RESEARCH ON THE REWARDS AND PUNISHMENTS OF URBAN SEWAGE TREATMENT UNDER THE UNCERTAINTY THEORY

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This paper fills a gap in urban sewage treatment decision-making by exploring reward and punishment mechanisms within the uncertainty theory framework. Addressing challenges in characterizing sewage treatment capacity due to incomplete information, the study integrates incentive and punishment mechanisms in a decision-making model.

Results indicate that government rewards encourage companies to treat more sewage, but a nuanced relationship exists between rewards, treatment capacity, and expected returns. The study emphasizes the need for a balanced approach, suggesting a focus on technology enhancement over sole reliance on rewards.

Examining a double-headed sewage market, the research establishes an uncertain Cournot game model, providing insights into sewage treatment dynamics under asymmetric information.

In conclusion, the paper contributes to practical reward and punishment strategies in sewage treatment policies, offering theoretical support for effective decision-making in urban sewage management within a concise framework..

Keywords: Urban sewage treatment; Uncertain Theory; reward mechanism; punishment mechanism

MSC2020: 49J30, 47H09, 47H10, 47J20.

1. Introduction

Urban sewage has a profound and detrimental impact on the delicate ecological environment[1]-[2]. Therefore, improving the efficiency of wastewater treatment through the study of urban wastewater treatment policies plays a vital role in improving the quality of the urban environment.[3]-[4]. In order to transfer some of the government's responsibilities to the enterprise, the government adopted the PPP (Public Private Partnership) model at the micro level, which was based on successful foreign practices.Some scholars have explored how the government effectively transfers the risk of sewage treatment to enterprises through Public-Private Partnership (PPP) models [5]. Others have highlighted the ability of PPP models to alleviate funding shortages in government-led sewage treatment projects. Risk management[6] models for urban sewage treatment projects have been proposed, as well as studies on incentive price control, optimal pricing standards for sewage treatment fees in China, and performance evaluations of sewage treatment enterprises[7].

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Describing the sewage treatment capacity of enterprises as a random variable has been discussed by some scholars[8]-[9]. However, this approach has limitations due to the difficulty in identifying the occurrence patterns of various uncertain factors leading to the failure of sewage treatment tasks. Collecting sufficient historical production data to estimate the probability distribution of sewage treatment capacity is challenging, as each sewage treatment enterprise has only a few production lines, and it takes time to accumulate a substantial number of samples. In addressing this issue, some scholars suggest using expert experience and probability theory or subjective probability theory to describe the actual sewage treatment volume of enterprises[10]. However, subjective probability theory lacks theoretical proof of the additivity of subjective experiences, raising concerns about its applicability [11]. Additionally, the use of fuzzy random variables has been proposed to describe sewage treatment control systems [12]. Furthermore, accurately characterizing the sewage treatment capacity of enterprises is challenging due to incomplete information, uncertainties, and limitations in data collection and modeling techniques.

This paper aims to integrate both incentive and punishment mechanisms and examine how the incentive mechanism influences urban sewage treatment decision-making within the framework of uncertainty theory. Experts are devoted to the evaluation of a wastewater treatment decision-making model which includes reward and punishment mechanism in the uncertain decision space and discussing the role of enterprises reward mechanism in tapping the potential wastewater treatment capacity. All in all, it has an important and real significance for deepening the government's idea which is "replacing compensation with reward" for improving the urban sewage treatment capacity. The following structure is arranged as follows: Section 2 is the theoretical framework and model . Section 3 gives the numerical simulation results. Section 4 gives a final conclusion.

2. **Preliminaries**

Definition 2.1. *[11] A number $M(\Lambda)$ indicate the reliability of Λ, which event Λ is an element on L and Λ^c is a complement of Λ on L. To treat this reliability rationally, Liu proposes three three axioms in the following.*
Axiom 1. (Normality) $M(\Lambda) = 1$;
Axiom 2.(Self-duality)$M(\Lambda) + M(\Lambda^c) = 1$ for any event Λ ;
Axiom 3.(Subadditivity)$M\bigcup_{i=1}^{\infty}\Lambda_i \leq \sum_{i=1}^{\infty} M\{\Lambda_i\}$ for any countable sequence of events$\{\Lambda_i\}$.

Definition 2.2. *[11] Let ξ be an uncertain variable. Then the expected value of ξ is defined as follows*

$$E[\xi] = \int_0^{+\infty} M\{\xi \geq x\}dx - \int_{-\infty}^{0} M\{\xi \leq x\}dx$$

provided that one of the two integrals is finite at least.

Theorem 2.1. *[11] Let ξ be an uncertain variable with an uncertain distribution Φ, then*

$$E[\xi] = \int_0^{+\infty}(1 - \Phi(x))dx - \int_{-\infty}^{0} \Phi(x)dx$$

Theorem 2.2. *[11] Let ξ be an uncertain variable with an uncertain distribution Φ, if its expected value exists, then the expected value is*

$$E[\xi] = \int_{-\infty}^{+\infty} xd\Phi(x)dx$$

If the uncertain distribution Φ is regularthen we have $E[\xi] = \int_0^1 \Phi^{-1}(\alpha)d\alpha$.

