U.P.B. Sci. Bull., Series A, Vol. 78, Iss. 4, 2016 ISSN 1223-7027

WEIGHTED GENERALIZATION OF SOME INTEGRAL INEQUALITIES FOR DIFFERENTIABLE CO-ORDINATED CONVEX FUNCTIONS

M. A. LATIF[1], S. S. DRAGOMIR[2], E. MOMONIAT[3]

In this paper, a new weighted identity for differentiable functions of two variables defined on a rectangle from the plane is established. By using the obtained identity and analysis, some new weighted integral inequalities for the classes of co-ordinated convex, co-ordinated wright-convex and co-ordinated quasi-convex functions on the rectangle from the plane are established which provide weighted generalization of some recent results proved for co-ordinated convex functions.

Keywords: Hermite-Hadamard's inequality, co-ordinated convex function, co-ordinated wright-convex function, co-ordinated quasi-convex function, Hölder's integral inequality, quadrature formula.

1. Introduction

A function $f : I \to \mathbb{R}$, $\varnothing \neq I \subseteq \mathbb{R}$, is said to be convex on I if the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y),$$

holds for all x, $y \in I$ and $\lambda \in [0,1]$.

The most celebrated inequality for convex functions is the Hermite-Hadamard's inequality (see for instance [7]). This double inequality is stated as:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1}$$

where $f : I \to \mathbb{R}$, $\varnothing \neq I \subseteq \mathbb{R}$ is a convex function, a, $b \in I$ with $a < b$. The inequalities in (1) are reversed if f is a concave function.

[1] School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa, e-mail: m_amer_latif@hotmail.com
[2] School of Engineering and Science, Victoria University, PO Box 14428 Melbourne City, MC 8001, Australia
[2] School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa, e-mail: sever.dragomir@vu.edu.au
[3] School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa, e-mail: ebrahim.momoniat@wits.ac.za

The inequalities (1) have various applications for generalized means, information measures, quadrature rules etc., and there is a growing literature providing its new proofs, extensions, refinements and generalizations, see for example [2, 4, 5, 6, 9, 21, 22] and the references therein.

Let us consider now a bidimensional interval $[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. A mapping $f:[a,b]\times[c,d]\to\mathbb{R}$ is said to be convex on $[a,b]\times[c,d]$ if the inequality

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)\le \lambda f(x,y)+(1-\lambda)f(z,w),$$

holds for all $(x,y),(z,w)\in[a,b]\times[c,d]$ and $\lambda\in[0,1]$.

A modification for convex functions on $[a,b]\times[c,d]$, which are also known as co-ordinated convex functions, was initiated by Dragomir [4, 6] as follows:

A function $f:[a,b]\times[c,d]\to\mathbb{R}$ is said to be convex on co-ordinates on $[a,b]\times[c,d]$ if the partial mappings $f_y:[a,b]\to\mathbb{R}, f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}, f_x(v)=f(x,v)$ are convex where defined for all $x\in[a,b], y\in[c,d]$.

A formal definition for co-ordinated convex functions may be stated as follows:

Definition 1.1. [13] A function $f:[a,b]\times[c,d]\to\mathbb{R}$ is said to be convex on co-ordinates on $[a,b]\times[c,d]$ if the inequality

$$\begin{aligned}&f(tx+(1-t)y,su+(1-s)w)\\&\le tsf(x,u)+t(1-s)f(x,w)+s(1-t)f(y,u)+(1-t)(1-s)f(y,w),\end{aligned}$$

holds for all $(t,s)\in 0,1]\times 0,1]$ and $(x,u),(y,w)\in[a,b]\times[c,d]$.

Clearly, every convex mapping $f:[a,b]\times[c,d]\to\mathbb{R}$ is convex on co-ordinates. Furthermore, there exist co-ordinated convex functions which are not convex, (see for example [4, 6]).

In a recent paper [20], Özdemir et al. give the notion of co-ordinated quasi-convex functions which generalize the notion of co-ordinated convex functions as follows:

Definition 1.2. [20] A function $f:[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ is said to be quasi-convex on $[a,b]\times[c,d]$ if the inequality

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)\le \max\{f(x,y),f(z,w)\},$$

holds for all $(x,y),(z,w)\in[a,b]\times[c,d]$ and $\lambda\in[0,1]$.

A function $f:[a,b]\times[c,d]\to \mathrm{R}$ is said to be quasi-convex on co-ordinates on $[a,b]\times[c,d]$ if the partial mappings $f_y:[a,b]\to \mathrm{R}, f_y(u)=f(u,y)$ and $f_x:[c,d]\to \mathrm{R}, f_x(v)=f(x,v)$ are quasi-convex where defined for all $x\in[a,b], y\in[c,d]$.

The definition of co-ordinated quasi-convex functions may be stated as follows.

