A CLASS OF CONCAVE OPERATORS AND RELATED OPTIMIZATION

Octav OLTEANU[1], Janina Mihaela MIHĂILĂ[2]

Using minimum principle for concave functions, we prove a related constrained inequality, firstly for finite sums. The case of infinite sums is deduced from the previously mentioned inequality, passing to the limit. In the end, a similar result for a class of concave operators taking values in the positive cone of a special space of self-adjoint operators is discussed. Two related types of examples are given.

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1. Introduction

The importance of the notion of an extreme point of a convex subset, of Carathéodory's and Krein-Milman theorems, finding the maximum point(s) of a real convex continuous function on such a subset and applications can be found in [1]. Generalizations to concave (or convex) operators can be got by means of the background contained in [2], [3]. Various further applications and/or generalizations are contained in [4]-[12] and references there. The proof of Carathéodory's theorem can be found in [1]. From optimization viewpoint, a main result is the maximum principle for convex continuous real functions f on a convex compact finite dimensional set K. It says that the maximum is attained at an extreme point of K. In the case of finite dimensional closed convex unbounded subsets A, a similar result remains valid for a convex continuous real function on A, provided that the set $Ex(A)$ of all extreme points is not empty, and the maximum of f on A is attained at some point(s) of A. Due to Krein-Milman theorem, for a convex compact subset K of an arbitrary (Hausdorff) locally convex space and $f: K \to \mathbb{R}$ convex and continuous, we have

$$\sup_{x \in K} f(x) = \sup_{e \in Ex(K)} f(e)$$

[1] Prof., Dept.of Mathematics-Informatics, University POLITEHNICA of Bucharest, Romania, e-mail: olteanuoctav@yahoo.ie; octav.olteanu50@gmail.com
[2] Associate Prof., Ecological University of Bucharest, Romania, e-mail: janinamihaelamihaila@yahoo.it

Applications of Krein-Milman theorem to the moment problem can be found in [9]. For optimization related to the moment problem see [8], [10]. Of course, to any maximum principle for convex functions there is a corresponding minimum principle for concave functions. The main purpose of the present work is to prove a constrained inequality related to an infinite sum of concave operators. Some of the results mentioned above are applied. The rest of the paper is organized as follows. In Section 2, the first aim is to apply the minimum principle for special concave continuous real functions on finite dimensional simplexes, in order to prove inequalities for such functions. Passing to the limit, from these inequalities one derives similar statements for infinite sums of special concave functions. Two classes of examples are pointed out. Section 3 presents an operatorial version of the first result of section 2. Section 4 concludes the paper.

2. An inequality related to a special class of concave functions

Let $n \geq 2$ be a natural number, $\{c_1, \dots, c_n\} \subset (0, \infty), h: [1, \infty) \to \mathbb{R}_+$ a concave strictly increasing continuous function, such that $h(1) = 0, h(t) > 0, \forall t > 1$. $h(1) = 0$.

Theorem 2.1. (a) *For any* $\{x_1, \dots, x_n\} \subset \mathbb{R}_+$, $\sum_{j=1}^{n} x_j = M > 0$, *the following relation holds true:*

$$\sum_{j=1}^{n} c_j h(1 + x_j) \geq \left(\min_{1 \leq j \leq n} c_j\right) h(1 + M) \tag{1}$$

If h *is strictly concave, then equality occurs in (1) if and only if there exists* $j_0 \in \{1, \dots, n\}$ *such that* $x_j = 0, j \in \{1, \dots, n\} \setminus \{j_0\}, x_{j_0} = M$ *and* $c_{j_0} = \min_{1 \leq j \leq n} c_j$.