Theorem 2.3. *[12] Let ξ be an uncertain variable with an uncertain distribution Φ. If $f(x)$ is a strictly monotonic (monotonically increasing or monotonically decreasing) function, then there exists an expectation*

$$E[f(\xi)] = \int_{-\infty}^{+\infty} f(x)d\Phi(x)$$

Theorem 2.4. *[12] Let ξ be an uncertain variable with an uncertain distribution Φ, then*

$$E[\xi] = \int_{0}^{+\infty}(1-\Phi(x))dx - \int_{-\infty}^{0}\Phi(x)dx$$

Theorem 2.5. *[12] Let ξ be an uncertain variable with an uncertain distribution Φ, if its expected value exists, then the expected value is*

$$E[\xi] = \int_{-\infty}^{+\infty} xd\Phi(x)dx$$

If the uncertain distribution Φ is regularthen we have $E[\xi] = \int_0^1 \Phi^{-1}(\alpha)d\alpha$.

Definition 2.3. *[12] An uncertain variable satisfies condition*

$$\Phi(x) = \begin{cases} 0, x \le a \\ \dfrac{x-a}{b-a}, a \le x \le b, 0, \\ 1, x \ge b \end{cases}$$

Then ξ is said to be linear and satisfies a linear distributiondenoted as $\xi \sim L[a,b]$.

Theorem 2.6. *[12] Let ξ be an uncertain variable with an uncertain distribution Φ. If $f(x)$ is a strictly monotonic (monotonically increasing or monotonically decreasing) function, then there exists an expectation*

$$E[f(\xi)] = \int_{-\infty}^{+\infty} f(x)d\Phi(x)$$

2.1. Decision-making model for urban wastewater treatment with a reward and punishment mechanism

In a monopolistic competitive wastewater treatment market, the government's incentive and punishment mechanism for businesses are set as two aspects respectively: (1) During the signing period, if the enterprise completes the contracted sewage treatment capacity in advance, the government will reward the net income of the excess treatment for the encourage of the enterprise continuing treating sewage; (2) During the contract period, if the enterprise fails to complete the contracted sewage treatment volume, the government will punish the value of the difference between the contracted volume and the actual treatment volume to restrain the enterprise's Moral hazard. Note that, under only conditions on the penalty mechanism, the enterprise can effectively avoid the risk of penalty by appropriately reducing the contracted volume. So the added value $k(k=0)$ of Marginal cost of the enterprise's sewage treatment capacity brought by the punishment set (The change in cost per additional unit of sewage treatment). However, in the reward and punishment mechanism, to obtain the expected reward, the sewage treatment volume signed by the enterprise will be accompanied by a higher punishment risk. Therefore, the Marginal cost caused by the punishment mechanism will increase to a constant $k(k>0)$.

1.Hypothesis

(1)The government provides sufficient sewage for treatment. If the actual sewage for treatment is insufficient, the fee shall be paid according to the contracted sewage treatment amount.

(2)The quality of the treated wastewater meets the requirements of the government;

(3) The sewage treatment agreement will not be affected by all external majeure factors;

2.Model establishment

Table 1. List of Variables, Definitions and Sources

Variables	**Definition/Measurement**
x	Contracted sewage volume (million tons), $x \geq 0$
ξ	Actual volume of sewage treated (million tons), $\xi \geq 0$
p	Unit price of treatment cost to be paid by the government to the enterprise for each ton of sewage treated (RMB/ton)
c	Unit price of the total cost of treating wastewater at the enterprise (RMB/ton) $p > c > 0$
w	Government incentive factor (percentage) for business over-achievement$w > 0$
v	Government Penalty Factor for non-completion by enterprises (percentage) $v > 0$
$\mu(x)$	Corporate income (million)
k	A constant for the increase in a firm's marginal cost due to the penalty mechanism$0 < k < \frac{p-c}{v}$

Suppose that ξ is an uncertain variable, which is defined in the uncertainty space (Γ, L, M) and is of an independent and regular uncertain distribution. Also, ξ is the actual amount of wastewater treated by the company. Since $\xi \geq 0$, then we have $M\{\xi < 0\}$. Therefore, when the firm's contracted treatment quantity is x million tons, then its revenue $\mu(x)$ can be viewed as an uncertain revenue function defined on the uncertain space (Γ, L, M), where min $\{x, \xi\}$ is a fractional function that compares the firm's contracted sewage treatment quantity with the actual treatment quantity.

$$\begin{aligned}\mu(x,\xi) &= (p-c)min\{x,\xi\} - p(x - min\{x,\xi\})v + (p - c(v))(\xi - min\{x,\xi\})(1+w)\\ &= (p-c)min\{x,\xi\} - p(x - min\{x,\xi\})v + (p - (c+kv))(\xi - min\{x,\xi\})(1+w)\\ &= [p - c + pv - (p - c - kv)(1+w)]min\{x,\xi\} - pvx + (p - c - kv)(1+w)\xi\end{aligned} \tag{1}$$

For the actual sewage treatment problem of enterprises, we solve it and consider the expected return of enterprises by use of definition 2.11, (2) and the uncertainty theory. Next, we build the following sewage treatment expected return decision model,

Definition 2.4. *[8] Let the income of the sewage treatment enterprise be less than or equal to μ^*, and its maximum risk index be less than or equal to β, then we follow the reward and punishment mechanism and the sewage treatment Decision model is the following*

$$\begin{cases} maxE[u(x,\xi)] \\ s.t. \\ M\{u(x,\xi) \leq u^*\} \leq \beta \\ 0 \leq x \end{cases} \tag{2}$$

Where μ^* is the conservative return of the firm, β is the maximum risk index that the firm can bear $(0 \leq \beta \leq 1)$. the maximization of the expected return of the wastewater treatment firm satisfies that: $\mu(x,\xi) \leq \mu^*$occurs with level of confidence M less than or equal to the maximum risk index β.

