# DISCUSSION ON GENERALIZED NONLINEAR CONTRACTIONS

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*In this paper, we present some fixed point theorems for generalized nonlinear contractions involving a new pair of auxiliary functions in a complete metric space. Our newly proved results generalize, extend, unify, enrich, sharpen and improve some well-known fixed point theorems existing in the literature. Finally, we also provide an example, which substantiates the utility of our results.*

**Keywords:** fixed points, ($\phi$-$\psi$)-contractions, complete metric space.

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## 1. Introduction

Metric fixed point theory is a relatively old but still a young area of research which occupies an important place in nonlinear functional analysis. In fact, the strength of fixed point theory lies in its wide range of applications within and outside mathematics. Fixed point theory has provided powerful tools in the existence and uniqueness theories of ordinary differential equations, partial differential equations, integral equations, functional equations, matrix equations, random differential equations, variational inequalities *etc.* besides facilitating various problems arising in different fields such as: functional analysis, topology, operator theory, differential geometry, eigen value problems, approximation theory *etc.* Many practical and research problems in various fields beyond mathematics can be reduced to fixed point problems, which include statistics, operations research, physics, engineering, computer science, chemistry, biology, economics, global analysis and several others in dealing with various mathematical models representing phenomena arising in probability theory, optimization theory, game theory, fractal theory, control theory, potential theory, electrical heating in Joule-Thomson effect, fluid flow, steady state temperature distribution, chemical equations, neutron transport theory, Nash equilibria, econometrics, equilibrium points in economy, epidemics *etc.*

Indeed, the most popular result of metric fixed point theory is the Banach contraction principle(BCP), which is essentially due to S. Banach [1] (proved in 1922). This classical result guarantees that a nonlinear operator admits a fixed point. After the appearance of the BCP, lots of generalizations, in many different frameworks, have been presented, e.g., [2, 3, 4, 5, 8, 11, 20, 12, 13, 14, 15, 16, 17, 18, 19] and reference therein. In particular, many authors extended Banach contraction principle employing relatively more general contractive conditions. Some novel fixed point theorems involve contractivity conditions depending on one or more auxiliary functions (such as: control functions, comparison functions, altering distance functions, Geraghty functions, simulation functions, L-functions etc). One

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noted generalization of this kind was given by Dutta and Choudhury [20], often referred as generalized nonlinear contraction involving two auxiliary functions which was later generalized and improved by many authors (see [21, 22, 23, 24]).

The aim of this manuscript is to extend the previous families of pair of auxiliary functions and utilize the same to extend the Banach contraction principle under generalized nonlinear contraction. An example is also presented, which attests to the credibility of our results.

## 2. Preliminaries

Throughout the manuscript, $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{R}$ denote the sets of positive integers, non-negative integers and real numbers respectively (i.e., $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$). In this section, we summarize several core fixed point theorems under generalized contractivity conditions involving auxiliary functions. Recall that a self-mapping $T$ defined on a metric space $(X, d)$ satisfying

$$d(Tx, Ty) \leq kd(x, y) \quad \forall\ x, y \in X \tag{1}$$

where $k \in [0, 1)$ is a constant, is called a linear contraction with respect to $k$ (or, in short, $k$-contraction). Here, it can be noticed that the non-negative constant $k < 1$ plays a key role. Many authors generalized Banach contraction principle by replacing the involved constant $k$ with an auxiliary mapping. One noted generalization of this kind is $\varphi$-contraction. Recall that a self-mapping $T$ defined on a metric space $(X, d)$ is called a nonlinear contraction with respect to $\varphi$, or in short, $\varphi$-contraction if $\varphi : [0, \infty) \to [0, \infty)$ is a mapping (called control function) such that $\varphi(t) < t$ for all $t > 0$, satisfying

$$d(Tx, Ty) \leq \varphi(d(x, y)) \quad \forall\ x, y \in X.$$

On setting $\varphi(t) = kt$ where $0 \leq k < 1$, $\varphi$-contraction reduces to linear contraction. Indeed, the idea of $\varphi$-contraction was initiated by Browder [2] in 1968, which was further improved by Boyd and Wong [3], Matkowski [4], Mukherjea [5] and Jotic [6]. Alam et al. [7] established possible inter-relations among different existing control functions.

Following Khan et al. [8], a function $\phi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied:

(i) $\phi(t) = 0$ iff $t = 0$,
(ii) $\phi$ is continuous,
(iii) $\phi$ is monotone increasing.

Notice that the condition (i) is equivalent to $\phi^{-1}(0) = \{0\}$. Let $\mathcal{F}_{\text{alt}}$ denotes the family of altering distance functions.

**Theorem 2.1.** [8] *Let $(X, d)$ be a complete metric space and $T$ a self-mapping on $X$. If there exists $\phi \in \mathcal{F}_{\text{alt}}$ and a constant $c \in [0, 1)$ such that*

$$\phi(d(Tx, Ty)) \leq c\phi(d(x, y)) \quad \forall\ x, y \in X,$$

*then $T$ has a unique fixed point.*

Under the restriction $\phi = I$, identity map on $[0, \infty)$, Theorem 2.1 reduces to Banach contraction principle. It is straightforward that the contraction condition (1) is equivalent to

$$d(Tx, Ty) \leq d(x, y) - \lambda d(x, y)\ (0 < \lambda \leq 1) \quad \forall\ x, y \in X, \tag{2}$$

which can be derived from (1) by setting $k = 1 - \lambda$. A nonlinear formulation of inequality (2) is called weak contraction. Thus far, a self-mapping $T$ defined on a metric space $(X, d)$ is called weak $\psi$-contraction if $\psi : [0, \infty) \to [0, \infty)$ is a mapping satisfying

$$d(Tx, Ty) \leq d(x, y) - \psi(d(x, y)) \quad \forall\ x, y \in X.$$

Under the restriction $\psi(t) = (1-k)t$, weak $\psi$-contraction reduces to linear contraction. The notion of weak contraction is initiated by Krasnosel'skiĭ et al. [9]. In 1997, Alber and Guerre-Delabriere [10] proved some fixed point theorems under weak contractions in the setting of Hilbert spaces. Later, Rhoades [11] observed that the results of Alber and Guerre-Delabriere [10] are still true for a class of Banach space. Also, Rhoades [11] extended the Banach contraction principle to weak contraction, which runs as follows:

