ON THE MEDIAN OF A JUMP TYPE DISTRIBUTION

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Atât în biologia moleculară cât şi în ştiinţele sociale sau la modelarea proceselor economice este necesară determinarea pragului θ, $a<\theta<b$, ce defineşte o schimbare a frecventei de răspuns pentru variabila aleatoare X. Vom presupune ca variabila X are densitatea de probabilitate constantă pe subdomeniile $[a\ ,\ \theta\,]$, respectiv $(\theta\,,\ b\,]$ şi, în plus, aceste constante sunt invers proporţionale cu lungimile intervalelor mentionate.

Luând în considerare aceste restricţii, vom demonstra ca estimatorul W a lui θ, estimator bazat pe mediana variabilei aleatoare X, este deplasat. Rezultatele rulării unei proceduri Monte Carlo de simulare stocastică confirmă deplasarea estimaţiilor.

In the molecular biology as well as for modeling the social and economical processes it is necessary to determine the change point θ, $a<\theta<b$, of the frequence response from a random variable X. We'll suppose that the variable X has a constant probability density function on the subdomains $[a\ ,\ \theta\,]$, respectively $(\theta\,,\ b\,]$ and more, these both constants are inverse proportional with the lengths of the mentioned intervals.

Respecting these restrictions we proved that the estimator W of θ, based on the median index of the random variable X, is biased. The running outputs of a stochastic simulation Monte Carlo procedure confirms the theoretical results.

Keywords : change point, estimation, Monte Carlo simulation, median index, probability density function.

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1. Introduction

Often in the molecular biology as well as in economy it is necessary to determine the threshold θ , $a<\theta<b$, which defines a change in the appearance frequencies of some observed events ([6], [8], [9]).

Taking into consideration the restrictions imposed by this kind of phenomena, we'll suppose that the random variable (r.v.) X with the support $[a\,,b]$ has a constant probability density function (p.d.f.) $f_0(x\,;\theta,a,b)$ on the

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subdomains $[a\,,\theta]$ and $(\theta\,,b]$. More, the mentioned constants are inverse proportional with the length of the subintervals $[a\,,\theta]$, respectively $(\theta\,,b]$.

Therefore, the r.v. X has the p.d.f. $f_0(x;\theta,a,b)$,

$$f_0(x;\theta,a,b)=\begin{cases}1/[2(\theta-a)] & ; \quad when \;\; a\le x\le\theta \\ 1/[2(b-\theta)] & ; \quad when \;\; \theta<x\le b\end{cases} \tag{1}$$

In this situation we'll write $X\sim CUT(\theta,a,b)$.

To simplify the exposure we designate by $CU(\lambda)$, $0<\lambda<1$, the standard distribution $CUT(\lambda,0,1)$.

So, the r.v. $Y\sim CU(\lambda)$, $0<\lambda<1$, has the p.d.f. $f(y;\lambda)$, $0\le y\le 1$, where

$$f(y;\lambda)=\begin{cases}1/(2\lambda) & ; \quad for \;\; 0\le x\le\lambda \\ 1/[2(1-\lambda)] & ; \quad for \;\; \lambda<x\le 1\end{cases} \tag{2}$$

In the subsequent, basing on the specific form of the median $Mdn(X)$, we'll propose an estimation $\hat{\theta}$ for the unknown threshold θ when there are known n independent observations $x_1,\, x_2,\, x_3,\; \ldots\;,\, x_n$ from X.

