphi_4(ax) \right] \end{aligned} \tag{18}$$

where

$$\begin{aligned} a^2 \varphi_1(\tau) &= \cos 3\tau - \cos \tau - 12\tau \sin \tau, \\ a^3 \varphi_2(\tau) &= +3 \sin 3\tau - 21 \sin \tau + 12\tau \cos \tau, \\ a^4 \varphi_3(\tau) &= 3 \cos \tau - 3 \cos 3\tau - 12\tau \sin \tau, \\ a^5 \varphi_4(\tau) &= -\sin 3\tau - 9 \sin \tau + 12\tau \cos \tau, \end{aligned} \tag{19}$$

and

2. for $\beta = -a^2$, similar formulae, where the trigonometric functions are replaced with corresponding hyperbolic functions.

Commentary. Voinea, [26], emphasized a very interesting analogy between two completely different physical phenomena: the rectilinear displacement in the relativistic frame under a constant force and the large deformations of a straight bar for a constant bending moment and constant rigidity. He showed that the corresponding governing equations differ by a sign and both the solutions for null Cauchy data may be put under a common form of a conic depending on a parameter a. The case $a < 0$ yields a hyperbola and represents the implicit solution of the relativistic Cauchy problem, while $a > 0$ corresponds to an ellipse (or circle) and gives the implicit solution of the standard cantilever bar problem.

Taking into account this analogy, we considered the relativistic model for time-dependent forces on the one hand, and the Bernoulli-Euler bar loaded with variable bending moments and rigidities on the other hand. We firstly reduced each model class to an intrinsic equation, which does not depend on the physical data, and then found the corresponding solutions for associated Cauchy problems with arbitrary data. A third term of comparison was also emphasized: the deflection of a relativistic electron beam (REB) under a magnetic field, previously associated to the Bernoulli-Euler bar deflection [27].

b) FORCED DAMPED OSCILLATOR

Let us take $\beta = 0$ with Ueda. Introducing three auxiliary functions $y = \dot{x}, u = \cos\omega t, v = \sin\omega t$, Düffing's equation may be written in the form of a homogeneous polynomial first order ODS

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\delta y - \alpha x^3 + Au, \\ \dot{u} &= -\omega v, \\ \dot{v} &= \omega u. \end{aligned} \tag{20}$$

The transposed of the associated LEM matrix is then

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11}^{\mathrm{T}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots \\ \mathbf{A}_{13}^{\mathrm{T}} & \mathbf{A}_{33}^{\mathrm{T}} & \mathbf{0} & \mathbf{0} & \dots \\ \mathbf{0} & \mathbf{A}_{35}^{\mathrm{T}} & \mathbf{A}_{55}^{\mathrm{T}} & \mathbf{0} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}, \tag{21}$$

where

$$\mathbf{A}_{11}^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -\delta & 0 & 0 \\ 0 & A & 0 & \omega \\ 0 & 0 & -\omega & 0 \end{bmatrix}, \quad \mathbf{A}_{13}^{\mathrm{T}} = \left[a_{jk} \right]_{\substack{j=\overline{1,20}, \\ k=1,4,}} \quad a_{jk} = -\alpha\delta_1^2 . \tag{22}$$

If we stick to third order effects, then we truncate the LEM matrix to

$$\mathbf{A}_3 = \begin{bmatrix} \mathbf{A}_{11}^{\mathrm{T}} & \mathbf{0} \\ \mathbf{A}_{13}^{\mathrm{T}} & \mathbf{A}_{33}^{\mathrm{T}} \end{bmatrix}. \tag{23}$$

The eigenvalues of $\mathbf{A}_{11}^{\mathrm{T}}$ are

$$0,\ \delta,\ \pm\mathrm{i}\omega; \tag{24}$$

according to the general LEM results [11][19], the eigenvalues of $\mathbf{A}_{33}^{\mathrm{T}}$ will be

$$\begin{aligned}&0,\ 0, -\delta, -\delta, -2\delta, -3\delta, \pm\mathrm{i}\omega, \pm\mathrm{i}\omega, \pm 2\mathrm{i}\omega,\\ &\pm 3\mathrm{i}\omega, \delta\pm\mathrm{i}\omega, \delta\pm 2\mathrm{i}\omega, 2\delta\pm\mathrm{i}\omega.\end{aligned} \tag{25}$$