**Theorem 2.2.** [11] *Let $(X, d)$ be a complete metric space and $T$ a self-mapping on $X$. If there exists $\psi \in \mathcal{F}_{\text{alt}}$ such that*

$$d(Tx, Ty) \leq d(x, y) - \psi(d(x, y)) \quad \forall\ x, y \in X,$$

*then $T$ has a unique fixed point.*

On combining the ideas involved in Theorem 2.1 and Theorem 2.2, Dutta and Choudhury [20] introduced the concept of generalized nonlinear contraction involving two auxiliary functions $\phi$ and $\psi$, which is also referred as $(\phi, \psi)$-contraction. Utilizing the idea of $(\phi, \psi)$-contraction, Dutta and Choudhury [20] obtained the following novel generalization of Banach contraction principle:

**Theorem 2.3.** [20] *Let $(X, d)$ be a complete metric space and $T : X \to X$ a mapping. If there exist $\phi, \psi \in \mathcal{F}_{\text{alt}}$ such that*

$$\phi(d(Tx, Ty)) \leq \phi(d(x, y)) - \psi(d(x, y)) \quad \forall\ x, y \in X,$$

*then $T$ has a unique fixed point.*

On particularizing at $\psi(t) = (1 - c)\phi(t)$ where $0 \leq c < 1$, Theorem 2.3 reduces to Theorem 2.1. Again, if we consider $\phi$ as identity map, then Theorem 2.3 reduces to Theorem 2.2. Jachymski [25] established a result regarding some existing contractive conditions, which are equivalent to $(\phi, \psi)$-contractions (see Theorem 3 [25]).

In 2009, Doric [21] observed that the auxiliary function $\psi$ utilized in Theorem 2.3 need not be an altering distance function. He removed monotonicity of $\psi$ and replaced its continuity by lower semicontinuity. Denote the family of auxiliary functions due to Doric [21] by

$$\Psi' = \{\psi : [0, \infty) \to [0, \infty) : \psi^{-1}(0) = \{0\} \text{ and } \psi \text{ is lower semicontinuous}\}.$$

In 2011, Popescu [22] further generalized the fixed point results of Doric [21] by considering the following family of auxiliary functions:

$$\Psi'' = \{\psi : [0, \infty) \to [0, \infty) : \psi^{-1}(0) = \{0\} \text{ and } \liminf_{n\to\infty} \psi(t_n) > 0 \text{ if } \lim_{n\to\infty} t_n = r > 0\}.$$

Luong and Thuan [23] proved some coupled fixed point theorems in an ordered metric space under $(\phi, \psi)$-contractions, wherein they independently defined the following family:

$$\Psi''' = \{\psi : [0, \infty) \to [0, \infty) :\ \psi(t) = 0 \Rightarrow t = 0,\ \lim_{t\to 0^+} \psi(t) = 0 \text{ and } \lim_{t\to r} \psi(t) > 0\ \forall\ r > 0\}.$$

Notice that Luong and Thuan [23] used the first condition (i.e., $\psi(t) = 0 \Rightarrow t = 0$) but failed to mention it. Second condition of $\Psi'''$ is relatively weaker as compared to the condition $\psi(0) = 0$ of $\Psi''$, while the last condition of $\Psi'''$ is relatively stronger than the last condition of $\Psi''$.

On the other hand, to remove the continuity requirement on $\phi$, Roldán-López-de-Hierro [24] considered the following two families:

$$\Phi' = \{\phi : [0, \infty) \to [0, \infty) : \phi^{-1}(0) = \{0\} \text{ and } \phi \text{ is upper semicontinuous and monotone increasing}\}$$

$$\Phi'' = \{\phi : [0, \infty) \to [0, \infty) : \phi^{-1}(0) = \{0\} \text{ and } \phi \text{ is right continuous and monotone increasing}\}.$$

Recall that a function $\phi : [0,\infty) \to [0,\infty)$ is right continuous at $r \in [0,\infty)$ if for any sequence $\{t_n\} \subset [0,\infty)$ such that $t_n > r$ for all $n \in \mathbb{N}$ and $t_n \xrightarrow{\mathbb{R}} r$, we have $\phi(r) = \lim\limits_{n\to\infty} \phi(t_n)$.

## 3. New Auxiliary Functions

As discussed in the previous section, various authors imposed a common conditions on both auxiliary functions, $\phi^{-1}(0) = \{0\}$ and $\psi^{-1}(0) = \{0\}$. We observe that such condition is unnecessary on $\phi$. Also, we must withdraw the condition $\psi(0) = 0$ as there is no need of this condition in proof, however this condition is automatically appeared under $(\phi,\psi)$-contractivity condition (as for $x = y$, the contractivty condition: $\phi(d(Tx,Ty)) \leq \phi(d(x,y)) - \psi(d(x,y))$ gives rise $\psi(0) = 0$). Thus far, we define the following new families of auxiliary functions:

$\Phi$ denotes the family of functions $\phi : [0,\infty) \to [0,\infty)$ satisfying the following axioms:

$\Phi_1$ : $\phi$ is right continuous,
$\Phi_2$ : $\phi$ is monotone increasing.