2. Estimating the threshold θ

2.1. Some properties of the distribution $CUT(\theta,a,b)$

Proposition 1. The cumulative distribution function (c.d.f.) $F_0(x;\theta,a,b)$ of the r.v. $X\sim CUT(\theta,a,b)$, $a<\theta<b$, is given by the expression

$$F_0(x;\theta,a,b)=\begin{cases}(x-a)/[2(\theta-a)] & ; \quad for \;\; a\le x\le\theta \\ (x+b-2\theta)/[2(b-\theta)] & ; \quad for \;\; \theta<x\le b\end{cases} \tag{3}$$

Proof. Indeed, for any $a\le x\le b$ it is obvious the equality

$$\frac{\partial F_0(x;\theta,a,b)}{\partial x}=f_0(x;\theta,a,b)$$

Remark 1. Particularly, the r.v. $Y\sim CU(\lambda)$, $0<\lambda<1$, has the c.d.f. $F(y;\lambda)$, $0\le y\le 1$,

$$F(y;\lambda)=\begin{cases}y/(2\lambda) & ; \quad when \;\; 0\le y\le\lambda \\ (y+1-2\lambda)/[2(1-\lambda)] & ; \quad when \;\; \lambda<y\le 1\end{cases} \tag{4}$$

Proposition 2. If $X\sim CUT(\theta,a,b)$ with $a<\theta<b$ and

$$Y=(X-a)/(b-a) \qquad\qquad \lambda=(\theta-a)/(b-a) \tag{5}$$

then $Y \sim CU(\lambda)$.

Proof. Indeed, for any $0 \le y \le 1$ we have $a \le a + (b-a)y \le b$ and therefore

$$Pr(Y \le y) = Pr((X-a)/(b-a) \le y) =$$
$$= Pr(X \le a + (b-a)y) = F_0(a + (b-a)y; \theta, a, b)$$

Since $a \le X \le b$, $a < \theta < b$ it results $0 \le Y \le 1$, $0 < \lambda < 1$.

Case 1. If $0 \le y \le \lambda$ then $a \le a + (b-a)y \le a + (b-a)\lambda = \theta$. So

$$Pr(Y \le y) = F_0(a + (b-a)y; \theta, a, b) = \frac{a + (b-a)y - a}{2(\theta - a)} = \frac{y}{2\lambda} = F(y; \lambda)$$

Case 2. For $\lambda < y \le 1$ we have $\theta = a + (b-a)\lambda < a + (b-a)y \le b$. Therefore

$$Pr(Y \le y) = F_0(a + (b-a)y; \theta, a, b) =$$
$$= \frac{a + (b-a)y + b - 2\theta}{2(b-\theta)} = \frac{y + 1 - 2\lambda}{2(1-\lambda)} = F(y; \lambda)$$

We designate by $Mdn(X)$ the median coefficient for the r.v. X .

Proposition 3. If $X \sim CUT(\theta, a, b)$, $a < \theta < b$, then

$$Mdn(X) = \theta \tag{6}$$

Proof. Indeed we have

$$Pr(X \le \theta) = F_0(\theta; \theta, a, b) = \frac{\theta - a}{2(\theta - a)} = \frac{1}{2}$$

Remark 2. In the subsequent, basing on Proposition 2, instead of the r.v. $X \sim CUT(\theta, a, b)$ we'll study the r.v. $Y \sim CU(\lambda)$.

More precisely, from Proposition 3 and formula (5) we obtain

$$Mdn(Y) = \frac{Mdn(X) - a}{b - a} \tag{7}$$

Instead of the independent observations $x_1, x_2, x_3, \ldots, x_n$ regarding the r.v. X we'll operate with the quantities

$$y_i = \frac{x_i - a}{b - a} \quad , \quad 1 \le i \le n \tag{8}$$

which describe the behavior of the r.v. Y .

2.2. The estimator W based on the median index

Applying Proposition 3 and considering $n = 2s + 1$ independent observations y_1, y_2, y_3, …, y_{2s+1} of the r.v. Y , $Y \sim CU(\lambda)$, we obtain the estimation $\hat{\lambda}$ for the parameter λ , $0 < \lambda < 1$,

$$\hat{\lambda} = y_{(s+1)} \tag{9}$$

where $y_{(i)}$, $1 \le i \le 2s + 1$, are just the quantities y_i arranged in an increased order, that is

$$y_{(1)} \le y_{(2)} \le y_{(3)} \le \ldots \le y_{(s+1)} \le \ldots \le y_{(2s+1)} \tag{10}$$

So we get the estimator W_s to evaluate the unknown value λ, $0 < \lambda < 1$,

$$W_s = Y_{(s+1)} \tag{11}$$

where $Y_{(i)}$, $1 \le i \le 2s + 1$, are the order statistic of degree i attached to the r.v. $Y \sim CU(\lambda)$.