Definition 1.3. [16] A function $f:[a,b]\times[c,d]\subset \mathrm{R}^2\to \mathrm{R}$ is said to be quasi-convex on co-ordinates on $[a,b]\times[c,d]$ if

$$f(tx+(1-t)z, sy+(1-s)w)\le \max\{f(x,y), f(x,w), f(z,y), f(z,w)\},$$

for all $(x,y),(z,w)\in[a,b]\times[c,d]$ and $(s,t)\in [0,1]\times[0,1]$.

The class of co-ordinated quasi-convex functions on $[a,b]\times[c,d]$ is denoted by $QC([a,b]\times[c,d])$. It has also been proved in [20] that every quasi-convex function on $[a,b]\times[c,d]$ is quasi-convex on co-ordinates on $[a,b]\times[c,d]$. It is to be noted that there exist quasi-convex functions on co-ordinates which are not quasi-convex, see for instance [16].

Another generalization of the notion of the co-ordinated convex functions is the concept of wright-convex functions which is given in the definition below.

Definition 1.4. [20] A function $f:[a,b]\times[c,d]\subset \mathrm{R}^2\to \mathrm{R}$ is said to be wright-convex on $[a,b]\times[c,d]$ if the inequality

$$f(\lambda x+(1-\lambda)z, \lambda y+(1-\lambda)w)+f((1-\lambda)x+\lambda z, (1-\lambda)y+\lambda w)$$
$$\le \max\{f(x,z), f(y,w)\},$$

holds for all $(x,z),(y,w)\in[a,b]\times[c,d]$ and $\lambda\in [0,1]$.

A function $f:[a,b]\times[c,d]\to \mathrm{R}$ is said to be wright-convex on co-ordinates on $[a,b]\times[c,d]$ if the partial mappings $f_y:[a,b]\to \mathrm{R}, f_y(u)=f(u,y)$ and $f_x:[c,d]\to \mathrm{R}, f_x(v)=f(x,v)$ are wright-convex where defined for all $x\in[a,b], y\in[c,d]$.

Definition 1.5. [20] A function $f:[a,b]\times[c,d]\subset \mathrm{R}^2\to \mathrm{R}$ is said to be wright-convex on co-ordinates on $[a,b]\times[c,d]$ if

$$f(tx+(1-t)z, sy+(1-s)w)+f((1-t)x+tz, (1-s)y+sw)$$
$$\le f(x,y)+f(z,y)+f(x,w)+f(z,w)$$

for all $(x,z),(y,w)\in[a,b]\times[c,d]$ and $(s,t)\in [0,1]\times[0,1]$.

The class of co-ordinated wright-convex functions on $[a,b]\times[c,d]$ is represented by $W([a,b]\times[c,d])$. It has also been proved in [20] that every wright-convex function on $[a,b]\times[c,d]$ is wright-convex on co-ordinates on $[a,b]\times[c,d]$.

For recent results on co-ordinated convex, co-ordinated quasi-convex, co-ordinated m-convex, co-ordinated (α ,m)-convex and co-ordinated s-convex functions on a rectangle $[a,b]\times[c,d]$ from the plane $\mathbb{R}^2$, we refer the readers to [1, 4, 5, 8], [10]-[20].

In the present paper, we establish a new weighted identity for differentiable mappings defined on a rectangle $[a,b]\times[c,d]$ from the plane $\mathbb{R}^2$ and by using the obtained identity and analysis, some new weighted integral inequalities for differentiable co-ordinated convex, co-ordinated wright-convex and co-ordinated quasi convex functions are proved. The results proved in the paper provide a weighted generalization of the results given in [15].

2. Main Results

The following notions will be used throughout this section for our convenience

$$U_1(a,b,t)=U_1(t)=\frac{1-t}{2}a+\frac{1+t}{2}b, L_1(a,b,t)=L_1(t)=\frac{1+t}{2}a+\frac{1-t}{2}b,$$

$$U_2(c,d,s)=U_2(s)=\frac{1-s}{2}c+\frac{1+s}{2}d, L_2(c,d,s)=L_2(s)=\frac{1+s}{2}c+\frac{1-s}{2}d,$$

$$\Psi(a,b,c,d;|f_{ts}|)=\frac{|f_{ts}(a,c)|+|f_{ts}(a,d)|+|f_{ts}(b,c)|+|f_{ts}(b,d)|}{4},$$

$$\lambda_1\left(b,d,\frac{a+b}{2},\frac{c+d}{2};|f_{ts}|\right)$$
$$=\max\left\{|f_{ts}(b,d)|,\left|f_{ts}\left(b,\frac{c+d}{2}\right)\right|,\left|f_{ts}\left(\frac{a+b}{2},d\right)\right|,\left|f_{ts}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\},$$