$\Psi$ denotes the family of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following axioms:

$\Psi_1$ : $\psi(t) > 0 \quad \forall\, t > 0$,
$\Psi_2$ : $\liminf\limits_{t\to r} \psi(t) > 0 \quad \forall\, r > 0$.

**Remark 3.1.** *Clearly, axiom $\Psi_1$ is equivalent to the following:*

$\Psi_1'$ : *If there exists $t_0 \in [0,\infty)$ such that $\psi(t_0) = 0$, then $t_0 = 0$.*

**Proposition 3.1.** [24] *If $\phi : [0,\infty) \to [0,\infty)$ is a monotonic increasing and upper semicontinuous function, then it is also right continuous.*

*Proof.* Let $\{t_n\} \subset [0,\infty)$ be a sequence such that $t_n > r$ for all $n \in \mathbb{N}$ and $t_n \to r$. Using upper semicontinuity of $\phi$, we get

$$\limsup_{n\to\infty} \phi(t_n) \leq \phi(r). \tag{3}$$

As $r < t_n$, due to monotonicity of $\phi$, we have $\phi(r) \leq \phi(t_n)$. Taking lower limit on both the sides, we get

$$\phi(r) \leq \liminf_{n\to\infty} \phi(t_n). \tag{4}$$

Combining (3) and (4), we obtain $\limsup\limits_{n\to\infty} \phi(t_n) \leq \phi(r) \leq \liminf\limits_{n\to\infty} \phi(t_n)$ yielding thereby $\lim\limits_{n\to\infty} \phi(t_n) = \phi(r)$, which shows that $\phi$ is right continuous. □

**Remark 3.2.** *Using the fact that continuity implies upper semicontinuity and by Proposition 3.1, the class $\Phi$ enlarges the classes $\mathcal{F}_{\rm alt}$, $\Phi'$ and $\Phi''$ under the following inclusion relation:*

$$\mathcal{F}_{\rm alt} \subset \Phi' \subset \Phi'' \subset \Phi.$$

**Proposition 3.2.** *If $\psi : [0,\infty) \to [0,\infty)$ is a continuous function, which satisfies axiom $\Psi_1$, then*

(i) $\lim\limits_{n\to\infty} \psi(t_n) > 0$ whenever $\lim\limits_{n\to\infty} t_n = r > 0$,
(ii) $\lim\limits_{t\to r} \psi(t) > 0 \ \forall\, r > 0$.

*Proof.* (i) Given that $t_n \xrightarrow{\mathbb{R}} r$ and $r > 0$. Using continuity of $\psi$, we get $\lim\limits_{n\to\infty} \psi(t_n) = \psi(r)$. But by axiom $\Psi_1$, $\psi(r) > 0$ and hence, we obtain $\lim\limits_{n\to\infty} \psi(t_n) > 0$.
(ii) Take an arbitrary $r > 0$ and using continuity of $\psi$, we get $\lim\limits_{t\to r} \psi(t) = \psi(r)$, which in lieu of $\Psi_1$ reduces to $\lim\limits_{t\to r} \psi(t) > 0$. □

**Proposition 3.3.** *If $\psi : [0, \infty) \to [0, \infty)$ is a lower semicontinuous function, which satisfies axiom $\Psi_1$, then*

(i) $\liminf_{n\to\infty} \psi(t_n) > 0$ whenever $\lim_{n\to\infty} t_n = r > 0$,

(ii) $\liminf_{t\to r} \psi(t) > 0 \ \forall\ r > 0$.

*Proof.* (i) Given that $t_n \xrightarrow{\mathbb{R}} r$ and $r > 0$. Using lower semicontinuity of $\psi$, we get $\liminf_{n\to\infty} \psi(t_n) \geq \psi(r)$. But by axiom $\Psi_1$, $\psi(r) > 0$ and hence, we obtain $\liminf_{n\to\infty} \psi(t_n) > 0$.

(ii) Using the sequential approach of lower limit, for all sequences $\{t_n\}$ converging to $r$, we have, $\liminf_{t\to r} \psi(t) = \liminf_{n\to\infty} \psi(t_n)$. As $r > 0$, in view of (i), above equation reduces to $\liminf_{t\to r} \psi(t) > 0$. □

**Remark 3.3.** *Using the fact that continuity implies lower semicontinuity and by Propositions 3.2 and 3.3, we conclude that the class $\Psi$ enlarges the classes $\mathcal{F}_{\text{alt}}$, $\Psi'$ and $\Psi''$ under the following inclusion relation:*

$$\mathcal{F}_{\text{alt}} \subset \Psi' \subset \Psi'' \subset \Psi$$

$$\mathcal{F}_{\text{alt}} \subset \Psi''' \subset \Psi.$$

**Proposition 3.4.** *If there exists a pair of auxiliary functions $\phi, \psi : [0, \infty) \to [0, \infty)$, wherein $\phi$ satisfies axiom $\Phi_2$ while $\psi$ satisfies axiom $\Psi_1$, such that for all $s \in [0, \infty)$ and $t \in (0, \infty)$,*

$$\phi(s) \leq \phi(t) - \psi(t),$$

*then $s < t$.*

*Proof.* Given that

$$\phi(s) \leq \phi(t) - \psi(t). \tag{5}$$

We have to show that $s < t$. On contrary suppose that $t \leq s$, then using property $\Phi_2$, we obtain $\phi(t) \leq \phi(s)$. Thus, the identity (5) reduces to

$$\phi(s) \leq \phi(s) - \psi(t)$$

so that $\psi(t) = 0$, which contradicts the property $\Psi_1$. Therefore, we conclude that $s < t$. □