(b) *Using the same notations und hypothesis, under the weaker constraint* $\sum_{j=1}^{n} x_j \geq M > 0$, *for a strictly concave and strictly increasing function* h, *the same relation (1) holds, and equality occurs in the same case as that mentioned at point (a).*

Proof. (a) Define the $n - 1$ dimensional simplex

$$S_{n-1,M} := \left\{(x_1, \dots, x_n) \in \mathbb{R}^n;\ x_j \geq 0, j \in \{1, \dots, n\}, \sum_{j=1}^{n} x_j = M \right\}$$

This is a simplex of vertices $e_1 := (M, 0, \dots, 0), \dots, e_n := (0, 0, \dots, 0, M)$ (the set $Ex(S_{n-1,M})$ equals the set $\{e_1, \dots, e_n\}$). $S_{n-1,M}$ is contained in the $n - 1$ dimensional linear variety defined by

$$V_{n-1} = \{e_{j_1}\} + Span\left\{e_j - e_{j_1}; j \in \{1, \dots, n\} \setminus \{j_1\}\right\}$$

for any $j_1 \in \{1, \dots, n\}$. On the other hand, the function

$$g(x) = g(x_1, \dots, x_n) := \sum_{j=1}^{n} c_j h\left(1 + x_j\right), x \in K := S_{n-1,M}$$

is concave and continuous, as a sum of n functions having these two properties. Application of the minimum principle for g leads to

$$g(x) \geq \min_{1 \leq j \leq n} g\left(e_j\right) = \min_{1 \leq j \leq n} \left(c_j h(1 + M)\right) = \left(\min_{1 \leq j \leq n} c_j\right) h(1 + M)$$

This proves the first assertion of the theorem. Assume now that h is strictly concave. Then so is g. If equality occurs in the relation from above for $x \in K \setminus \left\{e_j; j \in \{1, \dots, n\}\right\}$, then, due to Carathéodory's theorem, there exist a subset $E_1 = \left\{e_{j_1}, \dots, e_{j_k}\right\}, k \in \{1, \dots, n\}$ and $\{\alpha_1, \dots, \alpha_k\} \subset (0, \infty), \sum_{l=1}^{k} \alpha_l = 1$, such that $x = \sum_{l=1}^{k} \alpha_l e_{j_l}$. Now strict concavity of g and equality in the relation in discussion yield

$$g(x) > \sum_{l=1}^{k} \alpha_l g\left(e_{j_l}\right) \geq \left(\min_{1 \leq l \leq k} g\left(e_{j_l}\right)\right) \left(\sum_{l=1}^{k} \alpha_l\right) \geq \min_{1 \leq j \leq n} g\left(e_j\right) =$$
$$\left(\min_{1 \leq j \leq n} c_j\right) h(1 + M) = g(x),$$

which is a contradiction. This concludes the proof of (a). To prove (b), consider the unbounded set

$$A_M := \left\{(x_1, \dots, x_n) \in \mathbb{R}^n;\ x_j \geq 0, j \in \{1, \dots, n\}, \sum_{j=1}^{n} x_j \geq M \right\} = \bigcup_{\varepsilon \geq 0} S_{n-1,M+\varepsilon}$$

Application of the results of (a) leads to

$$\min_{x \in A} g(x) = \inf_{\varepsilon \geq 0} \inf_{x \in S_{n-1,M+\varepsilon}} g(x) = \left(\min_{1 \leq j \leq n} c_j\right) \inf_{\varepsilon \geq 0} h(1 + M + \varepsilon) =$$
$$\left(\min_{1 \leq j \leq n} c_j\right) h(1 + M)$$

Observe that the minimum of g on the unbounded closed subset A_M is attained at one of the extreme points of A_M. Latter points are exactly the extreme points of $S_{n-1,M}$. The conclusion follows. □

Corollary 2.1. *Let $n \geq 2$ be a natural number and*

$$\{c_1, \dots, c_n\} \subset (0,\infty), p_j \geq 0, j = 1, \dots, n, \ \sum_{j=1}^{n} p_j = 1.$$

Then the following relation holds

$$\sum_{j=1}^{n} c_j \, ln(1 + p_j) \geq \left(\min_{1 \leq j \leq n} c_j\right) ln(2)$$

where equality occurs if and only if $p_j = 0, j \in \{1, \dots, n\} \backslash \{j_0\}, p_{j_0} = 1,$ *and* $c_{j_0} = \min_{1 \leq j \leq n} c_j$ *for some* $j_0 \in \{1, \dots, n\}$.