3.Model solution

The given model (3) is an uncertain planning model and we assume that ξ obey the linear uncertainty distribution $L[a, b]$. According to Definition 2.10[2],Theorem 2.4[2],Theorem 2.7 [2] and (2), then we have

Theorem 2.7. *According to the uncertainty theory, Expected returns of wastewater treatment firms with incentive and penalty mechanisms is the following*

$$E[\mu_k(x,\xi)] = \begin{cases} [kv-(p-c-kv)w]x+\frac{1}{2}(p-c-kv)(1+w)(a+b), x\leq a \\ \frac{1}{2(b-a)}[((p-c-kv)w-kv-pv)x^2+2(pva+(kv-(p-c-kv)w)b)x \\ -((1-v)p-c)a^2+(p-c-kv)(1+w)b^2], a\leq x\leq b \\ -pvx+\frac{1}{2}[(1+v)p-c](a+b), x\geq b \end{cases}$$

Proof Substituting equation (1) into $E[\mu_{jc}(x,\xi)]$, and by the definition of uncertain expectation, we show that

$$E[\mu_{jc}(x,\xi)] = E[[p-c-pv-(p-c-kv)(1+w)]min\{x,\xi\}-pvx+(p-c-kv)(1+w)\xi] \\ = \int_{-\infty}^{+\infty}\mu_{jc}(x,t)d\Phi(t) \tag{3}$$

where ξ is an uncertain variable subject to a linear distribution $L[a,b]$ and has a regular uncertain distribution$\Phi(t),(t\in\Re^+)$. Also $\mu_{jc}(x,\xi)$ is a monotonically increasing function with respect to ξ, which is known from the practical background of wastewater treatment $x\geq 0,\xi\geq 0$. Also by Theorem 2.7 [6] and the nature of the linear distribution $L[a,b]$,x can be classified into the following three cases:

(1) When $x\leq a$, then $x<\xi$, then equation (3) will be rewritten as

$$E[\mu_k(x,\xi)] = E[[kv-(p-c-kv)w]x+(p-c-kv)(1+w)\xi] \\ = [kv-(p-c-kv)w]x+(p-c-kv)(1+w)E[\xi],$$

since$\xi\in L[a,b]$, then$E[\xi]=\int_0^1\Phi^{-1}(\alpha)d\alpha$, which means that

$$E[\mu_k(x,\xi)] = [kv-(p-c-kv)w]x+\frac{1}{2}(p-c-kv)(1+w)(a+b)$$

(2) When $a\leq x\leq b$, , then x less than ξ or greater than ξ, then equation (3) shows that

$$\begin{aligned} E[\mu_k(x,\xi)] &= \int_{-\infty}^{+\infty}\mu_{jc}(x,t)d\Phi(t) \\ &= \int_0^a\mu_{jc}(x,t)d\Phi(t)+\int_a^x\mu_{jc}(x,t)d\Phi(t) \\ &+ \int_x^b\mu_{jc}(x,t)d\Phi(t)+\int_b^{+\infty}\mu_{jc}(x,t)d\Phi(t) \\ &= \frac{1}{2(b-a)}[((p-c-kv)w-kv-pv)x^2+2(pva+(kv-(p-c-kv)w)b)x \\ &-((1+v)p-c)a^2+(p-c-kv)(1+w)b^2] \end{aligned}$$

(3) When $x\geq b$, then $x>\xi$, then equation (3) leads to the following conclusion

$$\begin{aligned} E[\mu_k(x,\xi)] &= \int_{-\infty}^{+\infty}\mu_{jc}(x,t)d\Phi(t) = \int_0^1\{((1+v)p-c)\Phi^{-1}(\alpha)-pvx\}d\alpha \\ &= \int_0^1\{((1+v)p-c)(1+\alpha)a+\alpha b\}d\alpha-pvx \\ &= -pvx+\frac{1}{2}[(1+v)p-c](a+b) \end{aligned}$$

All in all, we conclude that

$$E[\mu_k(x,\xi)] = \begin{cases} [kv-(p-c-kv)w]x+\frac{1}{2}(p-c-kv)(1+w)(a+b) \\ \frac{1}{2(b-a)}[((p-c-kv)w-kv-px)x^2+2(pva+(kv-(p-c-kv)w)b)x] \\ -((1+v)p-c)a^2+(p-c-kv)(1+w)b^2 \\ -pvx+\frac{1}{2}[(1+v)p-c](a+b) \end{cases}$$

Second, for the constraint $M\{\mu_{jc}(x,\xi)\leq\beta\}$,since ξ is an uncertain variable and $\mu_{jc}(x,\xi)$ is an uncertain variable function, which is continuous and monotonic according to Theorem 2.1, we denote the set of feasible solutions by $D(w,v,\beta)$. For the analysis of $D(w,v,\beta)$, we further get the following result.