The p.d.f. $g(w;\lambda,s)$, $0 \le w \le 1$, for the r.v. W_s has the expression (Ciucu, Craiu, [1], p.40)

$$g(w;\lambda,s) = \frac{(2s+1)!}{(s!)(s!)} \big(F(w;\lambda)\big)^s \big(1 - F(w;\lambda)\big)^s f(w;\lambda) \tag{12}$$

After a straightforward calculus we deduce

$$g(w;\lambda,s) = \begin{cases} K_s \left(\dfrac{w}{\lambda}\right)^s \left(\dfrac{2\lambda - w}{\lambda}\right)^s \dfrac{1}{\lambda} & ; \quad for\ \ 0 \le w \le \lambda \\ K_s \left(\dfrac{w + 1 - 2\lambda}{1 - \lambda}\right)^s \left(\dfrac{1 - w}{1 - \lambda}\right)^s \dfrac{1}{1 - \lambda} & ; \quad for\ \ \lambda < w \le 1 \end{cases} \tag{13}$$

where

$$K_s = \frac{(2s+1)!}{2^{2s+1}(s!)(s!)} \tag{14}$$

Proposition 4. Respecting the previous notations we have

$$Mean(W_s) = \lambda + (1 - \lambda)\, c_s \tag{15}$$

with

$$c_s = \frac{(s+2)(s+3)(s+4)\ldots(2s+1)}{2^{2s+2}\,(s!)} \tag{16}$$

Proof. Indeed

$$Mean(W_s) = K_s \left(J_s^{(1)} + J_s^{(2)}\right) \tag{17}$$

where

$$J_s^{(1)} = \int_0^{\lambda} w \left(\frac{w}{\lambda}\right)^s \left(\frac{2\lambda - w}{\lambda}\right)^s \frac{dw}{\lambda}$$

$$J_s^{(2)} = \int_{\lambda}^{1} w \left(\frac{w + 1 - 2\lambda}{1-\lambda}\right)^s \left(\frac{1-w}{1-\lambda}\right)^s \frac{dw}{1-\lambda} \tag{18}$$

Using the substitution $w = (1-t)\lambda$ we deduce

$$J_s^{(1)} = -\int_1^0 (1-t)\lambda(1-t)^s (1+t)^s \, dt = \lambda \int_0^1 \left(1-t^2\right)^s dt - \lambda \int_0^1 t\left(1-t^2\right)^s dt \tag{19}$$

Similarly, after the substitution $w = \lambda + (1-\lambda)v$ it results

$$J_s^{(2)} = \int_0^1 \left(\lambda + (1-\lambda)v\right)(1+v)^s (1-v)^s \, dv =$$

$$= \lambda \int_0^1 \left(1-v^2\right)^s dv + (1-\lambda)\int_0^1 v\left(1-v^2\right)^s dv \tag{20}$$

From the formulas (17), (19) and (20) we finally get

$$Mean(W_s) = K_s \left((2\lambda)\, J_s^{(3)} + (1-2\lambda)\, J_s^{(4)}\right) \tag{21}$$

where

$$J_s^{(3)} = \int_0^1 \left(1-t^2\right)^s dt \qquad\qquad J_s^{(4)} = \int_0^1 t\left(1-t^2\right)^s dt \tag{22}$$

The value of the expression $J_s^{(3)}$ can be computed recurrently. So

$$J_s^{(3)} = \int_0^1 \left(1-t^2\right)^s dt = t\left(1-t^2\right)^s \Bigg|_{t=0}^{t=1} + 2s\int_0^1 t^2\left(1-t^2\right)^{s-1} dt =$$

$$= -2s\int_0^1 \left(1-t^2\right)^s dt + 2s\int_0^1 \left(1-t^2\right)^{s-1} dt = -2s\, J_s^{(3)} + 2s\, J_{s-1}^{(3)}$$