Corollary 2.2. L*et* h *be a concave continuous strictly increasing function on* $[1,\infty), (c_n)_{n\geq 1}$ *a bounded sequence of positive numbers such that* $inf_{n\geq 1} c_n > 0$. *Assume that* $0 < h(1 + x) < x, \forall x > 0,$ $h(1) = 0$. *Then for any convergent series* $\sum_{n=1}^{\infty} x_n = M$ *of nonnegative numbers, we have*

$$\sum_{n=1}^{\infty} c_n h(1 + x_n) \geq \left(\inf_{n\geq 1} c_n\right) h(1 + M)$$

Proof. Observe the series in the left hand side of the above inequality is convergent, because of

$$\sum_{n=1}^{\infty} c_n h(1 + x_n) \leq \left(\sup_{n\geq 1} c_n\right) \sum_{n=1}^{\infty} x_n = M \left(\sup_{n\geq 1} c_n\right) < \infty$$

For each $n \geq 2$, from Theorem 2.1 we know that

$$\sum_{j=1}^{n} c_j \, h(1 + x_j) \geq \left(\min_{1\leq j\leq n} c_j\right) h\left(1 + \sum_{j=1}^{n} x_j\right).$$

Passing through the limit over $n \to \infty$, one obtains

$$\sum_{n=1}^{\infty} c_n h(1 + x_n) = \lim_{n\to\infty} \left(\sum_{j=1}^{n} c_j \, h(1 + x_j) \right) \geq \lim_{n\to\infty} \left(\min_{1\leq j\leq n} c_j\right) h \left(1 + \sum_{j=1}^{n} x_j \right) =$$
$$\left(\inf_{n\geq 1} c_n\right) h(1 + M).$$

This concludes the proof. □

Theorem 2.2. *Let h be a strictly increasing strictly concave continuous real* *function on* $[1,\infty)$, *such that* $h(1)=0$, *and the associated function* $u \to u\cdot h(1+1/u)$ *is strictly increasing on* $(0,\infty)$.

(a) *Let* $n\in\mathbb{N}, n\geq 2, \{c_1,\dots,c_n\}\subset(0,\infty), x_j\geq 0, j\in\{1,\dots,n\}$ *be such that* $\sum_{j=1}^{n} c_j x_j = M_n$, *where* $M_n > 0$ *is a given constant. Then we have*

$$\sum_{j=1}^{n} c_j\cdot h(1+x_j) \geq \left(\min_{1\leq j\leq n} c_j\right)\cdot h\left(1+\frac{M_n}{\min\limits_{1\leq j\leq n} c_j}\right)$$

and equality occurs if and only if $x=(x_1,\dots,x_n)=\left(0,\dots,0,\frac{M_n}{c_{j_0}},0,,,0\right)$, *where* $c_{j_0}=min_{1\leq j\leq n} c_j$, *and the non- null component(s) of* x *is (are)* x_{j_0}.