The associated LEM ODS for up to third order effects will have two blocks; solving it by using the Laplace transformation [21], we find the normal LEM solution for null Cauchy data in the form

$$\begin{aligned}x(t) \cong{}& \frac{A}{\delta^2+\omega^2}\left(\mathrm{e}^{-\delta t} - \cos\omega t + \frac{\delta}{\omega}\sin\omega t\right) - 6\alpha A^3\big[c_0 + c_1\mathrm{e}^{-\delta t}\\ &+ c_2 t\mathrm{e}^{-\delta t} + c_3\mathrm{e}^{-3\delta t} + c_4\cos\omega t + c_5\sin\omega t + c_6\cos 3\omega t\\ &+ c_7\sin 3\omega t + \mathrm{e}^{-\delta t}\left(c_8\cos 2\omega t + c_9\sin 2\omega t\right)\\ &+ \mathrm{e}^{-2\delta t}\left(c_{10}\cos\omega t + c_{11}\sin\omega t\right)\big],\end{aligned} \tag{26}$$

where, with the notation $\sigma^2 = \delta^2 + \omega^2$, the coefficients $c_j, j = \overline{1,10}$, have the following expressions

$$\begin{aligned}&c_0 = \frac{4\delta^2 + 11\omega^2}{9\delta^2\omega^4\left(\delta^2+4\omega^2\right)\left(4\delta^2+\omega^2\right)},\quad c_1 = -\frac{3}{\delta^2\sigma^4\left(\delta^2+9\omega^2\right)},\\ &c_2 = -\frac{1}{4\delta\omega^2\sigma^4},\quad c_3 = \frac{1}{36\delta^2\sigma^6},\quad c_4 = -\frac{\delta^2-\omega^2}{8\omega^4\sigma^6},\quad c_5 = -\frac{2\omega\delta}{8\omega^4\sigma^6},\\ &c_6 = \frac{\delta^4 - 12\delta^2\omega^2 + 3\omega^4}{72\omega^4\sigma^6\left(\delta^2+9\omega^2\right)},\quad c_7 = \frac{2\delta\omega\left(3\delta^2-5\omega^2\right)}{72\omega^4\sigma^6\left(\delta^2+9\omega^2\right)},\\ &c_8 = \frac{-1}{4\sigma^6\left(\delta^2+4\omega^2\right)},\quad c_9 = \frac{\delta\left(\delta^2+3\omega^2\right)}{8\omega^3\sigma^6\left(\delta^2+4\omega^2\right)},\\ &c_{10} = \frac{1}{2\sigma^6\left(4\delta^2+\omega^2\right)},\quad c_{11} = \frac{\delta}{\omega\sigma^6\left(4\delta^2+\omega^2\right)}.\end{aligned} \tag{27}$$

4. Numerical comparison

We compared the values given by the two above LEM formulae with the corresponding numerical solutions obtained by using the Runge-Kutta method, for various ranges of the involved parameters, also establishing the intervals of concordance (denoted by I) of both solutions. Let us note that, immaterial the previously established intervals of convergence of the LEM representations, such a comparison is more realistic, as it can show significant enlargements of the domain of validity of the LEM solutions.

Tables 1 and 2 show this comparison for the free undamped Düffing oscillator (formula (18)) and for the damped forced oscillator (formula (26)) accordingly.