$$\lambda_2\left(a,d,\frac{a+b}{2},\frac{c+d}{2};|f_{ts}|\right)$$
$$=\max\left\{|f_{ts}(a,d)|,\left|f_{ts}\left(a,\frac{c+d}{2}\right)\right|,\left|f_{ts}\left(\frac{a+b}{2},d\right)\right|,\left|f_{ts}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\},$$

$$\lambda_3\left(b,c,\frac{a+b}{2},\frac{c+d}{2};|f_{ts}|\right)$$

$$= \max\left\{ |f_{ts}(b,c)|, \left| f_{ts}\left(b, \frac{c+d}{2}\right)\right|, \left| f_{ts}\left(\frac{a+b}{2}, c\right)\right|, \left| f_{ts}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right| \right\}$$

and

$$\lambda_4\left(a, c, \frac{a+b}{2}, \frac{c+d}{2}; |f_{ts}|\right)$$
$$= \max\left\{ |f_{ts}(a,c)|, \left| f_{ts}\left(a, \frac{c+d}{2}\right)\right|, \left| f_{ts}\left(\frac{a+b}{2}, c\right)\right|, \left| f_{ts}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right| \right\}.$$

The following lemma is the key result to establish the results in this section.

Lemma 2.1. Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a twice partially differentiable mapping on Δ° and $p : [a,b] \times [c,d] \to [0,\infty)$ be continuous and symmetric about $\frac{a+b}{2}$ and $\frac{c+d}{2}$ for $[a,b] \times [c,d] \subset \Delta^\circ$ with $a < b$, $c < d$. If $f_{ts} \in L([a,b] \times [c,d])$, then

$$\Im(a,b,c,d;p,f) = f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\int_c^d\int_a^b p(x,y)dxdy + \int_c^d\int_a^b f(x,y)p(x,y)dxdy$$
$$-\int_c^d\int_a^b f\left(x, \frac{c+d}{2}\right)p(x,y)dxdy - \int_c^d\int_a^b f\left(\frac{a+b}{2}, y\right)p(x,y)dxdy$$
$$= \frac{(b-a)(d-c)}{4}\int_0^1\int_0^1\left[\int_c^{L_2(s)}\int_a^{L_1(t)} p(x,y)dxdy\right]\left[f_{ts}(U_1(t), U_2(s))\right.$$
$$\left. - f_{ts}(U_1(t), L_2(s)) - f_{ts}(L_1(t), U_2(s)) + f_{ts}(L_1(t), L_2(s))\right]dsdt. \qquad (2)$$

Proof. Let

$$I = \frac{(b-a)(d-c)}{4}\int_0^1\int_0^1\left[\int_c^{L_2(s)}\int_a^{L_1(t)} p(x,y)dxdy\right]\left[f_{ts}(U_1(t), U_2(s))\right.$$
$$\left. - f_{ts}(U_1(t), L_2(s)) - f_{ts}(L_1(t), U_2(s)) + f_{ts}(L_1(t), L_2(s))\right]dsdt$$

and

$$\int_c^{L_2(s)}\int_a^{L_1(t)} p(x,y)dxdy = q(t,s).$$

then

$$I = \frac{(b-a)(d-c)}{4}\int_0^1\int_0^1 q(t,s)\left[f_{ts}(U_1(t), U_2(s)) - f_{ts}(U_1(t), L_2(s))\right.$$
$$\left. - f_{ts}(L_1(t), U_2(s)) + f_{ts}(L_1(t), L_2(s))\right]dsdt.$$

