U.P.B. Sci. Bull., Series A, Vol. 84, Iss. 2, 2022 ISSN 1223-7027

AN ITERATIVE SCHEME FOR SOLVING FIXED POINT PROBLEM OF ASYMPTOTICALLY PSEUDOCONTRACTIVE OPERATORS AND VARIATIONAL INEQUALITIES

Haibin Shen[1], Yasong Chen[2], Zhangsong Yao[3]

Fixed point problems of asymptotically pseudocontractive operators and quasimonotone variational inequalities have been considered in Hilbert spaces. An iterative scheme for finding a common point of an asymptotically pseudocontractive operator and the solution of a quasimonotone variational inequality is presented. Convergence analysis of the proposed scheme is proved under several additional assumptions.

Keywords: fixed point, asymptotically pseudocontractive operators, quasimonotone variational inequalities, iterative scheme.

MSC2020: 47J25, 65K10, 90C99.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Let C be a nonempty closed and convex subset of H. Let $f : C \to H$ be an operator. Recall that the variational inequality is to find a point $\hat{u} \in C$ such that

$$\langle f(\hat{u}), u - \hat{u}\rangle \geq 0, \ \forall u \in C. \tag{1}$$

Let $\mathrm{Sol}(C, f)$ be the solution set of the variational inequality (1).

Variational inequality problems provide a unified mathematical framework for many practical problems arising in optimization ([12–14, 17, 23, 24, 27, 42, 46, 53, 57]). Variational inequality problems include transportation networks ([18]), signal processing ([5]), equilibrium problems ([47, 58]), fixed point problems ([29–31, 33, 34, 37, 39, 40, 49]), complementarity problem ([14, 28]), etc. There are numerous iterative algorithms for solving variational inequalities and related problems, see for examples, [1–4, 7, 11, 21, 25, 36, 41, 43, 44, 48, 50–52, 54, 57].

For solving (1), one may use projection methods ([15, 16, 22, 26]) that employ the metric projection onto the feasible set C. The projection-gradient method requires that f must be strongly (pseudo-) monotone (see [19]) or f is inverse strongly monotone (see [5]). However, if f is plain monotone, then the projection-gradient method does not necessarily converge. Consequently, many iterative methods have been investigated, such as proximal point method ([10]), Korpelevich's extragradient method [20, 35] and its variant forms ([9, 38]), subgradient extragradient method ([8, 45, 55]), Tseng's method ([32]) and

[1]Ningbo Institute of Dalian University of Technology, Ningbo, 315016, Zhejiang Province, China, e-mail: shenhaibin1976@163.com

[2]School of Mathematical Sciences, Tiangong University, Tianjin 300387, China, e-mail: yasongchen@126.com

[3]Corresponding author. School of Information Engineering, Nanjing Xiaozhuang University, Nanjing 211171, China, e-mail: yaozhsong@163.com

so on. Especially, Bot et al. [6] proposed the following Tseng-type algorithm

$$\begin{cases} v_n = P_C(u_n - \lambda f(u_n)), \\ u_{n+1} = \mu_k(v_n + \lambda(f(u_n) - f(v_n)) + (1-\mu_k)u_n, \ \forall n \geq 0. \end{cases} \tag{2}$$

Bot et al. ([6]) proved that the sequence $\{u_n\}$ generated by (2) converges weakly to an element in $\mathrm{Sol}(C, f)$ provided f is pseudomonotone and sequentially weakly continuous.

In this paper, we present an iterative algorithm for solving fixed point problem of an asymptotically pseudocontractive operator and a quasimonotone variational inequality in a real Hilbert space. Our algorithm combines Tseng-type method and self-adaptive method. We prove that the proposed algorithm converges weakly to a common element of the solution of a quasimonotone variational inequality and the fixed point of an asymptotically pseudocontractive operator under some additional conditions.