Theorem 2.8. *The set of feasible solutions satisfies (2), then*

$$D(w,v,\beta)=[\frac{\mu^*-(p-c-kv)(1+w)(a+(b-a)\beta)}{kv-(p-c-kv)w},\frac{[(1+v)p-c][a+(b-a)\beta]-\mu^*}{pv}] \tag{4}$$

Proof Substituting (1) into the constraint (2), then we have

$$M\{\mu_{jc}(x,\xi)\leq\mu^*\}\leq\beta\Leftrightarrow$$
$$M\{[(1+v)p-c-(p-c-kv)(1+w)]min\{x,\xi\}-pvx+(p-c-kv)(1+w)\xi\leq\mu^*\}\leq\beta\Leftrightarrow$$
$$M\{x\leq\frac{\mu^*+pvx-(p-c-kv)(1+w)\xi}{p-c+pv-(p-c-kv)(1+w)}\}\leq\beta, \tag{5}$$

or

$$M\{\xi\leq\frac{\mu^*+pvx-(p-c-kv)(1+w)\xi}{p-c+pv-(p-c-kv)(1+w)}\}\leq\beta \tag{6}$$

Analyzing (5), we can further obtain

$$M\{x\leq\frac{\mu^*+pvx-(p-c-kv)(1+w)\xi}{p-c+pv-(p-c-kv)(1+w)}\}\leq\beta\Leftrightarrow M\{\xi\leq\frac{\mu^*-[p-c-(p-c-kv)(1+w)]x}{(p-c-kv)(1+w)}\}\leq\beta$$
$$\Leftrightarrow x\geq\frac{\mu^*-(p-c-kv)(1+w)(a+(b-a)\beta)}{kv-(p-c-kv)w},$$

Similarly, analyzing (6), we can get the result as follow: Consider above, $D(w,v,\beta)$ satisfy the condition

$$D(w,v,\beta)=[\frac{\mu^*-(p-c-kv)(1+w)(a+(b-a)\beta)}{kv-(p-c-kv)w},\frac{[(1+v)p-c][a+(b-a)\beta-\mu^*}{pv}]$$

Based on the above analysis, model (2) can be translated into the following corollary:
Corollary 1 Based on the uncertainty theory, the model for wastewater treatment decisions with a reward and punishment mechanism is given in the following,

$$\max_{x\in D(w,v,\beta)} E[u(x,\xi)]$$

where $E[(x,\xi)]$ satisfies Theorem 2.14 and $x\in D(w,v,\beta)$ satisfies equation (3).

Next, the paper further analyses the existence of the solution of model (2). For ease of analysis, the expected return function $E[\mu(x,\xi)]$ segments of the objective function (3) are labelled as E_1, E_2, E_3 respectively, then we get

$$E[\mu(x,\xi)] = \begin{cases} E_1 \ x \le a \\ E_2 \ a \le x \le b \\ E_3 \ x \ge b \end{cases}$$

$$E_1 = [kv - (p-c-kv)w]x + \frac{1}{2}(p-c-kv)(1+w)(a+b),$$

$$E_2 = \frac{1}{2(b-a)}[((p-c-kv)w - kv - pv)x^2 + 2(pva + (kv - (p-c-kv)w)b)x - ((1+v)p-c)a^2 + (p-c-kv)(1+w)b^2],$$

$$E_3 = -pvx + \frac{1}{2}[(1+v)p-c](a+b),$$

Where E_1 and E_3 are linear functions of x on $[0,a]$ and $[b,+]$ respectively, and E_3 is decreasing, while E_2 is a quadratic parabolic function of x on $[a,b]$and the axis of symmetry xd of E_2 can be obtained by taking the first order derivative of E_2 on $[a,b]$ to its extreme value.

$$E_2 = \frac{1}{2(b-a)}[2((p-c-kv)w - kv - pv)x + 2(pva + (kv - (p-c-kv)w)b)] = 0$$

$$x_d = \frac{-pva - (kv - (p-c-kv)w)b}{(p-c-kv)w - kv - pv},$$

Further, we labelle the constraint interval with $D(v,,\mu^*)$ as $[x1, x2]$,then

$$[x] = [\frac{\mu^* - (p-c-kv)(1+w)(a+(b-a)\beta)}{kv-(p-c-kv)w}, \frac{[(1+v)p-c][a+(b-a)\beta - \mu^*}{pv}]$$