Now by integration by parts and by using the symmetry of $p(x,y)$ about

$x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$, we have

$$
\begin{aligned}
&\frac{(b-a)(d-c)}{4}\int_0^1\int_0^1 q(t,s)f_{ts}(U_1(t),U_2(s))dsdt\\
&=\frac{(b-a)}{2}\int_0^1\left[q(t,s)f_t(U_1(t),U_2(s))\Big|_0^1-\int_0^1 q_s(t,s)f_t(U_1(t),U_2(s))ds\right]dt\\
&=\frac{(b-a)}{2}\int_0^1\left[-f_t\left(U_1(t),\frac{c+d}{2}\right)\left(\int_{\frac{c+d}{2}}^d\int_{U_1(t)}^b p(x,y)dxdy\right)\right.\\
&\quad\left.+\int_0^1\left(\int_{U_1(t)}^b p(x,U_2(s))dx\right)f_t(U_1(t),U_2(s))ds\right]dt\\
&=-\frac{(b-a)}{2}\int_0^1 f_t\left(U_1(t),\frac{c+d}{2}\right)\left(\int_{\frac{c+d}{2}}^d\int_{U_1(t)}^b p(x,y)dxdy\right)dt\\
&\quad+\frac{(b-a)}{2}\int_{\frac{c+d}{2}}^d\int_0^1\left(\int_{U_1(t)}^b p(x,y)dx\right)f_t(U_1(t),y)dtdy\\
&=f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_{\frac{c+d}{2}}^d\int_{\frac{a+b}{2}}^b p(x,y)dxdy-\int_{\frac{c+d}{2}}^d\int_{\frac{a+b}{2}}^b f\left(x,\frac{c+d}{2}\right)p(x,y)dxdy\\
&\quad-\int_{\frac{c+d}{2}}^d\int_{\frac{a+b}{2}}^b f\left(\frac{a+b}{2},y\right)p(x,y)dxdy+\int_{\frac{c+d}{2}}^d\int_{\frac{a+b}{2}}^b p(x,y)f(x,y)dxdy. \qquad (3)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&-\frac{(b-a)(d-c)}{4}\int_0^1\int_0^1 q(t,s)f_{ts}(U_1(t),L_2(s))dsdt\\
&=f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_c^{\frac{c+d}{2}}\int_{\frac{a+b}{2}}^b p(x,y)dxdy-\int_c^{\frac{c+d}{2}}\int_{\frac{a+b}{2}}^b f\left(x,\frac{c+d}{2}\right)p(x,y)dxdy\\
&\quad-\int_c^{\frac{c+d}{2}}\int_{\frac{a+b}{2}}^b f\left(\frac{a+b}{2},y\right)p(x,y)dxdy+\int_c^{\frac{c+d}{2}}\int_{\frac{a+b}{2}}^b p(x,y)f(x,y)dxdy, \qquad (4)\\
&-\frac{(b-a)(d-c)}{4}\int_0^1\int_0^1 q(t,s)f_{ts}(L_1(t),U_2(s))dsdt\\
&=f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_{\frac{c+d}{2}}^d\int_a^{\frac{a+b}{2}} p(x,y)dxdy-\int_{\frac{c+d}{2}}^d\int_a^{\frac{a+b}{2}} f\left(x,\frac{c+d}{2}\right)p(x,y)dxdy\\
&\quad+\frac{1}{2}\int_{\frac{c+d}{2}}^d\int_a^{\frac{a+b}{2}} f\left(\frac{a+b}{2},y\right)p(x,y)dxdy+\int_{\frac{c+d}{2}}^d\int_a^{\frac{a+b}{2}} p(x,y)f(x,y)dxdy \qquad (5)
\end{aligned}
$$

and

$$\frac{(b-a)(d-c)}{4}\int_0^1\int_0^1 q(t,s) f_{ts}(L_1(t),L_2(s))dsdt$$
$$= f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_c^{\frac{c+d}{2}}\int_a^{\frac{a+b}{2}} p(x,y)dxdy - \int_c^{\frac{c+d}{2}}\int_a^{\frac{a+b}{2}} f\left(x,\frac{c+d}{2}\right)p(x,y)dxdy$$
$$-\int_c^{\frac{c+d}{2}}\int_a^{\frac{a+b}{2}} f\left(\frac{a+b}{2},y\right)p(x,y)dxdy + \int_c^{\frac{c+d}{2}}\int_a^{\frac{a+b}{2}} p(x,y)f(x,y)dxdy. \quad (6)$$

Adding (3)-(6), we get the desired result.

Remark 2.1. If we take $p(x,y)=\frac{1}{(b-a)(d-c)}$ for all $(x,y)\in[a,b]\times[c,d]$ in Lemma 2.1, we get Lemma 1 in [15, page 3]. Moreover, for $p(x,y)=\frac{1}{(b-a)(d-c)}$ for all $(x,y)\in[a,b]\times[c,d]$, it is natural to consider following notation

$$\Im(a,b,c,d;f)= f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)dxdy$$
$$\left.-\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx-\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right|.$$

Now by using lemma 2.1, we present the main results of this section.

Theorem 2.1. Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a twice differentiable mapping on Δ° and $p:[a,b]\times[c,d]\to[0,\infty)$ be continuous and symmetric about $\frac{a+b}{2}$ and $\frac{c+d}{2}$ for $[a,b]\times[c,d]\subset\Delta^\circ$ with $a<b$, $c<d$. If $f_{ts}\in L([a,b]\times[c,d])$ and $|f_{ts}|^q$ is convex on co-ordinates on $[a,b]\times[c,d]$ for $q\geq 1$, then

$$|\Im(a,b,c,d;p,f)|\leq (b-a)(d-c)\left[\Psi\left(a,b,c,d;|f_{ts}|^q\right)\right]^{\frac{1}{q}}$$
$$\times\int_0^1\int_0^1\int_c^{L_2(s)}\int_a^{L_1(t)} p(x,y)dxdydsdt. \quad (7)$$