(b) *If* $(c_n)_{n\geq 1}$ *is a bounded sequence of positive numbers with* $inf_{n\geq 1} c_n > 0$, *assuming that* $h(1+x) < x, \forall x > 0$, *then for any sequence* $(x_n)_{n\geq 1}$ *of nonnegative numbers such that the sum* $\sum_{n=1}^{\infty} c_n x_n = M \in (0,\infty)$,*we have*

$$\sum_{n=1}^{\infty} c_n h(1+x_n) \geq \left(\inf_{n\geq 1} c_n\right) h\left(1+\frac{M}{\inf\limits_{n\geq 1} c_n}\right)$$

Proof. To prove (a), one repeats the idea of the proof of Theorem 2.1, where we define the simplex

$$S_{n-1} := \left\{x=(x_1,\dots,x_n)\in\mathbb{R}^n;\ x_j\geq 0, j\in\{1,\dots,n\}, \sum_{j=1}^{n} c_j x_j = M_n\right\}$$

Then

$$Ex(S_{n-1}) = \{e_1,\dots,e_n\}, e_1 := \left(\frac{M_n}{c_1},0,\dots,0\right),\dots, e_n := \left(0,\dots,0,\frac{M_n}{c_n}\right),$$

$$g_n(x) := \sum_{j=1}^{n} c_j\cdot h(1+x_j), x=(x_1,\dots,x_n)\in S_{n-1} \Rightarrow g_n(e_j) = c_j h\left(1+\frac{M_n}{c_j}\right),$$
$$j=1,\dots,n$$

Due to concavity property of g, one has

$$\min_{x\in S_{n-1}} g_n(x) = \min_{1\leq j\leq n} g_n(e_j) = \min_{1\leq j\leq n} c_j h\left(1+\frac{M_n}{c_j}\right) =$$

$$\left(\min_{1\le j\le n} c_j\right)\cdot h\left(1+\frac{M_n}{\min_{1\le j\le n} c_j}\right)$$

where the last inequality follows from the hypothesis on the function $u \to u \cdot h(1+1/u), u > 0$, which was assumed to be strictly increasing. The last assertion from (a) as well as that from (b), can be deduced in a similar way to the proof of Theorems 2.1 and Corollary 2.2, also using what we have already discussed of in the present proof. □

Example 2.1. All assumptions in the statement of Theorem 2.2 are valid in the particular case when $h(t) = ln(t), t > 0$.

Example 2.2. The function $h(t) = t^p - 1, t \ge 0, 0 < p < 1$ satisfies all requirements of Theorem 2.2. Verifying these assertions is an elementary task. Therefore, we shall omit the proof. In particular, Theorem 2.1 can be applied for these functions.

Remark 2.1. The inequality of Theorem 2.2 point (a) remains valid under the weaker constraints $x_j \ge 0, j \in \{1,\dots,n\}, \sum_{j=1}^{n} c_j x_j \ge M_n > 0$.

The next result refers to some properties of the function $\rho(u) := u \cdot ln(1+1/u), u > 0$. The proof will be omitted because is very elementary.

Proposition 2.1. (a) *The function ρ is strictly increasing, strictly concave on $(0,\infty)$, and the horizontal line of equation $y = 1$ is an asymptote for the graph of the function ρ at infinity.*

(b) *The unique fixed point $\tilde{u} \in (0,\infty)$ of the function ρ is $\tilde{u} = \frac{1}{e-1}$. The function ρ is a contraction from*

$$[1/(e-1),\infty) \text{ to } [1/(e-1),1) \subset [1/(e-1),\infty),$$

of contraction constant $\rho'\left(\frac{1}{e-1}\right) = \frac{1}{e}$. Choosing $u_0 = 1, u_{n+1} = \rho(u_n), n \in \mathbf{N}$, we have the following well-known relation controlling the rapidity of convergence $u_n \to \tilde{u}$

$$\left|u_n - \frac{1}{e-1}\right| \le \frac{e^{-n}}{1-e^{-1}}\left(1 - ln(2)\right), n \in \mathbf{N}$$

(c) *Denote by $\mathcal{U}$ the collection of all functions $u: (0,\infty) \to (0,\infty)$, and let*

$$\breve{\rho}: \mathcal{U} \to \mathcal{U}, \breve{\rho}(u) := \rho \circ u, \left(\rho(u(x)) = u(x) ln\left(1 + \frac{1}{u(x)}\right), x \in (0,\infty)\right).$$

Then the unique fixed point of $\breve{\rho}$ is the constant function $u \equiv \frac{1}{e-1}$.