After the following comparison ofa, x_1, x_2, x_d, b , we get

$$x_d - a = \frac{-pva - (kv - (p-c-kv)w)b}{(p-c-kv)w - kv - pv} - a = \frac{[(p-c-kv)w - kv](b-a)}{(p-c-kv)w - kv - pv}, \quad (7)$$

$$x_d - b = \frac{-pva - (kv - (p-c-kv)w)b}{(p-c-kv)w - kv - pv} - b = \frac{pv(b-a)}{(p-c-kv)w - kv - pv}, \quad (8)$$

$$\begin{aligned} x_1 - x_2 &= \frac{\mu^* - (p-c-kv)(1+w)(a+(b-a)\beta)}{kv-(p-c-kv)w} - \frac{[(1+v)p-c][a+(b-a)\beta - \mu^*}{pv} \\ &= \frac{[pv + kv - (p-c-kv)w][\mu^* - (p-c)(a+(b-a)\beta)]}{(kv-(p-c-kv)w)pv}, \end{aligned} \quad (9)$$

$$\begin{aligned} x_1 - a &= \frac{\mu^* - (p-c-kv)(1+w)(a+(b-a)\beta)}{kv-(p-c-kv)w} - a \\ &= \frac{\mu^* - (p-c)(a+(b-a)\beta) + [kv-(p-c-kv)w](b-a)\beta}{kv-(p-c-kv)w}, \end{aligned} \quad (10)$$

$$x_2 - x_d = \frac{[(1+v)p-c][a+(b-a)\beta] - \mu^*}{pv} - \frac{-pva - (kv - (p-c-kv)w)b}{(p-c-kv)w - kv - pv} \quad (11)$$

$$x_1 - x_d = \frac{\mu^* - (p-c-kv)(1+w)(a+(b-a)\beta)}{kv-(p-c-kv)w} + \frac{pva + (kv - (p-c-kv)w)b}{(p-c-kv)w - kv - pv} \quad (12)$$

Theorem 2.9. *If $w < \frac{kv}{p-c-kv}$, then the model in Corollary 1 has a unique solution.*
(1) If (11) is less than 0, then we have $x_2 < x_d$, then the model in Corollary 1 achieves the maximum expected return at $x = x_2$,

$$\begin{aligned} E[\mu] = & \frac{1}{2(b-a)}[(p-c-kv)(1+w)b^2 - ((1+v)p-c)a^2 \\ & + \frac{2(pva + (kv-(p-c-kv)w)b)((1+v)p-c)\beta) - \mu^*}{pv} \\ & + \frac{((p-c-kv)w - kv - pv)((1+v)p-c)(a+(b-a)\beta) - \mu^*)^2}{p^2v^2}], \end{aligned} \tag{13}$$

(2) if (11) equals 0, then we have $x_2 = x_d$, and the model in Corollary 1 achieves the maximum expected return at $x = x_d$

$$E[\mu] = \frac{1}{2}[((1-v)p-c)b + ((1+v)p-c)a - \frac{p^2v^2(b-a)}{(p-c-kv)w - kv - pv}] \tag{14}$$

(3) if the (11) is greater than 0, then we get $x_2 > x_d$,

If (12) is less than or equal to 0, then the model in Corollary 1 achieves the maximum expected return at $x = x_d$

$$E[\mu] = \frac{1}{2}[((1-v)p-c)b + ((1+v)p-c)a - \frac{p^2v^2(b-a)}{(p-c-kv)w - kv - pv}] \tag{15}$$

If the (12) is greater than and equal to 0, then we have $x_d < x_1 < x_2$, the model in Corollary 1 achieves the maximum expected return at $x = x_1$,

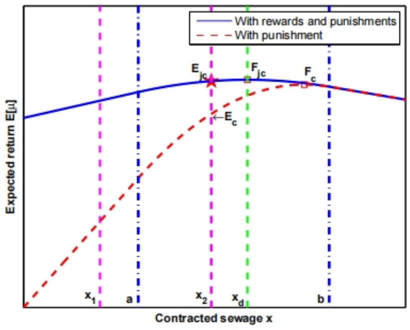
$$\begin{aligned} E[\mu] = & \frac{1}{2(b-a)}[b^2(p-c-kv)(1+w) - a^2((1+v)p-c) \\ & + \frac{2(pva + (kv-(p-c-kv)w)b)(\mu^* - (1+w)(p-c-kv)(a+(b-a)\beta))}{kv-(p-c-kv)w} \\ & + \frac{((p-c-kv)w - kv - pv)(\mu^* - (1+w)(p-c-kv)(a+(b-a)\beta))^2}{(kv-(p-c-kv)w)^2}] \end{aligned} \tag{16}$$

Proof Since $w < \frac{kv}{p-c-kv} < \frac{(p+k)v}{p-c-kv}$, we get $kv - (p-c-kv)w > 0, (p-c-kv)w - kv - pv < 0$. According to (7) and(8) we get $x_d - a > 0, x_d - b < 0$, which means that E_2 curve is inverted U-shaped, and the axis of symmetry satisfies $a < x_d < b$. Therefore, the objective function of model is an increasing and decreasing image on R.