Proof. Taking absolute value on both sides of (2), by using the properties of absolute value and the Hölder inequality, we have

$$|\Im(a,b,c,d;p,f)|\leq\frac{(b-a)(d-c)}{4}\left(\int_0^1\int_0^1\left[\int_c^{L_2(s)}\int_a^{L_1(t)}p(x,y)dxdy\right]dsdt\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\int_0^1\int_0^1\left[\int_c^{L_2(s)}\int_a^{L_1(t)}p(x,y)dxdy\right]|f_{ts}(U_1(t),U_2(s))|^q dsdt\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^1\int_0^1\left[\int_c^{L_2(s)}\int_a^{L_1(t)}p(x,y)dxdy\right]\left|f_{ts}(U_1(t),L_2(s))\right|^q dsdt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1\int_0^1\left[\int_c^{L_2(s)}\int_a^{L_1(t)}p(x,y)dxdy\right]\left|f_{ts}(L_1(t),U_2(s))\right|^q dsdt\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_0^1\int_0^1\left[\int_c^{L_2(s)}\int_a^{L_1(t)}p(x,y)dxdy\right]\left|f_{ts}(L_1(t),L_2(s))\right|^q dsdt\right)^{\frac{1}{q}}\right]. \tag{8}$$

By the power-mean inequality ($a_1^r + a_2^r + a_3^r + a_4^r \le 4^{1-r}(a_1+a_2+a_3+a_4)^r$ for a_1, a_2, a_3, $a_4 > 0$ and $r<1$) and using the convexity of $|f_{ts}|^q$ on co-ordinates on $[a,b]\times[c,d]$ for $q\ge 1$, we have

$$\left(\int_0^1\int_0^1\left[\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)}p(x,y)dxdy\right]\left|f_{ts}(U_1(t),U_2(s))\right|^q dsdt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1\int_0^1\left[\int_c^{L_2(s)}\int_a^{L_1(t)}p(x,y)dxdy\right]\left|f_{ts}(U_1(t),L_2(s))\right|^q dsdt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1\int_0^1\left[\int_c^{L_2(s)}\int_a^{L_1(t)}p(x,y)dxdy\right]\left|f_{ts}(L_1(t),U_2(s))\right|^q dsdt\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_0^1\int_0^1\left[\int_c^{L_2(s)}\int_a^{L_1(t)}p(x,y)dxdy\right]\left|f_{ts}(L_1(t),L_2(s))\right|^q dsdt\right)^{\frac{1}{q}}\right]$$

$$\le 4^{1-\frac{1}{q}}\left[\left|f_{ts}(a,c)\right|^q+\left|f_{ts}(a,d)\right|^q+\left|f_{ts}(b,c)\right|^q+\left|f_{ts}(b,d)\right|^q\right]^{\frac{1}{q}}$$

$$\times\left(\int_0^1\int_0^1\int_c^{L_2(s)}\int_a^{L_1(t)}p(x,y)dxdydsdt\right)^{\frac{1}{q}}. \tag{9}$$

A usage of (9) in (8) yields the desired result.

Remark 2.2. If we take $p(x,y)=\dfrac{1}{(b-a)(d-c)}$ for all $(x,y)\in[a,b]\times[c,d]$ in Theorem 2.1, we get Theorem 4 in [15, page 8].

A different approach leads to the following result.

Theorem 2.2. Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a twice differentiable mapping on Δ° and $p:[a,b]\times[c,d]\to[0,\infty)$ be continuous and symmetric about $\dfrac{a+b}{2}$ and $\dfrac{c+d}{2}$ for $[a,b]\times[c,d]\subset\Delta^\circ$ with $a<b$, $c<d$. If $f_{ts}\in L([a,b]\times[c,d])$ and $|f_{ts}|^q$ is convex on

co-ordinates on $[a,b]\times[c,d]$ for $q>1$, then

$$\left|\Im(a,b,c,d;p,f)\right|\le(b-a)(d-c)\left[\Psi\left(a,b,c,d;\left|f_{ts}\right|^{q}\right)\right]^{\frac{1}{q}}$$
$$\times\left(\int_0^1\int_0^1\left[\int_c^{L_2(s)}\int_a^{L_1(t)}p(x,y)dxdy\right]^{p}dsdt\right)^{\frac{1}{p}}, \tag{10}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.1 and the Hölder inequality, we have

$$\left|\Im(a,b,c,d;p,f)\right|\le\frac{(b-a)(d-c)}{4}\left(\int_0^1\int_0^1\left[\int_c^{L_2(s)}\int_a^{L_1(t)}p(x,y)dxdy\right]^{p}dsdt\right)^{\frac{1}{p}}$$
$$\times\left[\left(\int_0^1\int_0^1\left|f_{ts}(U_1(t),U_2(s))\right|^{q}dsdt\right)^{\frac{1}{q}}+\left(\int_0^1\int_0^1\left|f_{ts}(U_1(t),L_2(s))\right|^{q}dsdt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\int_0^1\left|f_{ts}(L_1(t),U_2(s))\right|^{q}dsdt\right)^{\frac{1}{q}}+\left(\int_0^1\int_0^1\left|f_{ts}(L_1(t),L_2(s))\right|^{q}dsdt\right)^{\frac{1}{q}}\right]. \tag{11}$$