2. **Preliminaries**

Let C be a nonempty closed convex subset of a real Hilbert space H. An operator $f : C \to H$ is said to be

(i) monotone if

$$\langle f(x^\dagger) - f(\hat{x}), x^\dagger - \hat{x}\rangle \geq 0, \ \forall x^\dagger, \hat{x} \in C.$$

(ii) strongly monotone if there exists a constant $\mu > 0$ such that

$$\langle f(x^\dagger) - f(\hat{x}), x^\dagger - \hat{x}\rangle \geq \mu\|x^\dagger - \hat{x}\|^2, \ \forall x^\dagger, \hat{x} \in C.$$

(iii) inverse-strongly monotone if there exists a constant $\gamma > 0$ such that

$$\langle f(x^\dagger) - f(\hat{x}), x^\dagger - \hat{x}\rangle \geq \gamma\|f(x^\dagger) - f(\hat{x})\|^2, \ \forall x^\dagger, \hat{x} \in C.$$

(iv) pseudomonotone if for all $u, u^\dagger \in C$, we have

$$\langle f(u^\dagger), u - u^\dagger\rangle \geq 0 \text{ implies that } \langle f(u), u - u^\dagger\rangle \geq 0.$$

(v) quasimonotone if for all $u, u^\dagger \in C$, the following relation holds

$$\langle f(u^\dagger), u - u^\dagger\rangle > 0 \text{ implies that } \langle f(u), u - u^\dagger\rangle \geq 0.$$

(vi) L-Lipschitz if there exists a constant $L > 0$ such that

$$\|f(u) - f(u^\dagger)\| \leq L\|u - x^\dagger\|, \ \forall u, u^\dagger \in C.$$

Let $\mathrm{Sol}^d(C, f)$ be the solution set of the dual variational inequality of (1), that is,

$$\mathrm{Sol}^d(C, f) := \{u \in C | \langle f(x), x - u\rangle \geq 0, \ \forall x \in C\}.$$

Note that $\mathrm{Sol}^d(C, f)$ is closed convex. If C is convex and f is continuous, then $\mathrm{Sol}^d(C, f) \subset \mathrm{Sol}(C, f)$. To show the convergence of the sequence $\{u_n\}$, a common condition $\mathrm{Sol}(C, f) \subset \mathrm{Sol}^d(C, f)$ has been added, that is,

$$\langle f(x), x - u\rangle \geq 0, \ \forall u \in \mathrm{Sol}(C, f) \text{ and } x \in C,$$

which is a direct consequence of the pseudomonotonicity of f. But this conclusion (that is, $\mathrm{Sol}(C, f) \subset \mathrm{Sol}^d(C, f)$) is false, if f is quasimonotone.

Let $T : H \to H$ be an operator. Let $Fix(T)$ be the set of fixed points of T, i.e., $Fix(T) := \{x \in H | x = Tx\}$. Recall that T is said to be

(i) τ_n-asymptotically pseudocontractive if for all $x, x^\dagger \in C$, we have

$$\langle T^n(x) - T^n(x^\dagger), x - x^\dagger\rangle \leq \tau_n\|x - x^\dagger\|^2, \forall n \geq 1, \tag{3}$$

where $\{\tau_n\} \subset [1, \infty)$ satisfies $\lim_{n\to\infty} \tau_n = 1$.
Note that the definition (3) is equivalent to

$$\|T^n(x) - T^n(x^\dagger)\|^2 \leq (2\tau_n - 1)\|x - x^\dagger\|^2 + \|(I - T^n)x - (I - T^n)x^\dagger\|^2. \tag{4}$$

(ii) uniformly L-Lipschitz if there exists a positive constant L such that

$$\|T^n(x) - T^n(x^\dagger)\| \leq L\|x - x^\dagger\|, \ \forall n \geq 1,$$

for all $x, x^\dagger \in C$.