Since (9) and $x_1x_2 < 0$, then we have $x_1 < x_2$, and according to (11), we show that

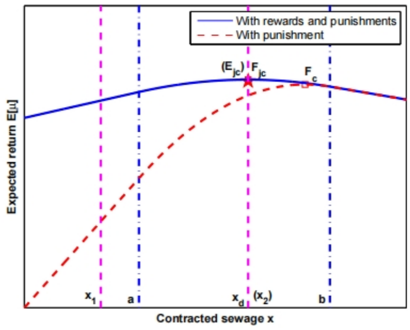
As shown in Figure a, if $x_2 < x_d < 0$,then $x_2 < x_d$, the model in Corollary 1 achieves the maximum expected return at $x = x_2$.

$$\begin{aligned} E[\mu] = & \frac{1}{2(b-a)}[(p-c-kv)(1+w)b^2 - ((1+v)p-c)a^2 \\ & + \frac{2(pva + (kv-(p-c-kv)v)b)(((1+v)p-c)(a+(b-a)\beta) - \mu^*)}{pv} \\ & + \frac{((p-c-kv)w - kv - pv)(((1+v)p-c)(a+(b-a)\beta) - \mu^*)^2}{p^2v^2}] \end{aligned}$$



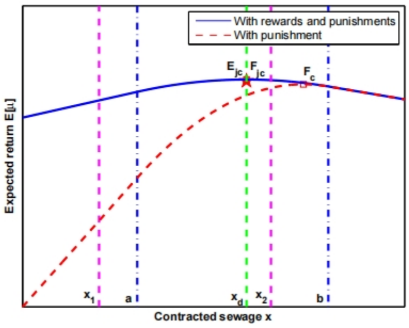
(a) $(x_2 < x_d)$

(a) Fig 1a



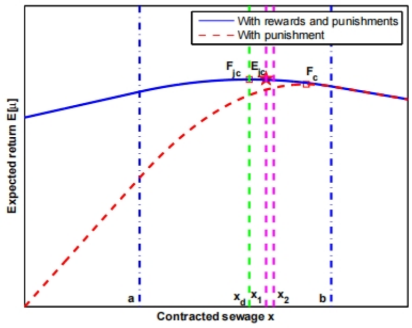
(b) $(x_2 = x_d)$

(b) Fig 1b



(c) $(x_1 \le x_d < x_2)$

(c) Fig 1c



(d) $(x_d < x_1 < x_2)$

(d) Fig 1d

As shown in Figure b , if $x_2 - x_d = 0$,then$x_2 = x_d$, the model in Corollary 1 achieves the maximum expected return at $x = x_d$

$$
\begin{aligned}
E[\mu] =& \frac{1}{2(b-a)}[((p-c-kv)w-kv-pv)(\frac{(-pva+(kv-(p-c-kv)w)b)}{((p-c-kv)w-kv-pv)})^2 \\
&+2(pva+(kv-(p-c-kv)w)b)(\frac{-pva+(kv-(p-c-kv)w)b}{(p-c-kv)w-kv-pv}) \\
&-((1+v)p-c)a^2+(p-c-kv)(1+w)b^2] \\
&-\frac{1}{2}[((1-v)p-c)b+((1+v)p-c)a-\frac{p^2v2(b-a)}{(p-c-kv)w-kv-pv}]
\end{aligned}
$$

As shown in Figure c, d, if $x_2 - x_d > 0$, then we have $x_2 > x_d$, If (12) is less than re equal to 0, then we have $x_1 \le x_d \le x_2$, the model in Corollary 1 achieves the maximum expected return at $x = x_d$.

$$\begin{aligned}E[\mu] =& \frac{1}{2(b-a)}[((p-c-kv)w-kv-pv)(\frac{(-pva+(kv-(p-c-kv)w)b)}{((p-c-kv)w-kv-pv)})^2\\ &+2(pva+(kv-(p-c-kv)w)b)(\frac{-pva+(kv-(p-c-kv)w)b}{(p-c-kv)w-kv-pv})\\ &-((1+v)p-c)a^2+(p-c-kv)(1+w)b^2]\\ &=\frac{1}{2}[((1-v)p-c)b+((1+v)p-c)a-\frac{p^2v2(b-a)}{(p-c-kv)w-kv-pv}]\end{aligned}$$

If (12) is greater than 0, then we have $x_1 \leq x_d \leq x_2$, the model in Corollary 1 achieves the maximum expected return at $x = x_1$.

$$\begin{aligned}E[\mu] = =& \frac{1}{2(b-a)}[b^2(p-c-kv)(1+w)-a^2((1+v)p-c)\\ &+\frac{2(pva+(kv-(p-c-kv)w)b)(\mu^*-(1+w)(p-c-kv)(a+(b-a)\beta))}{kv-(p-c-kv)w}\\ &+\frac{((p-c-kv)w-kv-pv)(\mu^*-(1+w)(p-c-kv)(a+(b-a)\beta))^2}{(kv-(p-c-kv)w)^2}]\end{aligned}$$

the proof is completed.

4.Numerical simulation

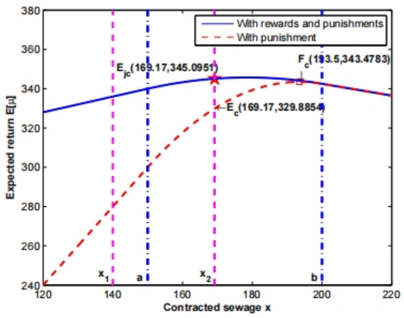
For the given model, we assume that the government pays the enterprise's sewage treatment amount according to the price P (yuan/ton) in practice, and the total cost of treating sewage by the enterprise is $c(yuan/ton)$. We also assume that (1) if the enterprise does not complete the sewage contract amount x (tons), then the government punish the value penalty v (%) to the difference of [x-ξ](tons) between the contract amount and the actual treatment amount of the contract amount; (2) if the enterprise completes the sewage contract quantity $x(tons)$ ahead of schedule, the government reward $w(\%)$ to the net proceeds $[\xi - x](tons)$ of the excess treatment portion of the amount of the contract, which is for encouraging the enterprise to continue to treat sewage. We also assume that the actual amount of sewage treatment ξ given by the assessment of sewage treatment experts is subject to the linear uncertain distribution of $L[a, b]$, and each variable of reward mechanism and the reward and punishment mechanism after joining the incentive mechanism is valued equally. The difference is that only the penalty mechanism when the enterprise effectively reduces the number of contracts to avoid the increase in marginal costs $K = 0$, but in the reward and punishment mechanism, there is a marginal cost increase constant $K > 0$ caused by the penalty mechanism, the set of each variable in the model (2) is as Table 2.