(d) *The unique self-adjoint operator* A *with the spectrum* $\sigma(A) \subset (0, \infty)$ *which verifies the equality*

$$A \cdot ln(I + A^{-1}) = A$$

is $A = \frac{1}{e-1} I$.

(e) *The convex cone of all strictly increasing, strictly concave continuous functions from* $(0, \infty)$ *to itself, is closed with respect to the operation of composition of functions. The set of all continuous strictly increasing strictly concave functions from* $(0, \infty)$ *to itself, having a common fixed point, is convex and closed with respect to composition of functions operation.*

Similar results to (a)-(d) hold true for the function

$$\rho_p(u) = u((1 + 1/u)^p - 1), u > 0, 0 < p < 1$$

The unique fixed point of this function is $u_p = 1/(2^{1/p} - 1) \in (0,1)$. Observe that

$$p \downarrow 0 \Leftrightarrow u_p \downarrow 0, \qquad p \uparrow 1 \Leftrightarrow u_p \uparrow 1$$

3. An operatorial variant

We start this section by recalling some known results on self-adjoint (linear) operators acting on an arbitrary complex Hilbert space H. Let $\mathcal{A}$ be the real vector space of self-adjoint operators from H to itself. Then $\mathcal{A}$ is an ordered vector space, endowed with the order relation defined by

$$U \leq V \Leftrightarrow < U(h), h > \leq < V(h), h >, \forall h \in H, U, V \in \mathcal{A}$$

Unfortunately, for arbitrary $U, V \in \mathcal{A}$, the supremum $sup\{U, V\} = U \vee V$ or/and the infimum $inf\{U, V\} = U \wedge V$ might not exist in $\mathcal{A}$. However, the following main result holds true.

Theorem 3.1. *Let* $(U_n)_{n\geq 0}$ *be a nondecreasing upper bounded sequence of operators in* $\mathcal{A}$. *Then there exists* $U = sup_{n\geq 0} U_n$ *in* $\mathcal{A}$ *and* $lim_{n\to\infty} \|U_n(h) - U(h)\| = 0, \forall h \in H$ (U *is the pointwise limit of the sequence* $(U_n)_{n\geq 0}$).

The proof of this theorem can be found in [2]. Obviously, a similar conclusion follows for decreasing bounded below sequences $(U_n)_{n\geq 0}$ of elements of $\mathcal{A}$.

Remark 3.1. A converse of Theorem 3.1 holds true in a much more general setting (its proof is obvious). Namely, let Y be an ordered vector space, which is also a topological vector space such that the positive cone Y_+ is topologically closed. If $(U_n)_{n\geq 0}$ is an increasing sequence in Y such that there exists $U = \lim_{n\to\infty} U_n \in Y$, then there exists $sup_{n\geq 0} U_n$ and $sup_{n\geq 0} U_n = U$.

To avoid the fact that $\mathcal{A}$ is not a vector lattice, as well as the non-commutativeness of multiplication of elements from $\mathcal{A}$, for any $A \in \mathcal{A}$ one uses the construction of the following space $Y = Y(A)$.

Theorem 3.2. *Let*

$$A \in \mathcal{A}, Y_1 := \{U \in \mathcal{A}; AU = UA\}, Y = Y(A) := \{V \in Y_1; VU = UV, \forall U \in Y_1\}.$$

Then Y is a commutative (real) Banach algebra and an order-complete Banach lattice, where

$$|V| := sup\{V, -V\} = \sqrt{V^2}, \forall V \in Y$$

($|V|$ *equals the positive square root of the positive self-adjoint operator* V^2*).*

The proof of Theorem 3.2 can be found in [3]. Having in mind these background-type results and the above notations, we can prove the next main new theorem of this work. In the sequel, for $A \in \mathcal{A}$, the space $Y = Y(A)$ will be that defined in Theorem 3.2. For a continuous real function h on the spectrum $\sigma(A)$ of an operator $A \in \mathcal{A}$, we will denote also by h the mapping obtained from h by means of functional calculus attached to A.