**Proposition 3.5.** *Let $(X, d)$ be a metric space and $T$ a self-mapping on $X$. If there exists a pair of auxiliary functions $\phi, \psi : [0, \infty) \to [0, \infty)$, wherein $\phi$ satisfies axiom $\Phi_2$ while $\psi$ satisfies axiom $\Psi_1$, such that $T$ is $(\phi, \psi)$-contraction, then $T$ is contractive and hence is continuous.*

*Proof.* Take two distinct elements $x, y \in X$ so that $d(x, y) > 0$. Applying contractivity condition on this pair, we get

$$\phi(d(Tx, Ty)) \leq \phi(d(x, y)) - \psi(d(x, y)).$$

By Proposition 3.4, we have

$$d(Tx, Ty) < d(x, y).$$

Thus, the conclusion is immediate. □

**Definition 3.1.** *A function $\alpha : [0, \infty) \to [0, 1)$ is a Geraghty function if for any sequence $\{t_n\} \subset [0, \infty)$,*

$$\alpha(t_n) \xrightarrow{\mathbb{R}} 1 \Rightarrow t_n \xrightarrow{\mathbb{R}} 0.$$

The class of Geraghty functions is introduced by Geraghty [28].

**Proposition 3.6.** *Let $\alpha$ be a Geraghty function. Define $\phi(t) = t$ and $\psi(t) = \big(1 - \alpha(t)\big)t$ for all $t \geq 0$. Then $\phi \in \Phi$ and $\psi \in \Psi$.*

*Proof.* Obviously $\phi$ being identity function belongs to the family $\Phi$. Henceforth, we have to show only that $\psi \in \Psi$. If there is some $t_0 \in [0, \infty]$, such that $\psi(t_0) = 0$ then, we have

$$\big(1 - \alpha(t_0)\big)t_0 = 0 \Longrightarrow \alpha(t_0) = 1 \text{ or } t_0 = 0 \Longrightarrow t_0 = 0 \text{ (as } \alpha(t_0) = 1 \text{ is not defined)}$$

implying thereby $\psi(t) > 0 \quad \forall\ t > 0$. Clearly, $\liminf\limits_{t\to r} \psi(t) \geq 0$ for any $r \geq 0$. Now, we show that $\liminf\limits_{t\to r} \psi(t) > 0$ for all $r > 0$. Suppose on contrary that $\liminf\limits_{t\to r'} \psi(t) = 0$ for some $r' > 0$. Then, there exists a sequence $\{t_n\} \subset [0, \infty)$ converging to $r'$ such that

$$\lim_{n\to\infty} \psi(t_n) = 0. \tag{6}$$

As $r' > 0$, there exists $N$ such that $t_n \neq 0$ for all $n \geq N$. Now, we can write

$$1 - \alpha(t_n) = \frac{\psi(t_n)}{t_n} \quad \forall\ n \geq N.$$

Taking limit $n \to \infty$ on both the sides, we get

$$\lim_{n\to\infty} (1 - \alpha(t_n)) = \lim_{n\to\infty} \frac{\psi(t_n)}{t_n}$$

which on using (6) gives rise $\lim\limits_{n\to\infty} (1 - \alpha(t_n)) = 0$ implying thereby $\lim\limits_{n\to\infty} \alpha(t_n) = 1$. Now, using the property of $\alpha$, we get

$$\lim_{n\to\infty} t_n = 0.$$

By the uniqueness of limit, we have $r' = 0$, which is a contradiction. Therefore, $\psi \in \Psi$, which concludes the proof. □

**Definition 3.2.** *A function $\varphi : [0, \infty) \to [0, \infty)$ is a BW function if*

(i) $\varphi(t) < t$ *for each* $t > 0$
(ii) $\limsup\limits_{t\to r^+} \varphi(t) < r$ *for each* $t > 0$.

*The class of BW functions is introduced by Boyd and Wong* [3].

**Proposition 3.7.** *Let $\varphi : [0, \infty) \to [0, \infty)$ be a BW function. Define $\phi(t) = t$ and $\psi(t) = t - \varphi(t)$. Then $\phi \in \Phi$ and $\psi \in \Psi$.*

*Proof.* Notice that $\phi \in \Phi$ is obvious. Thus, we have to show only that $\psi \in \Psi$. As $\varphi(t) < t$ for all $t > 0$, we have $\psi(t) = t - \varphi(t) > 0$ for all $t > 0$. Thus, $\Psi_1$ is verified. To verify $\Psi_2$, suppose on contrary that $\liminf\limits_{t\to r'} \psi(t) = 0$ for some $r' > 0$. Now, we have

$$\liminf_{t\to r'^+} \psi(t) = r' - \limsup_{t\to r'^+} \varphi(t),$$

which gives $0 = r' - \limsup\limits_{t\to r'^+} \varphi(t)$, implying thereby $\limsup\limits_{t\to r'^+} \varphi(t) = r'$, which contradicts to the property $(ii)$ of $\varphi$. Therefore, $\liminf\limits_{t\to r} \psi(t) > 0$ for all $r > 0$, which concludes the proof. □

Finally, we indicate the following classical and well-known result, which is needed in the proof of our main result.