More precisely

$$J_s^{(3)} = \frac{2s}{2s+1}\, J_{s-1}^{(3)} \tag{23}$$

Applying the formula (23) we'll prove inductively the equality

$$J_s^{(3)} = \frac{2^{2s}\,(s!)(s!)}{(2s+1)!} = \frac{1}{2K_s} \tag{24}$$

First, the formula (24) is true for $s = 1$,

$$J_1^{(3)} = \int_0^1 \left(1 - t^2\right) dt = \frac{2}{3} = \frac{2^2\,(1!)(1!)}{3!}$$

Now, supposing that $J_s^{(3)}$ has the form (24) and applying the recurrent relation (23) we'll verify the inductive hypothesis. So

$$J_{s+1}^{(3)} = \frac{2s+2}{2s+3} J_s^{(3)} = \frac{2s+2}{2s+3} \cdot \frac{2^{2s}\,(s!)(s!)}{(2s+1)!} = \frac{2^{2(s+1)}\,(s+1)!\,(s+1)!}{(2s+3)!}$$

After using the variable transform $v = 1 - t^2$ we also have

$$J_s^{(4)} = \int_0^1 t\left(1 - t^2\right)^s dt = -\frac{1}{2}\int_1^0 v^s dv = \frac{1}{2(s+1)} \tag{25}$$

From the formulas (24)-(25) and respecting the notation (16), the expression (21) becomes finally

$$Mean(W_s) = K_s\left(2\lambda\frac{1}{2K_s} + (1-2\lambda)\frac{1}{2(s+1)}\right) =$$

$$= \lambda + (1-2\lambda)\frac{K_s}{2(s+1)} = \lambda + (1-2\lambda)c_s$$

Remark 3. Since $c_s > 0$ and $0 < \lambda < 1$, from the relation (15) it results that the estimator W_s overestimates the real value of the parameter λ when $0 < \lambda < 0.5$. Similarly, in the case $0.5 < \lambda < 1$, the parameter λ is subestimated. The estimator W_s is unbiased for $\lambda = 0.5$.

The bias value $B(s,\lambda) = (1-2\lambda)c_s$ of the estimator W_s depends on the value of λ and also on the size $n = 2s+1$ of the sample used in the estimation process.

More, the r.v. W_s is an absolutely correct estimator ([2], [5], [7]) since

Proposition 5. For any $0 < \lambda < 1$ we have

$$\lim_{s \to \infty} B(s,\lambda) = 0 \tag{26}$$

Proof. Since $B(s,\lambda) = (1-2\lambda)c_s$ it is sufficient to show that

$$\lim_{s \to \infty} c_s = c = 0 \tag{27}$$

But for every natural number s are true the relations

$$c_s > 0 \qquad \frac{c_{s+1}}{c_s} = \frac{2s+3}{2s+4} < 1 \tag{28}$$

which imply the strict inequalities

$$c_1 > c_2 > c_3 > \ldots > c_s > c_{s+1} > \ldots > 0 \tag{29}$$

Therefore the string $\{c_s\}_s$, being bounded and monotone, is convergent to a value $c \geq 0$.

It remains to prove that $c = 0$.

Indeed, applying successively k times the equality (28) we deduce

$$c_{2k} = \frac{(4k+1)(4k-1)(4k-3)\ldots(2k+3)}{(4k+2)(4k)(4k-2)\ldots(2k+4)} c_k$$

So it results the inequality

$$c_{2k} < \left(\frac{4k+1}{4k+2}\right)^k c_k$$

and hence

$$\lim_{k\to\infty} c_{2k} \leq \lim_{k\to\infty} \left(\frac{4k+1}{4k+2}\right)^k . \lim_{k\to\infty} c_k$$

that is

$$c \leq e^{-1/4} . c$$

This last inequality don't accept the variant $c > 0$ since in this hypothesis we obtain the incompatible result $1 \leq e^{-1/4}$. Having always $c \geq 0$ it remains only the alternative $c = 0$, that is the relation (27) is true.