By the power-mean inequality ($a_1^r+a_2^r+a_3^r+a_4^r\le 4^{1-r}(a_1+a_2+a_3+a_4)^r$ for a_1, a_2, a_3, $a_4>0$ and $r<1$) and using the convexity of $\left|f_{ts}\right|^{q}$ on co-ordinates on $[a,b]\times[c,d]$ for $q>1$, we have

$$\left(\int_0^1\int_0^1\left|f_{ts}(U_1(t),U_2(s))\right|^{q}dsdt\right)^{\frac{1}{q}}+\left(\int_0^1\int_0^1\left|f_{ts}(U_1(t),L_2(s))\right|^{q}dsdt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1\int_0^1\left|f_{ts}(L_1(t),U_2(s))\right|^{q}dsdt\right)^{\frac{1}{q}}+\left(\int_0^1\int_0^1\left|f_{ts}(L_1(t),L_2(s))\right|^{q}dsdt\right)^{\frac{1}{q}}$$
$$\le 4^{1-\frac{1}{q}}\left[\int_0^1\int_0^1\left|f_{ts}(U_1(t),U_2(s))\right|^{q}dsdt+\int_0^1\int_0^1\left|f_{ts}(U_1(t),L_2(s))\right|^{q}dsdt\right.$$
$$\left.+\int_0^1\int_0^1\left|f_{ts}(L_1(t),U_2(s))\right|^{q}dsdt+\int_0^1\int_0^1\left|f_{ts}(L_1(t),L_2(s))\right|^{q}dsdt\right]^{\frac{1}{q}}$$
$$\le 4\left[\frac{\left|f_{ts}(a,c)\right|^{q}+\left|f_{ts}(a,d)\right|^{q}+\left|f_{ts}(b,c)\right|^{q}+\left|f_{ts}(b,d)\right|^{q}}{4}\right]^{\frac{1}{q}}. \tag{12}$$

Using (12) in (11), we get (10).

Remark 2.3. If we take $p(x,y)=\frac{1}{(b-a)(d-c)}$ for all $(x,y)\in[a,b]\times[c,d]$ in Theorem 2.2, we get Theorem 3 in [15, page 6].

Remark 2.4. Theorem 2.1 and Theorem 2.2 continue to hold true if in their statements we replace the condition " convex on the co-ordinates" with the condition " wright-convex on co-ordinates". However, the details are left to the interested reader.

In what follows we give our results for the quasi-convex mappings on co-ordinates on $[a,b]\times[c,d]$.

Theorem 2.3. Suppose the assumptions of Theorem 2.1 are satisfied. If the mapping $|f_{ts}|^q$ is quasi-convex on co-ordinates on $[a,b]\times[c,d]$ for $q\geq 1$, then the following inequality holds

$$|\Im(a,b,c,d;p,f)|\leq\frac{(b-a)(d-c)}{4}\left\{\left[\lambda_1\left(b,d,\frac{a+b}{2},\frac{c+d}{2};|f_{ts}|^q\right)\right]^{\frac{1}{q}}\right.$$

$$+\left[\lambda_2\left(a,d,\frac{a+b}{2},\frac{c+d}{2};|f_{ts}|^q\right)\right]^{\frac{1}{q}}+\left[\lambda_3\left(b,c,\frac{a+b}{2},\frac{c+d}{2};|f_{ts}|^q\right)\right]^{\frac{1}{q}}$$

$$\left.+\left[\lambda_4\left(a,c,\frac{a+b}{2},\frac{c+d}{2};|f_{ts}|^q\right)\right]^{\frac{1}{q}}\right\}\int_0^1\int_0^1\int_c^{L_2(s)}\int_a^{L_1(t)}p(x,y)dxdydtds. \tag{13}$$

Proof. We continue inequality (8) in the proof of Theorem 2.1. Now, by the quasi-convexity on co-ordinates of $|f_{ts}|^q$ on $[a,b]\times[c,d]$ for $q\geq 1$ and the power-mean inequality, we obtain

$$|f_{ts}(U_1(t),U_2(s))|^q\leq\lambda_1\left(b,d,\frac{a+b}{2},\frac{c+d}{2};|f_{ts}|^q\right) \tag{14}$$

$$|f_{ts}(L_1(t),U_2(s))|^q\leq\lambda_2\left(a,d,\frac{a+b}{2},\frac{c+d}{2};|f_{ts}|^q\right) \tag{15}$$

$$|f_{ts}(U_1(t),L_2(s))|^q\leq\lambda_3\left(b,c,\frac{a+b}{2},\frac{c+d}{2};|f_{ts}|^q\right) \tag{16}$$

and

$$|f_{ts}(L_1(t),L_2(s))|^q\leq\lambda_4\left(a,c,\frac{a+b}{2},\frac{c+d}{2};|f_{ts}|^q\right). \tag{17}$$

for all $(t,s)\in[0,1]\times[0,1]$. Using (14)-(17) in (13) we get the desired result.