**Lemma 3.1.** *Let $(X, d)$ be a metric space and $\{x_n\}$ a sequence in $X$. If $\{x_n\}$ is not a Cauchy sequence, then there exist $\epsilon > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that*

(i) $k \leq m_k < n_k\ \forall\ k \in \mathbb{N}$,
(ii) $d(x_{m_k}, x_{n_k}) > \epsilon$,
(iii) $d(x_{m_k}, x_{n_k-1}) \leq \epsilon$.

*In addition to this, if $\{x_n\}$ also verifies $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = 0$, then*

(iv) $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon$,

(v) $\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k-1}) = \epsilon$.

## 4. **Main Results**

Now, we prove our main result regarding the existence and uniqueness of fixed points under generalized nonlinear contractions employing newly introduced auxiliary functions.

**Theorem 4.1.** *Let $(X, d)$ be a complete metric space and $T$ a self-mapping on $X$. If there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that $T$ is $(\phi, \psi)$-contraction, then $T$ has a unique fixed point.*

*Proof.* As $T$ is $(\phi, \psi)$-contraction, we have

$$\phi(d(Tx, Ty)) \leq \phi(d(x, y)) - \psi(d(x, y)) \ \forall\ x, y \in X. \tag{7}$$

Pick up a point $x_0 \in X$. We construct the sequence $\{x_n\}$ of Picard iteration based at the initial point $x_0$ such that

$$x_{n+1} = T(x_n) \ \forall\ n \in \mathbb{N}_0. \tag{8}$$

If there exists some $n_0 \in \mathbb{N}_0$ such that $d(x_{n_0}, x_{n_0+1}) = 0$, then using (8), we conclude that $x_{n_0}$ is a fixed point of $T$. Otherwise, suppose that $d(x_n, x_{n+1}) \neq 0$ for all $n \in \mathbb{N}_0$. Denoting $d_n := d(x_n, x_{n+1})$ and applying the contractivity condition (7), for each $n \in \mathbb{N}_0$, we get

$$\begin{aligned}\phi(d(x_{n+1}, x_{n+2})) &= \phi(d(Tx_n, Tx_{n+1}))\\ &\leq \phi(d(x_n, x_{n+1})) - \psi(d(x_n, x_{n+1}))\end{aligned}$$

or

$$\phi(d_{n+1}) \leq \phi(d_n) - \psi(d_n). \tag{9}$$

In view of Proposition 3.4, (9) gives rise

$$d_{n+1} < d_n \ \forall\ n \in \mathbb{N}_0$$

which yields that the sequence $\{d_n\}$ is a decreasing sequence of positive real numbers. Since it is bounded below (as $d_n > 0$), there is an element $r \geq 0$ such that

$$\lim_{n\to\infty} d_n = r.$$

Now, we show that $r = 0$. On contrary, suppose that $r > 0$. Using the facts that $d_n \xrightarrow{\mathbb{R}} r$ and $d_n > r$ for all $r \in \mathbb{N}_0$ and by the right continuity of $\phi$, we have

$$\limsup_{n\to\infty} \phi(d_n) = \phi(r). \tag{10}$$

Taking upper limit in (9), we get

$$\begin{aligned}\limsup_{n\to\infty} \phi(d_{n+1}) &\leq \limsup_{n\to\infty} \phi(d_n) + \limsup_{n\to\infty}[-\psi(d_n)]\\ &\leq \limsup_{n\to\infty} \phi(d_n) - \liminf_{n\to\infty} \psi(d_n).\end{aligned}$$

Making use of identity (10) above inequality reduces to

$$\phi(r) \leq \phi(r) - \liminf_{n\to\infty} \psi(d_n)$$

implying thereby

$$\liminf_{d_n\to r} \psi(d_n) = \liminf_{n\to\infty} \psi(d_n) \leq 0,\ r > 0$$

which contradicts to the property of $\Psi_2$. Therefore, we have

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{11}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence. On contrary, suppose that $\{x_n\}$ is not a Cauchy sequence. Therefore, by Lemma 3.1, there exist $\epsilon > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that

$$k \leq m_k < n_k,\ d(x_{m_k}, x_{n_k}) > \epsilon \geq d(x_{m_k}, x_{n_{k-1}})\ \ \forall\ k \in \mathbb{N}. \tag{12}$$

Denote $\delta_k := d(x_{m_k}, x_{n_k})$ and $\mu_k := d(x_{m_k-1}, x_{n_k-1})$. Due to availability of (11), Lemma 3.1 assures us that

$$\lim_{k\to\infty} \delta_k = \lim_{k\to\infty} \mu_k = \epsilon. \tag{13}$$

Hence, applying contractivity condition ((7)), we get for each $k \in \mathbb{N}$

$$\begin{aligned} \phi(d(x_{m_k}, x_{n_k})) &= \phi(d(Tx_{m_k-1}, Tx_{n_k-1})) \\ &\leq \phi(d(x_{m_k-1}, x_{n_k-1})) - \psi(d(x_{m_k-1}, x_{n_k-1})) \end{aligned}$$

so that

$$\phi(\delta_k) \leq \phi(\mu_k) - \psi(\mu_k). \tag{14}$$

As from (12), we have $\delta_k > \epsilon$ for each $k \in \mathbb{N}$, therefore using (13) and right continuity of $\phi$, we obtain

$$\limsup_{k\to\infty} \phi(\delta_k) = \phi(\epsilon). \tag{15}$$

We asserts that $\mu_k > 0$. To substantiate our claim, assume that there exists some $k_0 \in \mathbb{N}$ such that $\mu_{k_0} = d(x_{m_{k_0}-1}, x_{n_{k_0}-1}) = 0$. Then, we have $\delta_{k_0} = d(Tx_{m_{k_0}-1}, Tx_{n_{k_0}-1}) = 0$, which contradicts to $0 < \epsilon < \delta_{k_0}$. Now, making use of Proposition 3.4 in (14), we obtain $\delta_k < \mu_k$ yielding thereby $\epsilon < \mu_k$, for each $k \in \mathbb{N}$. Therefore by using again (13) and right continuity of $\phi$, we get