Remark 4. The string $\{c_s\}_s$ decreases very slowly to the limit $c = 0$ (see Table 1).

Table 1

The values of the coefficients c_s (formula (16))

s	1	2	3	4	5	6	7	8
c_s	0.1875	0.1563	0.1367	0.1231	0.1128	0.1047	0.0982	0.0927
s	9	10	15	20	25	30	35	40
c_s	0.0881	0.0841	0.0700	0.0612	0.0551	0.0505	0.0469	0.0439
s	45	50	75	100	125	150	175	200
c_s	0.0415	0.0394	0.0323	0.0280	0.0251	0.0229	0.0212	0.0199
s	250	300	350	400	500	600	700	800
c_s	0.0178	0.0163	0.0151	0.0141	0.0126	0.0115	0.0107	0.0100
s	900	1000	1500	2000	2500	3000	4000	5000
c_s	0.0094	0.0089	0.0073	0.0063	0.0056	0.0051	0.0045	0.0040

2.3. Monte Carlo simulations

Let U and V be two independent r.v.-s, each of them having an uniform distribution over the interval $[0\,,1]$.

Fixing an arbitrary value $0<\lambda<1$ we define the r.v. Z_λ ,

$$Z_\lambda = \begin{cases} \lambda U & ; \quad daca \;\; V \le 0.5 \\ \lambda + (1-\lambda)U & ; \quad daca \;\; V > 0.5 \end{cases} \tag{30}$$

Proposition 6. For any $0<\lambda<1$ we have $Z_\lambda \sim CU(\lambda)$.

Proof. Keeping the previous notations we'll show that for every $0 \le z \le 1$ it is true the equality

$$Pr(Z_\lambda \le z) = F(z\,;\lambda)$$

Indeed

$$\begin{aligned} Pr(Z_\lambda \le z) &= Pr((\lambda U \le z\,, V \le 0.5) \cup (\lambda + (1-\lambda)U \le z\,, V > 0.5)) = \\ &= Pr(\lambda U \le z\,, V \le 0.5) + Pr(\lambda + (1-\lambda)U \le z\,, V > 0.5) = \\ &= Pr(\lambda U \le z)\; Pr(V \le 0.5) + Pr(\lambda + (1-\lambda)U \le z)\; Pr(V > 0.5) = \\ &= \frac{1}{2} Pr\left(U \le \frac{z}{\lambda} \right) + \frac{1}{2} Pr\left(U \le \frac{z-\lambda}{1-\lambda} \right) \end{aligned}$$

Case 1. If $0 \le x \le \lambda$ then

$$Pr(Z_\lambda \le z) = \frac{1}{2} Pr\left(0 \le U \le \frac{z}{\lambda} \right) + \frac{1}{2} Pr(U \le 0) = \frac{1}{2}\left(\frac{z}{\lambda} \right) = F(z\,;\lambda)$$

Case 2. In the situation $\lambda < x \le 1$ we have

$$Pr(Z_\lambda \le z) = \frac{1}{2} Pr\left(U \le \frac{z}{\lambda} \right) + \frac{1}{2} Pr\left(U \le \frac{z-\lambda}{1-\lambda} \right) =$$

$$= \frac{1}{2} Pr(0 \le U \le 1) + \frac{1}{2} Pr\left(0 \le U \le \frac{z-\lambda}{1-\lambda} \right) = \frac{1}{2} + \frac{1}{2}\left(\frac{z-\lambda}{1-\lambda} \right) = F(z\,;\lambda)$$

Proposition 6 can be used to generate an arbitrary sample $y_1, y_2, y_3, \ldots, y_{2s+1}$ from the r.v. $Y \sim CU(\lambda)$ (see also [3], [4]).

The following Monte Carlo algorithm permits us an empirical validation of the biased property for W_s (according to Proposition 4) and more to evaluate the dispersion of the $\hat{\lambda}$ estimations.

Algorithm AMC (Monte Carlo procedure to obtain an estimation $\hat{\lambda}$).