Corollary 2.1. Suppose the assumptions of Theorem 2.3 are fulfilled and if $p(x,y)=\dfrac{1}{(b-a)(d-c)}$ for all $(x,y)\in[a,b]\times[c,d]$, then the following inequality holds valid

$$|\Im(a,b,c,d;f)|\le\frac{(b-a)(d-c)}{64}$$
$$\times\left\{\left[\lambda_1\left(b,d,\frac{a+b}{2},\frac{c+d}{2};|f_{ts}|^q\right)\right]^{\frac{1}{q}}+\left[\lambda_2\left(a,d,\frac{a+b}{2},\frac{c+d}{2};|f_{ts}|^q\right)\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\lambda_3\left(b,c,\frac{a+b}{2},\frac{c+d}{2};|f_{ts}|^q\right)\right]^{\frac{1}{q}}+\left[\lambda_4\left(a,c,\frac{a+b}{2},\frac{c+d}{2};|f_{ts}|^q\right)\right]^{\frac{1}{q}}\right\}. \tag{18}$$

Corollary 2.2. Suppose the assumptions of Theorem 2.3 are satisfied and additionally

1. If $|f_{ts}|^q$ is non-decreasing on co-ordinates on $[a,b]\times[c,d]$, then

$$\left|\Im(a,b,c,d;p,f)\right|\le\frac{(b-a)(d-c)}{4}\left\{|f_{ts}(b,d)|+\left|f_{ts}\left(\frac{a+b}{2},d\right)\right|\right.$$
$$\left.+\left|f_{ts}\left(b,\frac{c+d}{2}\right)\right|+\left|f_{ts}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\}\int_0^1\int_0^1\int_c^{L_2(s)}\int_a^{L_1(t)}p(x,y)\,dxdydtds. \tag{19}$$

holds valid.

2. If $|f_{ts}|^q$ is non-increasing on co-ordinates on $[a,b]\times[c,d]$, then

$$\left|\Im(a,b,c,d;p,f)\right|\le\frac{(b-a)(d-c)}{4}\left\{|f_{ts}(a,c)|+\left|f_{ts}\left(\frac{a+b}{2},c\right)\right|\right.$$
$$\left.+\left|f_{ts}\left(a,\frac{c+d}{2}\right)\right|+\left|f_{ts}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\}\int_0^1\int_0^1\int_c^{L_2(s)}\int_a^{L_1(t)}p(x,y)\,dxdydtds. \tag{20}$$

holds true.

Corollary 2.3. In Corollary 2.1

1. If $|f_{ts}|^q$ is non-decreasing on co-ordinates on $[a,b]\times[c,d]$, then

$$\left|\Im(a,b,c,d;f)\right|\le\frac{(b-a)(d-c)}{64}$$

$$\times\left\{\left|f_{ts}(b,d)\right|+\left|f_{ts}\left(\frac{a+b}{2},d\right)\right|+\left|f_{ts}\left(b,\frac{c+d}{2}\right)\right|+\left|f_{ts}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\}. \quad (21)$$

holds valid.

2. If $|f_{ts}|^q$ is non-increasing on co-ordinates on $[a,b]\times[c,d]$, then

$$\left|\Im(a,b,c,d;f)\right|\le\frac{(b-a)(d-c)}{64}$$

$$\times\left\{\left|f_{ts}(a,c)\right|+\left|f_{ts}\left(\frac{a+b}{2},c\right)\right|+\left|f_{ts}\left(c,\frac{c+d}{2}\right)\right|+\left|f_{ts}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\}. \quad (22)$$

holds true.

3. Conclusions

A new weighted identity involving a twice differentiable mapping defined on a rectangle from the plane and a continuous positive valued mapping which is symmetric on co-ordinates is established. The identity proved in this paper is more general than the results proved in earlier works. Some new weighted Hermite-Hadamard type inequalities are obtained using the achieved identity, analysis, the notion of convexity, quasi convexity and wright convexity on co-ordinates on a rectangle from the plane. The results can be used to refine previous related results since the notion of qausi convexity and wright convexity on co-ordinates are more general than notion of convexity on co-ordinates and hence the findings are believed to be very useful for further research in this filed.

Acknowledgments

The authors would like to say thank to the anonymous referees for their valuable comments to improve the final version of the manuscript.