$$\limsup_{k\to\infty} \phi(\mu_k) = \phi(\epsilon). \tag{16}$$

Taking upper limit in inequality (14), we get

$$\limsup_{k\to\infty} \phi(\delta_k) \leq \limsup_{k\to\infty} \phi(\mu_k) + \limsup_{k\to\infty} [-\psi(\mu_k)],$$

which using (15) and (16) becomes

$$\phi(\epsilon) \leq \phi(\epsilon) - \liminf_{k\to\infty} \psi(\mu_k)$$

so that

$$\liminf_{\mu_k\to\epsilon} \psi(\mu_k) = \liminf_{k\to\infty} \psi(\mu_k) \leq 0,$$

which contradicts to the property $\Psi_2$. It follows that $\{x_n\}$ is a Cauchy sequence. Consequently, completeness of $X$ guarantees the existence of $\overline{x} \in X$ such that $x_n \xrightarrow{d} \overline{x}$. By Proposition 3.5, $T$ being $(\phi, \psi)$-contraction is continuous, which yields that $x_{n+1} = T(x_n) \xrightarrow{d} T(\overline{x})$. Owing to the uniqueness of limit, we obtain $T(\overline{x}) = \overline{x}$ so that $\overline{x}$ is a fixed point of $T$.

To prove uniqueness of fixed points, suppose that $\overline{x}$ and $\overline{y}$ are two fixed points of $T$. Applying contractivity condition (7), we obtain

$$\phi(d(\overline{x}, \overline{y})) = \phi(d(T\overline{x}, T\overline{y})) \leq \phi(d(\overline{x}, \overline{y})) - \psi(d(\overline{x}, \overline{y}))$$

yielding thereby

$$\psi(d(\overline{x}, \overline{y})) = 0$$

which using axiom $\Psi'_1$ gives rise $\overline{x} = \overline{y}$. Therefore, $T$ has a unique fixed point. □

## 5. **Consequences**

In view of Remarks 3.2 and 3.3, we conclude that Theorem 4.1 generalizes, extends, enrichs and sharpens the existing fixed point results due to Dutta and Choudhury [20], Doric [21], Popescu [22], Luong and Thuan [23] and Roldán-López-de-Hierro [24]. In the following lines, we deduce several more well-known results using Theorem 4.1.

Taking $\phi(t) = t$ and $\psi(t) = (1-k)t$ where $0 \le k < 1$ in Theorem 4.1, we get the classical Banach contraction principle, which runs as follows:

**Corollary 5.1.** *Let $(X, d)$ be a complete metric space and $T$ a self-mapping on $X$. If there exists a constant $k \in [0, 1)$ such that*

$$d(Tx, Ty) \le kd(x, y) \quad \forall\ x, y \in X,$$

*then $T$ has a unique fixed point.*

Taking $\psi(t) = (1-c)\phi(t)$ where $0 \le c < 1$ in Theorem 4.1, we get a sharpened version of Theorem 2.1, which runs as follows:

**Corollary 5.2.** *Let $(X, d)$ be a complete metric space and $T$ a self-mapping on $X$. If there exists $\phi \in \Phi$ and a constant $c \in [0, 1)$ such that*

$$\phi(d(Tx, Ty)) \le c\phi(d(x, y)) \quad \forall\ x, y \in X,$$

*then $T$ has a unique fixed point.*

Under the restriction $\phi = I$, the identity mapping, Theorem 4.1 reduces to the following enriched version of Theorem 2.2.

**Corollary 5.3.** *Let $(X, d)$ be a complete metric space and $T$ a self-mapping on $X$. If there exists $\psi \in \Psi$ such that*

$$d(Tx, Ty) \le d(x, y) - \psi(d(x, y)) \quad \forall\ x, y \in X,$$

*then $T$ has a unique fixed point.*

Setting $\phi(t) = t$ and $\psi(t) = (1 - \alpha(t))t$ in Theorem 4.1 and using Proposition 3.6, we deduce the following fixed point theorem of Geragthy [28].

**Corollary 5.4.** [28] *Let $(X, d)$ be a complete metric space and $T$ a self-mapping on $X$. If there exists a Geragthy function $\alpha$ such*

$$d(Tx, Ty) \le \alpha(d(x, y))d(x, y) \quad \forall\ x, y \in X,$$

*then $T$ has a unique fixed point.*

Setting $\phi(t) = t$ and $\psi(t) = t - \varphi(t)$ in Theorem 4.1 and using Proposition 3.7, we deduce the following fixed point theorem of Boyd and Wong [3].

**Corollary 5.5.** [3] *Let $(X, d)$ be a complete metric space and $T$ a self-mapping on $X$. If there exists a BW function $\varphi$ such that*

$$d(Tx, Ty) \le \varphi(d(x, y)) \quad \forall\ x, y \in X,$$

*then $T$ has a unique fixed point.*

The following result due to Eslamian and Akbar [27] can also be deducible from our main result.