TABLE 2. **Parameter assignment**

Machine made	p	c	a	b	μ^*	k	v	β
Punishment mechanism	3	1	150	200	300		10%	10%
Incentive mechanism	3	1	150	200	300	4	10%	10%

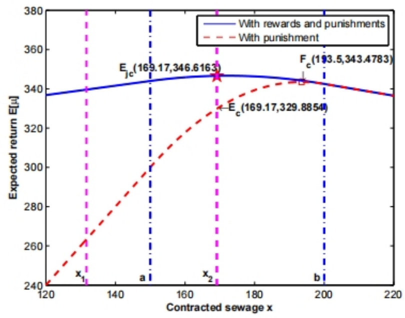
Next, we discuss different reward ranges w, namely, $(1) w = 0$; $(2) 0 < w < kv/(p-c-kv)$; $(3) kv/(p-c-kv) < w < (p+k)v/(p-c-kv)$; $(4) w > (p+k)v/(p-c-kv)$. Simulation of the expected return function of sewage treatment enterprises are as follows. The simulation results of Matlab are shown in next Figures (where the solid line represents the reward and punishment mechanism, and the dashed line represents the punishment mechanism), in which (a), (b), (c) and (d) represent the expected return function values of sewage treatment enterprises displayed by the reward and punishment mechanism and the punishment mechanism when w=0, $0 < w$=10%<25%, 25%< w=35%<43.75%, w=69%>43.75%, respectively.

Numerical results show that the maximum expected earnings of enterprises under different incentive ranges and theoretical results of the Theorem 2.7, Theorem 2.8, Theorem 2.9 are the same, with the increasing range of incentives, enterprises expect income is gradually increasing, and reach the maximum value under a certain amount of contracts, when the incentive range continues to increase, enterprises can achieve higher expected returns in a smaller number of contracts, see (a)-(d) in Figure 2. However, the incentive range increases, firms can achieve higher expected returns under smaller contracts, as shown in Figure 2 (d). Under the same number of contracts, the reward and punishment mechanism generates higher expected earnings.



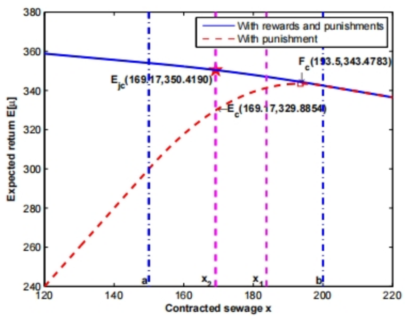
(a) $(\frac{w}{v} = 0)$

(e) Fig 2a



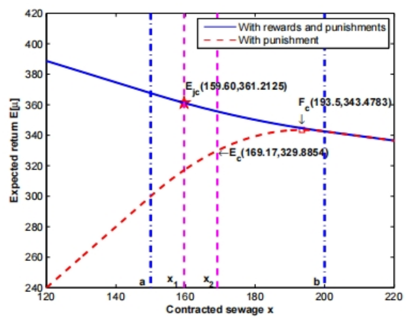
(b) $(\frac{w}{v} = 1)$

(f) Fig 2b



(d) $(\frac{w}{v} = 3.5)$

(g) Fig 2d



(f) $(\frac{w}{v} = 6.9)$

(h) Fig 2f

As shown in Table 3, when the incentive range increases from 0% to 43.75%, the expected earning of the enterprise is gradually increasing, and the maximum expected return of 351.7500 is achieved in the enterprise with a contracted sewage volume of x2 = 169.17, while the actual amount of sewage treated is increasing by 175.0000. However, the incentive range increased from 43.75 % to 69 %, companies achieved higher expected returns of 361.2125 and treated more sewage by 175.1378 with a lower number of contracts of 159.60.