REFERENCES

[1]. *M. Alomari and M. Darus*, Fejer inequality for double integrals, Facta Universitatis (NIŠ): Ser. Math. Inform., **24** (2009), 15-28.

[2]. *M. Alomari, M. Darus, U.S. Kirmaci*, Refinements of Hadamard-type inequalities for quasi-convex functions with applications to trapezoidal formula and to special means, Computers & Mathematics with Applications, **59** (1) (2010), 225-232

[3]. *S. S. Dragomir and R. P. Agarwal*, Two inequalities for differentiable mappings and applications to special means of real numbers and to Trapezoidal formula, Appl. Math. Lett., **11** (5) (1998), 91-95.

[4]. *S.S. Dragomir*, On Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane, Taiwanese Journal of Mathematics, **4** (2001), 775-788.

[5]. *S. S. Dragomir*, Two mappings in connection to Hadamard's inequalities, Journal of Mathematical Analysis and Applications, **167** (1992), 49-56.

[6]. *S.S. Dragomir and C.E.M. Pearce*, Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University, 2000.

[7]. *J. Hadamard*, Étude sur les Propriétés des Fonctions Entières en Particulier d'une Fonction Considérée par Riemann. Journal de Mathématiques Pures et Appliquées, **58**, 171-215.

[8] *D. Y. Hwang, K. L. Tseng, and G. S. Yang*, Some Hadamard's inequalities for co-ordinated convex functions in a rectangle from the plane, Taiwanese Journal of Mathematics, **11**(2007), 63-73.

[9]. *D. Y. Hwang*, Some inequalities for differentiable convex mapping with application to weighted trapezoidal formula and higher moments of random variables, Applied Mathematics and Computation, **217** (2011) 9598–9605.

[10]. *D. -Y. Hwang, K.-C. Hsu and K.-L. Tseng*, Hadamard-Type inequalities for Lipschitzian functions in one and two variables with applications, Journal of Mathematical Analysis and Applications, **405** (2013), 546-554.

[11]. *K.-C. Hsu*, Some Hermite-Hadamard type inequalities for differentiable co-ordinated convex functions and applications, Advances in Pure Mathematics, **4** (2014), 326-340.

[12]. *K.-C. Hsu*, Refinements of Hermite-Hadamard type inequalities for differentiable co-ordinated convex functions and applications, Taiwanese Journal of Mathematics, **19** (1) (2015), 133-157.

[13]. *M. A. Latif and M. Alomari*, Hadamard-type inequalities for product of two convex functions on the co-ordinates, Int. Math. Forum, **4**(47) (2009), 2327-2338.

[14]. *M. A. Latif and M. Alomari*, On the Hadamard-type inequalities for h-convex functions on the co-ordinates, Int. J. of Math. Analysis, **3** (33)(2009), 1645-1656.

[15]. *M. A. Latif, S. S. Dragomir*, On some new inequalities for differentiable co-ordinated convex functions, Journal of Inequalities and Applications, **2012** (2012), 2012:28.

[16]. *M. A. Latif, S. Hussain and S. S. Dragomir*, Refinements of Hermite-Hadamard type inequalities for co-ordinated quasi-convex functions, International Journal of Mathematical Archive, **3**(1) (2012), 161-171.

[17]. *S.-L. Lyu*, On the Hermite-Hadamard inequality for convex functions of two variable, Numerical Algebra, Control and Optimization, **4** (1) (2013), 1-8.

[18]. *M.E. Özdemir, E. Set and M.Z. Sarikaya*, New some Hadamard's type inequalities for co-ordinated m-convex and (α, m)-convex functions, Hacettepe Journal of Mathematics and Statistics, **40** (2) (2011), 219-229.

[19]. *M.E. Özdemir, M. A. Latif and A. O. Akdemir*, On some Hadamard-type inequalities for product of two s-convex functions on the co-ordinates, Journal of Inequalities and Applications, **2012** (2012).

[20]. *M.E. Özdemir, A. O. Akdemir, A* ğ *ri, C. Yildiz and Erzurum*, On co-ordinated quasi-convex functions, Czechoslovak Mathematical Journal, **62** (137) (2012), 889-900.

[21] *C. M. E. Pearce and J. E. Pečarič*,, Inequalities for differentiable mappings with applications to special means and quadrature formula, Appl. Math. Lett., **13** (2000) 51-55.

[22]. *J. E. Pečarič, F. Proschan and Y. L. Tong*, Convex Functions, Partial Ordering and Statistical Applications, Academic Press, New York, 1991.

[23]. *M.Z. Sarikaya, E. Set, M.E. Özdemir and S. S. Dragomir*, New some Hadamard's type inequalities for co-ordinated convex functions, Tamsui Oxford Journal of Information and Mathematical Sciences, **28**(2) (2012), 137-152.