Step 0. Input : $s \in N_+$ (sample size $n = 2s+1$) , $0 < \lambda < 1$

Step 1. $i = 0$

Step 2. $i = i + 1$

Step 3. Generate two independent variates U, V which are uniformly
Distributed over the interval $[0, 1]$

Step 4. (Generate independent observations y_i for the r.v. $Y \sim CU(\lambda)$)
If $V \le 0.5$ then $y_i = \lambda U$
else $y_i = \lambda + (1-\lambda)U$

Step 5. If $i < 2s+1$ then Goto Step 2

Step 6. (Sort in an increasing order the values y_i , $1 \le i \le 2s+1$)
Obtain the sorted string $y_{(1)} \le y_{(2)} \le y_{(3)} \le \ldots \le y_{(s+1)} \le \ldots \le y_{(2s+1)}$

Step 7. (Determine the estimation $\hat{\lambda}$ for the parameter λ)
$\hat{\lambda} = y_{(s+1)}$

Step 8. Output $\hat{\lambda}$. STOP.

Considering $s = 3$ and $\lambda = 0.25$, in Table 2 were listed the estimations $\hat{\lambda}$ obtained by running 20 times the stochastic simulation algorithm ***AMC***.

Graphic G1 represents the variation of the estimation values $\hat{\lambda}$ listed in Table 2.

Remark 5. Graphic G1 emphasizes the overevaluation of the estimations $\hat{\lambda}$ when $\lambda = 0.25$ (see Remark 3 and formula (15) too). The overevaluation property is also confirmed practically by the inequality

$$\frac{\hat{\lambda}_1 + \hat{\lambda}_2 + \hat{\lambda}_3 + \ldots + \hat{\lambda}_{20}}{20} = 0.29148 > 0.25 = \lambda \tag{31}$$

where $\hat{\lambda}_i$, $1 \le i \le 20$, are the values presented in Table 2.

Table 2

The estimations $\hat{\lambda}$ deduced by using the median of the sample y_i, $1 \le i \le n = 7$
($\lambda = 0.25$, 20 simulations).

sim	Y_1	y_2	y_3	y_4	y_5	Y_6	y_7	$\hat{\lambda}$
1	0.0894	0.8453	0.3327	0.0331	0.0337	0.0991	0.4308	0.0991
2	0.5433	0.0232	0.3697	0.2198	0.9935	0.2178	0.1240	0.2198
3	0.0152	0.3897	0.0308	0.5273	0.9170	0.0392	0.0583	0.0583
4	0.5477	0.3138	0.9773	0.1955	0.0489	0.7853	0.7278	0.5477
5	0.8860	0.7512	0.8655	0.6152	0.1604	0.1652	0.2345	0.6152
6	0.3182	0.1287	0.1813	0.2410	0.3708	0.0001	0.0688	0.1813
7	0.0235	0.8627	0.2669	0.8860	0.9663	0.1856	0.2210	0.2669
8	0.2946	0.2396	0.1041	0.5875	0.0979	0.5158	0.7381	0.2946
9	0.2082	0.1575	0.1355	0.8973	0.6042	0.1640	0.0328	0.1640
10	0.0096	0.4959	0.0784	0.5747	0.3884	0.1131	0.5351	0.3884
11	0.8209	0.5931	0.0335	0.0938	0.6130	0.0855	0.6887	0.5931
12	0.0409	0.6220	0.8546	0.7073	0.1530	0.0396	0.1401	0.1530
13	0.1961	0.2732	0.1396	0.3156	0.0648	0.2869	0.1366	0.1961
14	0.1592	0.0166	0.4368	0.7221	0.7313	0.5763	0.3220	0.4368
15	0.1364	0.0927	0.6586	0.0363	0.3010	0.3505	0.6361	0.3010
16	0.0301	0.0950	0.1003	0.5327	0.7526	0.3722	0.0935	0.1003
17	0.2789	0.5292	0.1988	0.0632	0.2420	0.5262	0.5329	0.2789
18	0.0459	0.9381	0.3177	0.2535	0.2392	0.7987	0.2446	0.2535
19	0.6950	0.4452	0.1591	0.6150	0.0315	0.5282	0.2340	0.4452
20	0.0323	0.2364	0.4964	0.0743	0.1727	0.5933	0.9111	0.2364

3. Concluding remarks

We analyzed the behavior of the estimator W_s proposed to evaluate the threshold λ of the r.v. $Y \sim CU(\lambda)$, $0 < \lambda < 1$.