**Corollary 5.6.** [27] *Let $(X, d)$ be a complete metric and $T : X \to X$ be such that*

$$\phi(d(Tx, Ty)) \le \alpha(d(x, y)) - \beta(d(x, y)) \quad \forall\ x, y \in X,$$

*where $\psi, \alpha, \beta : [0, \infty) \to [0, \infty)$ are such that $\psi$ is continuous and increasing, $\alpha$ is continuous, $\beta$ is lower semi-continuous, $\psi(t) = 0$ if and only if $t = 0$, $\alpha(0) = \beta(0) = 0$ and $\psi(t) - \alpha(t) + \beta(t) > 0$ for all $t > 0$. Then $T$ has a unique fixed point.*

*Proof.* Let $\phi \in \Phi$ be an arbitrary function. Define an auxiliary function $\psi : [0,\infty) \to [0,\infty)$ by

$$\psi(t) = \beta(t) - \alpha(t) + \phi(t).$$

It can be easily verified that $\psi \in \Psi$. Now, for all $x, y \in X$, we have

$$\begin{aligned} \phi(d(Tx,Ty)) &\leq \alpha(d(x,y)) - \beta(d(x,y)) \\ &= \phi(d(x,y)) - [\beta(d(x,y)) - \alpha(d(x,y)) + \phi(d(x,y))] \end{aligned}$$

which becomes

$$\phi(d(Tx,Ty)) \leq \phi(d(x,y)) - \psi(d(x,y)).$$

Hence, using Theorem 4.1, our result follows. □

## 6. An Illustrative Example

Finally, we furnish the following example to demonstrate the validity and utility of our newly proved result.

**Example 6.1.** *Consider $X = [0,1] \cup \{2,3,4,...\}$ with the metric $d$ defined by*

$$d(x,y) = \begin{cases} |x-y|, & \text{if } x,y \in [0,1],\ x \neq y, \\ x+y, & \text{if at least one of } x \text{ or } y \text{ does not belong to } [0,1] \text{ and } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

*Then $X$ is complete metric space, see* [3].

*Define the auxiliary functions $\psi, \phi : [0,\infty) \to [0,\infty)$ as follows:*

$$\phi(t) = \begin{cases} t+1, & \text{if } 0 \leq t < 1 \\ t^2, & \text{if } t \geq 1 \end{cases} \quad \text{and} \quad \psi(t) = \begin{cases} \frac{t^2}{4}, & \text{if } 0 \leq t \leq 1, \\ \frac{1}{3}, & \text{if } t > 1. \end{cases}$$

*Clearly, $\phi \in \Phi$ and $\psi \in \Psi$. Here $\phi$ and $\psi$ both are not continuous and $\phi(0) = 1$.*

*Let $T : X \to X$ be a mapping defined by*

$$T(x) = \begin{cases} x - \frac{x^2}{4}, & \text{if } 0 \leq x \leq 1, \\ x-1, & \text{if } x \in \{2,3,4,...\}. \end{cases}$$

*Notice that the $(\phi,\psi)$-contractivity condition holds trivially when $x = y$. Suppose $x \neq y$. Then, without loss of generality, we can assume that $x > y$. Now the following cases arise:*

***Case 1:*** *When $x \in [0,1]$, then by routine calculation it can be observed that $d(Tx,Ty) \in [0,1)$. Therefore, we have*

$$\begin{aligned} \phi(d(Tx,Ty)) &= (x - \frac{1}{4}x^2) - (y - \frac{1}{4}y^2) + 1 \\ &\leq (x-y) + 1 - \frac{1}{4}(x-y)^2 \\ &= \phi(d(x,y)) - \psi(d(x,y)). \end{aligned}$$

***Case 2:*** *When $x \in \{3,4,....\}$, then there are two possibilities of choosing $y$. Firstly, we take $y \in [0,1]$, then we have*

$$d(Tx,Ty) = d\Big(x-1, y - \frac{1}{4}y^2\Big) = x - 1 + y - \frac{y^2}{4} \leq x + y - 1.$$

*Otherwise, if* $y \in \{2, 3, 4, ...\}$*, then we have*

$$d(Tx, Ty) = d(x - 1, y - 1) = x + y - 2 < x + y - 1.$$

*Therefore, in both the cases, we have*

$$\begin{aligned}\phi(d(Tx, Ty)) &= (d(Tx, Ty))^2 < (x + y - 1)^2 \\ &< (x + y - 1)(x + y + 1) = (x + y)^2 - 1 \\ &< (x + y)^2 - \frac{1}{3} = \phi(d(x, y)) - \psi(d(x, y)).\end{aligned}$$

***Case 3:*** *When* $x = 2$*, then there are two possibilities of choosing* $y$*. Firstly, we take* $y = 0$*, then*

$$\phi(d(Tx, Ty)) = \phi(1) = 1 < 4 - \frac{1}{3} = \phi(d(x, y)) - \psi(d(x, y)).$$

*Secondly, if we take* $y \in (0, 1]$*, then*

$$d(Tx, Ty) = d(1, y - \frac{y^2}{4}) = 1 - (y - \frac{y^2}{4}) < 1$$

*implying thereby*

$$\begin{aligned}\phi(d(Tx, Ty)) &< \frac{11}{3} + y^2 + 4y = y^2 + 4y + 4 - \frac{1}{3} \\ &= (y + 2)^2 - \frac{1}{3} = \phi(d(x, y)) - \psi(d(x, y)).\end{aligned}$$

*In view of all the possible cases, we conclude that all the conditions of Theorem 4.1 are satisfied. Therefore, by Theorem 4.1,* $T$ *has a unique fixed point (namely:* $x = 0$*).*

## 7. Conclusion

In this paper, a new family of pair of auxiliary functions for generalized nonlinear contractions has been introduced and some new fixed point theorems are proved using our newly defined auxiliary functions. In process, several existing fixed point theorems can be deduced from our main result, which attests the importance of new auxiliary functions. For possible problems, we can improve other existing fixed point theorems using our newly introduced auxiliary functions.