Theorem 3.3. *Let $A \in \mathcal{A}$ be a positive operator, with the spectrum $\sigma(A) \subset (0, \infty)$, $(T_n)_{n\geq 1}$ a sequence of elements in $Y(A)$ such that there exist $a, b \in \mathbb{R}, 0 < a < b$, with the property that the spectrums $\sigma(T_n) \subset [a, b]$ for all $n \in \mathbb{N}\setminus\{0\}$. Let h be a concave continuous increasing function on $[1, \infty)$ such that*

$$0 \leq h(1 + x) \leq x, \forall x \geq 0, h(1) = 0, h(I) = \mathbf{0},$$

and $\sum_{n=1}^{\infty} x_n$ a convergent numerical series of positive terms. Denote $s := \sum_{n=1}^{\infty} x_n$. Suppose there exists $L \in (0, \infty)$ such that

$$|h(1 + u_1 t) - h(1 + u_2 t)| \leq L|u_1 - u_2|, \forall \{u_1, u_2\} \subset \mathbb{R}_+, \forall t \in \sigma(A).$$

Then the following inequalities hold

$$\sum_{n=1}^{\infty} T_n h(I + x_n A) \geq \left(\inf_{n\geq 1} T_n\right) h(I + sA) \geq a \cdot h(I + sA) \in Y_+\setminus\{0\} \qquad (2)$$

($I: H \to H$ is the identity operator).

Proof. Observe the conditions on the spectrums of T_n imply:

$$T_n = \int_{\sigma(T_n)} t dE_{T_n} \geq a \int_{\sigma(T_n)} dE_{T_n} = aI, \forall n \in \mathbb{N}\setminus\{0\} \Rightarrow \inf_{n\geq 1} T_n \geq aI \in Y_+\setminus\{0\},$$

where E_U is the (positive) spectral measure attached to the self-adjoint operator $U \in Y$. Similarly, we have $T_n \leq bI$ for all natural numbers $n \geq 1$, hence $sup_{n\geq 1} T_n \leq bI \in Y_+$. On the other hand, we have

$$h(I + x_n A) = \int_{\sigma(A)} h(1 + x_n t)\, dE_A \leq \int_{\sigma(A)} x_n t d E_A =$$

$$x_n \int_{\sigma(A)} t dE_A = x_n A, n \geq 1, n \in \mathbf{N}$$

Consequently, also using Theorem 3.1, we derive that the series

$$\sum_{n=1}^{\infty} T_n h(I + x_n A)$$

is convergent in Y_+, because of

$$\sum_{n=1}^{\infty} T_n h(I + x_n A) \leq \left(\sup_{n \geq 1} T_n\right)\left(\sum_{n=1}^{\infty} x_n\right) A \leq bIsA = bsA \in Y_+$$

The next step is to prove a result similar to that from Theorem 2.1, in the operatorial setting. The conclusion of the present theorem will follow via a passing to the limit operation. Let $n \geq 2$ be an arbitrary natural number,

$$s_n := \sum_{j=1}^{n} x_j, S_{n-1} := \left\{ y = (y_1, \dots, y_n) \in \mathbb{R}_+^n; \sum_{j=1}^{n} y_j = s_n \right\},$$

$$g_n(y) := \sum_{j=1}^{n} T_j h(I + y_j A), y \in S_{n-1} \qquad (3)$$

Obviously, for any fixed $t > 0$, the real function $u \to h(1 + ut)$ is concave on $\mathbb{R}_+$, so that $y \to \varphi(y) := h(1 + y_j t)$ is concave on S_{n-1}. Let $\{\alpha_1, \alpha_2\} \subset \mathbb{R}_+, \alpha_1 + \alpha_2 = 1, y^{(k)} \in S_{n-1}, k \in \{1,2\}$. The following relations hold