Table 1

Comparison between the LEM formula (18) **and the numerical solution**

β	x_0	$\dot{x}_0$	relative error/step	Interval of concordance (I)
1	0.1	0.1	0.0354	[0,100]
5	0.1	0.1	0.0871	[0, 500]
10	0.1	0.1	0.0867	[0,700]
100	0.1	0.1	0.0710	[0,2300]
1	0.01	0.01	0.0108	[0, 4000]
5	0.01	0.01	0.0584	[0,10000]
10	0.01	0.01	0.0240	[0, 30000]
100	0.01	0.01	0.0543	[0, 1000]

Table 2

Comparison between the LEM formula (26) **and the numerical solution**

δ	ω	A	α	relative error/step	Interval of concordance
0.05	1	7.5	1	0.0883	[0, 1.25]
0.05	5	7.5	0.05	0.0833	[0, 10]
0.05	5	7.5	0.5	0.0727	[0, 3]
0.05	10	7.5	0.5	0.0855	[0, 15]
0.05	10	7.5	1	0.0401	[0,7]
0.05	50	7.5	1	0.0546	[0, 6000]
0.05	100	7.5	1	0.0274	[0, 9000]
0.05	1	5	1	0.0183	[0, 1.3]
0.05	1	1	1	0.0468	[0, 2.5]
0.05	10	1	1	0.08869	[0, 150]
0.05	100	1	1	0.0284	[0, 10000] →
0.5	5	7.5	1	0.0686	[0, 1.3]
0.5	10	7.5	1	0.0725	[0, 40]
0.5	100	1	1	0.0436	[0, 100000] →
10	1	7.5	1	0.0051	[0, 3]

In table 1, the initial values $x_0, \dot{x}_0$; are also considered they are taken around the equilibrium point (0,0) in the phase space. We observe that the greater the value of β , the larger the interval of concordance; yet for large β we note that I is smaller, because the influence of secular terms in the LEM formula is significantly larger in this case. However, one cannot yet speak of long term concordance, because of the secular terms in the LEM formula (18).

Table 2 contains on its first row the Ueda values of the parameters [6]; we see that I is small, even if it contains around 1500 Runge-Kutta steps. For large ω, I is large enough to yield long term concordance; note that, unlike (18), formula (26) contains only terms bounded at infinity; the arrows on the right mean that I can still be larger. Larger damping coefficients seem to have less effect on I. The relative error per step was taken to be no greater that 9% , while in many of the cases it does not exceed 5%.

5. Conclusions

In this paper, we applied the linear equivalence method (LEM), previously introduced by Toma, to Düffing oscillator. We chose the normal LEM representations because they are, in fact, good analytic approximates depending on the involved parameters of the exact solution. Here, we got the normal LEM solution for a particular free undamped and for the forced damped oscillator. The comparison with the numerical solutions obtained by using the Runge-Kutta method showed that the two solutions are concordant on large time intervals for certain ranges of the parameters. This emphasizes the normal LEM solution for Düffing's oscillator as a qualitative tool for studying the long term behaviour of the phenomenon.

R E F E R E N C E S

[1]. *G.Düffing*, Erzwungene Schwingungen bei verändlicher Eigenfrequenz, Vieweg, Braunschweig, 1918.

[2]. *J.M.T. Thompson, H.B. Stewart*, Nonlinear dynamics and chaos, 2nd edition, John Wiley & Sons, 2002.

[3]. *R. Voinea, I. Stroe*, Introducere în teoria sistemelor dinamice, Ed. Academiei Române, Bucureşti, 2000.

[4]. *Christopher Zeeman*, Düffing's equation: catastrophic jumps of amplitude and phase, conference, Univ. of Texas at San Antonio, 31.03.2000.

[5]. *Takashi Kanamaru* , Duffing oscillator, Scholarpedia, **3**(3):6327, 2008.

[6]. *Y. Ueda*, The road to chaos, Aerial Press, 1992.

[7]. *H.P.W. Gottlieb*, Question #38. What is the simplest jerk function that gives chaos?, Am.J. Phys., **64**, 525, 1996.

[8]. *J.C. Sprott*, Some simple chaotic jerk functions, Am. J. Phys. **65**, 6, June 1997, pp.537-543.