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## REFERENCES

[1] *S. Banach*, Sur les opérations dans les ensembles abstraits et leur application aux équations intgrales, Fund. Math., **3**(1922), 133-181.

[2] *F. E. Browder*, On the convergence of successive approximations for nonlinear functional equations, Proc. K. Ned. Akad. Wet. Ser. A, Indag. Math., **71**(1968), 27-35.

[3] *D. W. Boyd and J. S. W. Wong*, On nonlinear contractions, Proc. Amer. Math. Soc., **20**(1969), No. 2, 458-464.

[4] *J. Matkowski*, Integrable solutions of functional equations, Dissertationes Math., **127**(1975), 68pp.

[5] *A. Mukherjea*, Contractions and completely continuous mappings, Nonlinear Analysis: Theory, Methods & Applications., **1**(1977), No. 3.

[6] *N. Jotić*, Some fixed point theorems in metric spaces, Indian J. Pure Appl. Math., **26**(1995), 947-952.

[7] *A. Alam, A. R. Khan and M. Imdad*, Some coincidence theorems for generalized nonlinear contractions in ordered metric spaces with applications, Fixed Point Theory Appl., **2014**(2014), Article No. 216.

[8] *M.S. Khan, M. Swaleh, and S. Sessa*, Fixed point theorems by altering distances between the points, Bulletin of Australian Mathematical Society, **30**(1984), No. 1, 1-9.

[9] *M. A. Krasnosel'skiĭ, G. M. Vaĭnikko, P. P. Zabreĭko, Ya. B. Rutitskiĭ and V. Ya. Stetsenko*, Approximate Solution of Operator Equations, Wolters–Noordhoff Publishing, Groningen, 1972.

[10] *Ya. I. Alber and S. Guerre-Delabriere*, Principles of weakly contractive maps in Hilbert spaces, New results in operator theory and its applications, **98**(1997), 7-22.

[11] *B. E. Rhoades*, Some theorems on weakly contractive maps, Nonlinear Analysis: Theory, Method & Applications, **47**(2001), No. 4, 2683-2693.

[12] *K. Abodayeh, E. Karapınar, A. Pitea and W. Shatanawi*, Hybrid contractions on Branciari type distance spaces, **7**(2019), No. 10, p.994.

[13] *E. Karapınar, A. Pitea and W. Shatanawi*, Function weighted quasi-metric spaces and fixed point results, **7**(2019), 89026–89032.

[14] *W. Shatanawi, A. Pitea and R. Lazović*, Contraction conditions using comparison functions on b-metric spaces, Fixed Point Theory Appl., **2014**(2014), Article No. 135.

[15] *W. Shatanawi and M. Postolache*, Coincidence and fixed point results for generalized weak contractions in the sense of Berinde on partial metric spaces, Fixed Point Theory Appl., **2013**(2013), Article No. 54.

[16] *H. Aydi, M. Postolache and W. Shatanawi*, Coupled fixed point results for ($\psi$, $\phi$)-weakly contractive mappings in ordered G-metric spaces, Computers & Mathematics with Applications, **63**(2012), No. 1, 298-309.

[17] *H. Aydi, W. Shatanawi, M. Postolache, Z. Mustafa and N. Tahat*, Theorems for Boyd-Wong-type contractions in ordered metric spaces, Abstract and Applied Analysis, **2012**(2012), Article ID 359054.

[18] *M. A. Miandaragh, M. Postolache and S. Rezapour*, Approximate fixed points of generalized convex contractions, Fixed Point Theory Appl., **2013**(2013), Article No. 255.

[19] *W. Shatanawi and M. Postolache*, Common fixed point results for mappings under nonlinear contraction of cyclic form in ordered metric spaces, Fixed Point Theory Appl., **2013**(2013), Article No. 60.

[20] *P. N. Dutta and B. S. Choudhury*, A generalization of contraction principle in metric spaces, Fixed Point Theory Appl., **2008**(2008), Article ID. 406368.

[21] *D. Dorić*, Common fixed point for generalized ($\psi$-$\phi$)-weak contractions, Applied Mathematics Letters, **22**(2009), 1896-1900.

[22] *O. Popescu*, Fixed points for ($\psi$-$\phi$)-weak contractions, Applied Mathematics Letters, **24**(2011), 1-4.

[23] *N. V. Luong and N. X. Thuan*, Coupled fixed points in partially ordered metric spaces and application, Nonlinear analysis, **74**(2011), 983-992.

[24] *A. F. Roldán-López-de-Hierro*, A unified version of Ran and Reuring's theorem and Nieto and R. Rodríguez-López's theorem and low-dimensional generalizations, Appl. Math. Inf. Sci., **10**(2016), No. 2, 383-393.

[25] *J. Jachymski*, Equivalent conditions for generalized contractions on (ordered) metric spaces, Nonlinear Anal., **74**(2011), 768-774.

[26] *R. Agarwal, E. Karapinar and A. F. Roldán-López-de-Hierro*, Fixed point theorems in quasi-metric space and applications to multidimensional fixed point theorems on $G$-metric space, Journal of Nonlinear and Convex Analysis, **16**(2015), No. 9, 1787-1816.

[27] *M. Eslamian, A. Abkar*, A fixed point theorems for generalized weakly contractive mappings in complete metric space, Ital. J. Pure Appl. Math, **2009**(2009).

[28] *M. A. Geraghty*, On Contractive Mappings, Proc. Amer. Math. Soc., **40**(1973), No. 2, 604-608.