$$\tilde{\varphi}\left(\alpha_1 y^{(1)} + \alpha_2 y^{(2)}\right) := h\left(I + \left(\alpha_1 y_j^{(1)} + \alpha_2 y_j^{(2)}\right) A\right) =$$

$$\int_{\sigma(A)} h\left(1 + \left(\alpha_1 y_j^{(1)} + \alpha_2 y_j^{(2)}\right) t\right) dE_A =$$

$$\int_{\sigma(A)} h\left(\alpha_1\left(1 + y_j^{(1)} t\right) + \alpha_2\left(1 + y_j^{(2)} t\right)\right) dE_A \geq$$

$$\alpha_1 \int_{\sigma(A)} h\left(1 + y_j^{(1)} t\right) dE_A + \alpha_2 \int_{\sigma(A)} h\left(1 + y_j^{(2)} t\right) dE_A =$$

$$\alpha_1 h\left(I + y_j^{(1)} A\right) + \alpha_2 h\left(I + y_j^{(2)} A\right) = \alpha_1 \tilde{\varphi}\left(y^{(1)}\right) + \alpha_2 \tilde{\varphi}\left(y^{(2)}\right)$$

Hence $\tilde{\varphi}: S_{n-1} \to Y$ is concave. It follows that $\psi: S_{n-1} \to Y$, $\psi(y) = T\tilde{\varphi}(y)$ is concave too, for any $T \in Y_+$ (one can multiply by a positive operator T both members of an inequality, preserving the sense of that inequality, because the product of two self-adjoint positive permutable operators is self-adjoint and positive). On the other hand, it is straightforward that a finite sum of concave operators from a convex subset of a vector space to an ordered vector space is concave. It results that the operator $g_n: S_{n-1} \to Y_+$ defined by (3) is concave. On the other hand, the set of extreme points of S_{n-1} is $Ex(S_{n-1}) = \{e_1, \dots, e_n\}$, where

$$e_1 = (s_n, 0, \dots, 0), \dots, e_n = (0, \dots, 0, s_n)$$

Let $x \in S_{n-1}$. Carathéodory's theorem leads to the existence of $\{\alpha_1, \dots, \alpha_n\} \subset \mathbb{R}_+$, $\sum_{j=1}^{n} \alpha_j = 1$, such that $x = \sum_{j=1}^{n} \alpha_j e_j$. Now we apply Jensen inequality for the concave operator g_n (which can be proved by induction, as in the case of concave real functions). Obviously, we have

$$h(I) = \mathbf{0} \Rightarrow g_n(e_j) = T_j h(I + s_n A), j \in \{1, \dots, n\}.$$

It results

$$g_n(x) \geq \sum_{j=1}^{n} \alpha_j\, g_n(e_j) = \sum_{j=1}^{n} \alpha_j T_j h(I + s_n A) \geq \left(\inf_{1 \leq j \leq n} T_j\right) h(I + s_n A)$$

To conclude the proof, observe that the Lipchitz condition on $y \to h(1 + yt)$ in variable y, uniformly with respect to parameter $t \in \sigma(A)$ leads to

$$s_n \to s \Rightarrow \|h(I + s_n A) - h(I + sA)\| =$$

$$\sup_{t \in \sigma(A)} |h(1 + s_n t) - h(1 + st)| \leq L|s_n - s| \to 0, n \to \infty$$

This proves that

$$s_n \to s \Rightarrow \lim_{n \to \infty} h(I + s_n A) = h(I + sA) \qquad (4)$$