TABLE 3. **Change of contract quantity,expected income and actual sewage quantity under different reward range**

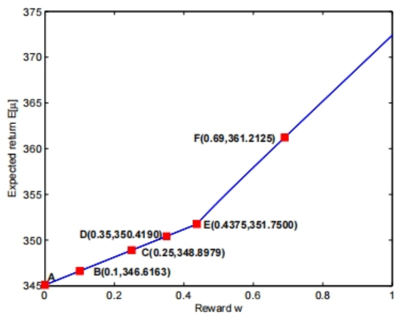
Legend	Rewards w	Punishment v	a, b, x_1, x_2, x_d	x	$E[\mu]$	ξ
		10%	$x_1 < a < x_2 < x_d < b$	169.17	329.8854	165.4961
(a)	0	10%	$x_1 < a < x_2 < x_d < b$	169.17	345.0951	173.3928
(b)	10	10%	$x_1 < a < x_2 < x_b < d$	169.17	346.6163	173.8729
(c)	35	10%	$x_d < a < x_2 < x_1 < b$	169.17	350.4190	174.7619
(d)	69	10%	$a < x_1 < x_2 < b < x_d$	159.60	361.2125	175.1378

5.Sensitivity analysis

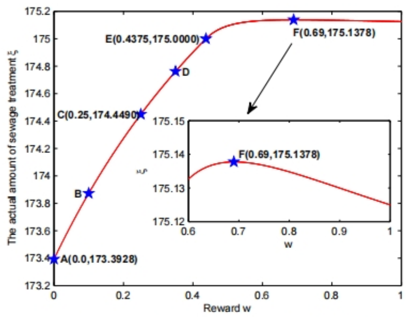
The sewage contract amount x and the actual sewage treatment amount ξ are obtained by Matlab under the different incentive factors w, as shown in Table 4. It is shown from Figure 3a-Figure 3c, the increasing in the range of rewards, the incentive effect is gradually reduced. If the reward reaches a certain degree, the incentive effect tends to 0. The incentive effect basically disappears, if the reward continues to increase.

TABLE 4. **The relationship between contract volume, expected income, actual sewage treatment volume and incentive coefficient**

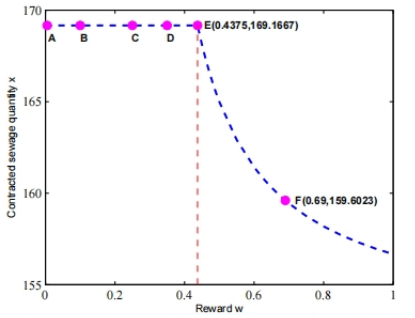
w	x(tons)	$E[\mu]$	$\xi(x \leq \xi)$	w	x(tons)	$E[\mu]$	$\xi(x \leq \xi)$
0.00	169.1667	349.0951	173.3928	0.50	165.0000	354.2250	175.0938
0.10	169.1667	346.6163	173.8729	0.60	161.4286	357.9396	175.1327
0.20	169.1667	348.1374	174.2729	0.62	160.9459	358.6699	175.1350
0.30	169.1667	349.6585	174.6114	0.64	160.5128	359.3981	175.1365
0.40	169.1667	351.1796	174.9016	0.66	160.1220	360.1247	175.1373
0.45	168.1250	352.2657	175.0283	0.80	158.1818	365.1883	175.1348
0.49	165.5208	353.8424	175.0849	1.00	156.6667	372.4000	175.1250



(i) Fig3a



(j) Fig3b



(k) Fig3c

(1) Under a certain reward and punishment mechanism, If $0 < w \leq 43.75\%$ and when the gradual increase in the incentive range w, the expected earning of sewage treatment enterprises E $[\mu]$ slowly increased, the actual amount of sewage treatment increased rapidly, sewage contract amount x remained unchanged. The bonus range of w=43.75% is the inflection point between the expected earnings E $[\mu]$ (point E in Figure 3a) and the increase in the actual sewage treatment (point E in Figure 3a), at which point the largest amount of contracts is made (point E in Figure 3b).

(2) When the incentive range w is added to 43.75 (43.75 %$< w \leq$ 69 %), the expected earnings E $[\mu]$ increase faster, the overall linear grows, the actual treatment of sewage began to increase slowly, the amount of signed sewage began to decrease rapidly. When the incentive range of w is 69%, the actual amount of sewage treated by the enterprise reaches a maximum of 175.1378 (point F in Figure 3b).

(3) When the incentive range 69%, the expected earning is still increasing rapidly, the actual amount of sewage treatment is beginning to decrease slowly. Meanwhile, the amount of signed sewage decreases slowly (point F in Figure 3c).

4. Conclusion and discussion

In this paper, the wastewater treatment volume of a company is portrayed as an uncertain variable that does not depend on historical statistical data but only depends on the evaluation of experts in the wastewater treatment industry. The research results show that the government introduces a reward and punishment mechanism afterward, certain rewards enable companies to actively disclose a reasonable amount of contracts to obtain potential government rewards and treat as much sewage as possible. However, as the rewards continue to increase, and if the company's sewage treatment capacity still has a surplus, the company hopes to achieve a higher expected return through a lower contract volume, so as to treat more sewage. But if the company's sewage treatment capacity has been fully tapped, A higher reward rate only encourage companies to sign a lower sewage treatment volume for obtaining the higher expected returns. The actual sewage treatment volume also loses its treatment power due to the higher expected returns.

Acknowledgments

This work was supported by the major project of North Minzu University(No.ZDZX201805), governance and social management research center of northwest Ethnic regions and First Class Disciplines Foundation of Ningxia (Grant No.NXYLXK2017B09).Yang was supported by National Social Science Fund Project: Research on the Co-creation of Ecosystem Value of Green Brands for Characteristic Agricultural Products in Northwest China Under the Dual Carbon Goals (23BMZ062).Tajul was supported by Fundamental Research Grant Scheme (FRGS) by Ministry of Higher Education Malaysia (MOHE)

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