[9]. *J.C.Sprott*, Some simple chaotic flows, Phys. Rev. E, **50**,2, 1994, pp. R647-R650.

[10]. *Ileana Toma*, On polynomial differential equations, Bull. Math. Soc. Sci. Math. de la Roumanie, **24**(**72**), *4*, 1980, pp. 417-424.
[11]. *Ileana Toma*, Metoda echivalenței lineare şi aplicațiile ei, Ed. Flores, Bucureşti, 1995.
[12]. *P.P. Teodorescu, N.-A Nicorovici,* Applications of the theory of groups in mechanics and physics, Kluwer, Acad. Publ., Boston/ Dordrecht/ London, 2004.
[13]. *L. Munteanu, Şt. Donescu,* Introduction to the soliton theory, Applications to mechanics, Book Series: Fundamental Theories of Physics, Kluwer Academic Publishers., 2004.
[14]. *P.P. Teodorescu*, Mechanical Systems. Classical Models, Springer, Dordrecht, vol.III, 2009.
[15]. *M.V. Soare, P.P. Teodorescu*, *Ileana Toma*, Ordinary Differential Systems with applications to mechanics, Springer, Dordrecht, 2006.
[16]. *L Munteanu, D. Popa, C. Secara, V. Chiroiu,* The analysis of a system of rigid bodies's dynamics by linear equivalence method, Proc. of the Roumanian Academy, series A, **8**, 2, 2007.
[17]. *L Munteanu, T. Badea, V. Chiroiu,* Linear equivalence method for the analysis of the double pendulum's motion, Complexity International Journal, **9**, pp. 26-43.
[18]. *Ileana Toma*, LEM solutions in mechanics and engineering, Proc. of ICTCAM 2007, 20-23.06.2007, pp.123-128.
[19]. *Ileana Toma*, Metoda echivalenței lineare şi aplicațiile ei în mecanică, Ed. Tehnică, Bucureşti, 2008.
[20]. *Ileana Toma*, Normal LEM representations for the non-linear forced pendulum, II NNMAE, (Nonsmooth/Nonconvex Mechanics with Applications in Engineering, Proc. of the International Conference in Memoriam of Professor P.D. Panagiotopoulos, ed. C.C. Baniotopoulos, Thessaloniki, Greece, 7-8.07.2006, pp.329-332.
[21]. *Ileana Toma*, Specific LEM techniques for some polynomial dynamical systems, Topics in Applied Mechanics, Ed. Academiei Române, Bucharest, eds. V. Chiroiu, T. Sireteanu, vol.III, pp.427-459.
[22]. *Ileana Toma*, The nonlinear pendulum from a LEM perspective, Research Trends in Mechanics, Ed. Academiei Române, Bucharest, eds. D.Popa, V. Chiroiu, I.Toma, vol.I., 2008, pp. 395-422.
[23]. *P.P Teodorescu, Ileana Toma,* Nonlinear elastic deformations treated by LEM, Topics in Applied Mechanics, Ed. Academiei Române, Bucharest, eds. V. Chiroiu, T. Sireteanu, vol.II, 2004, pp.391-442.
[24]. *W.T. Koiter*, Elastic stability and postbuckling behaviour, Proc. Symp. Nonlinear Problems, Ed. By R.E. Langer, Univ. Wesconsin Press, pp. 257, 1963.
[25]. *P.P Teodorescu, Ileana Toma,* A class of elastic structures with the same mathematical core, Honorary volume dedicated to professor emeritus Ioannis D. Mittas, Aristotle Univ. of Thessaloniki, Fac. of Eng., Dept. of Math, Phys. Sci., Division of Math., 200, pp. 499-508.
[26]. *R.P. Voinea,* An analogy, Proc. Ro. Acad. Series A, **1**,1, pp.51-53, 2000.
[27]. *P.P Teodorescu, Ileana Toma* On Voinea's analogy, Proc. of the Romanian Acad., series A, **4**, 3, 2003, pp. 157-166.