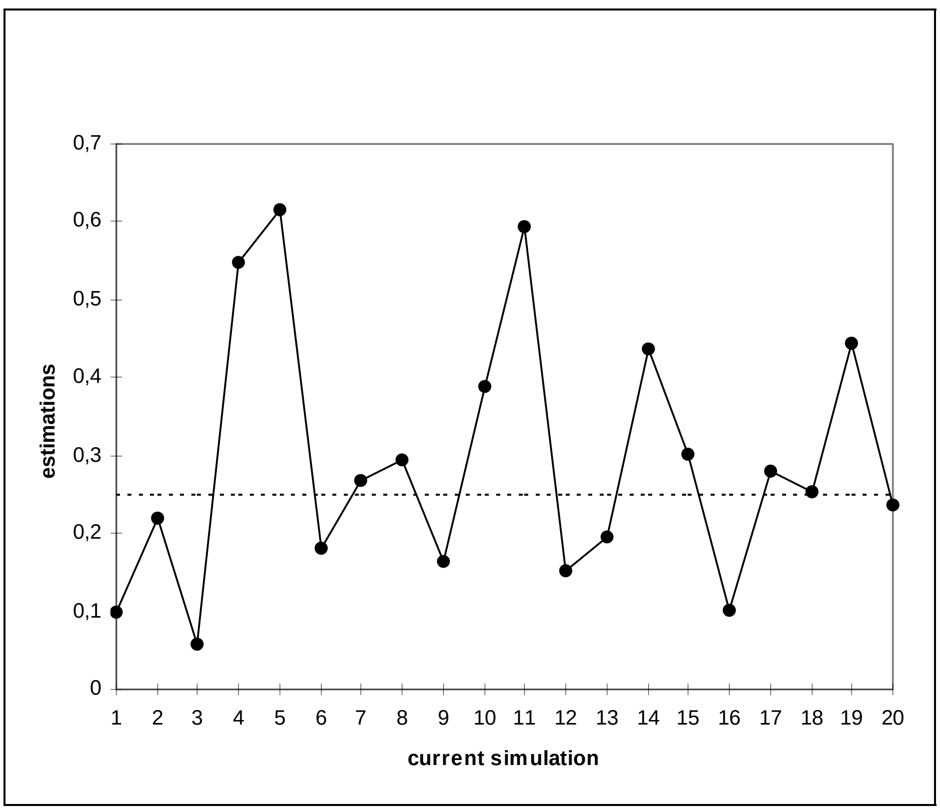
Proposition 4 brings supplementary informations regarding the size of the bias $B(s,\lambda)$ for the r.v. W_s. Depending on the effective value of λ we'll have an overevaluation process (when $0 < \lambda < 0.5$) or an underevaluation in the case $0.5 < \lambda < 1$ (Remark 3).

More, the bias $B(s;\lambda)$ tends to zero when is increased the volume $n=2s+1$ of the sample $\{y_i\}_{1\le i\le 2s+1}$ used to estimate the parameter λ (see Proposition 5).

As function of the index s the string $\{B(s;\lambda)\}_{s\in N}$ decreases very slowly to zero (Table 1).

The practical application of a Monte Carlo simulation procedure validates the theoretical results (to compare the estimation points in the graphic G1 and view the inequality (31)).

The distribution $CUT(\theta,a,b)$, $a<\theta<b$, has multiple practical interpretations in the study of some molecular processes and also for modeling complex aspects from the economy.



Graphic G1. Twenty estimations for the parameter λ
(n = 7 , λ = 0.25 , 20 simulations)

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