Passing to the limit, also using (4), we get

$$\sum_{n=1}^{\infty} T_n h(I + x_n A) = \lim_n \sum_{j=1}^{n} T_j h(I + x_j A) = \lim_n g_n(x) \geq$$

$$\lim_n \left(\inf_{1 \leq j \leq n} T_j\right) h(I + s_n A) \geq \left(\inf_{n \geq 1} T_n\right)\left(\lim_n h(I + s_n A)\right) =$$

$$\left(\inf_{n \geq 1} T_n\right) h(I + sA)$$

where the limits are considered in the pointwise convergence topology. Notice that convergence in the space Y defined in Theorem 3.2 implies pointwise convergence. The last relation in the statement follows too, because of $inf_{n \geq 1} T_n \geq aI$. □

Remark 3.2. With the notations and under hypothesis of Theorem 3.3, consider the function $u \to ln(1 + ut), u \in \mathbb{R}_+, t \in \sigma(A) \subset (0, \infty)$. Application of Lagrange theorem yields

$$|ln(1+u_1t)-ln(1+u_2t)| \le \sup_{\theta>0} \max_{t\in\sigma(A)} \frac{t}{1+\theta t}|u_1-u_2| \le \|A\|\,|u_1-u_2|$$

Similarly, for $0<p<1$, considering the function

$$u \to h(1+ut) := (1+ut)^p - 1, u \in \mathbb{R}_+, t \in \sigma(A) \subset (0,\infty),$$

one obtains

$$|h(1+u_1t)-h(1+u_2t)| = |(1+u_1t)^p-(1+u_2t)^p| = \left|\frac{p}{(1+\theta t)^{1-p}}t(u_1-u_2)\right| \le p\|A\|\,|u_1-u_2| = L|u_1-u_2|,$$

where the Lipchitz constant $L := p\|A\|$ related to the variable u does not depend on the parameter $t \in \sigma(A)$. Thus both examples 2.1 and 2.2 for h are suitable for the operatorial version of the first inequality (2) proved in Theorem 3.3.

Corollary 3.1. *Under the hypothesis and using the notations from Theorem 3.3, the following relations hold true*

$$\sum_{n=1}^{\infty} T_n ln(1+x_nA) \ge asA(I+sA)^{-1} \ge \frac{as\omega}{1+s\|A\|}I$$

where $\omega := \inf \sigma(A) > 0$.

Proof. Observe that for any $t \in \sigma(A)$ one has

$$ln(1+st) > st(1+st)^{-1}, \qquad st \le s \cdot \sup \sigma(A) = s\|A\|$$

We have $\sigma(I+sA) = 1 + s\sigma(A) \subset [1+s\omega, 1+s\|A\|]$. In particular, 0 is not an element of $\sigma(I+sA)$, so that $I+sA$ is invertible. Moreover, the following relations hold

$$\sigma((I+sA)^{-1}) = \left(\sigma(I+sA)\right)^{-1} \subset \left[\frac{1}{1+s\|A\|}, \frac{1}{1+s\omega}\right]$$

Integrating with respect to the spectral measure E_A one obtains

$$ln(I+sA) = \int_{\sigma(A)} ln(1+st)dE_A \ge \int_{\sigma(A)} st(1+st)^{-1}dE_A = sA(I+sA)^{-1}$$

Thus, the first inequality in the statement follows from Theorem 3.3. For the second one, observe that

$$st(1+st)^{-1} \ge s\omega(1+s\|A\|)^{-1}, \forall t \in \sigma(A) \Rightarrow sA(I+sA)^{-1} \ge \frac{s\omega}{1+s\|A\|}I$$

This concludes the proof. □

4. Conclusion

This work deals with inequalities related to a class of concave functions, by means of finite and respectively infinite sums of concave functions having additional properties. These relations can be adapted to an operatorial version as well. An example involving the function $ln(1+x), x \geq 0$ is pointed out. Another example is briefly discussed. Applications to minimization problems could be obtained for particular other functions and operators verifying the required conditions. The linear constraints from Theorems 2.1 and 2.2 could be replaced by other linear constraints, accompanied by appropriate modifications.

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