Let $x \in H$ be a fixed point. There exists a unique $x^\dagger \in C$ such that $\|x - x^\dagger\| = \inf\{\|x - \tilde{x}\| : \tilde{x} \in C\}$. Denote $x^\dagger$ by $P_C[x]$. It is well known that P_C satisfies ([47])

$$x \in H, \ \langle x - P_C[x], y - P_C[x]\rangle \leq 0, \ \forall y \in C. \tag{5}$$

By (5), we have

$$x^\dagger \in \mathrm{Sol}(C, f) \Leftrightarrow x^\dagger = P_C[x^\dagger - \alpha f(x^\dagger)], \forall \alpha > 0. \tag{6}$$

It is well known that in H, we have the following equality

$$\|\varsigma u + (1-\varsigma)u^\dagger\|^2 = \varsigma\|u\|^2 + (1-\varsigma)\|u^\dagger\|^2 - \varsigma(1-\varsigma)\|u - u^\dagger\|^2, \tag{7}$$

$\forall u, u^\dagger \in H$ and $\forall \varsigma \in [0, 1]$.

In the sequel, we use " $\rightharpoonup$ " and " $\rightarrow$ " to denote weak convergence and strong convergence, respectively. Let $\{u_n\}$ be a sequence in H. Let $\omega_w(u_n)$ be the set of all weak cluster points of $\{u_n\}$, i.e., $\omega_w(u_n) = \{u^\dagger : \exists \{u_{n_i}\} \subset \{u_n\}$ *such that* $u_{n_i} \rightharpoonup u^\dagger (i \to \infty)\}$.

Lemma 2.1 ([56]). *Let H be a real Hilbert space. Let $T\colon H \to H$ be a uniformly L-Lipschtzian and asymptotically pseudocontractive operator. Then, $I - T$ is demiclosed at zero.*

3. Main results

In this section, we present our main results.

Let H be a real Hilbert space and $C \subset H$ be a nonempty closed convex set. Let f be a quasimonotone and κ_1-Lipschitz operator on H. Let T be a uniformly κ_2-Lipschitz and τ_n-asymptotically pseudocontractive operator on H. Assume that the following two conditions hold: (C1): $\Gamma := Fix(T) \cap \mathrm{Sol}^d(C, f) \neq \emptyset$ and $\{x \in C : f(x) = 0\} \setminus \mathrm{Sol}^d(C, f)$ is a finite set; (C2): For any a sequence $\{x_n\}$ in H, if $x_n \rightharpoonup x^\dagger$ and $\underline{\lim}_{n\to+\infty}\|f(x_n)\| = 0$, then $f(x^\dagger) = 0$. Let $\{\varrho_n\}$, $\{\sigma_n\}$ and $\{\varsigma_n\}$ be three sequences in $(0, 1)$. Let $\zeta \in (0, 1)$ and $\alpha_0 > 0$ be two constants.

Now, we introduce an iterative algorithm for solving fixed point problems and variational inequalities.

Algorithm 3.1. *Select an initial point u_0 in H and set $n = 0$.*
Step 1. Let the current iterate u_n be given. Compute

$$\begin{cases} \hat{v}_n = (1-\varrho_n)u_n + \varrho_n T^n(u_n), \\ v_n = (1-\sigma_n)u_n + \sigma_n T^n(\hat{v}_n). \end{cases} \tag{8}$$

Step 2. Let the step-size α_n be given. Compute

$$w_n = P_C[v_n - \alpha_n f(v_n)], \tag{9}$$

and the next iterate

$$u_{n+1} = (1-\varsigma_n)v_n + \varsigma_n w_n + \varsigma_n \alpha_n[f(v_n) - f(w_n)]. \tag{10}$$

Step 3. Compute the step-size α_{n+1} via the following manner

$$\alpha_{n+1} = \begin{cases} \min\left\{\alpha_n, \frac{\zeta\|w_n - v_n\|}{\|f(w_n) - f(v_n)\|}\right\}, & \text{if } f(w_n) \neq f(v_n), \\ \alpha_n, & \text{else.} \end{cases} \tag{11}$$

Set $n := n + 1$ and return to step 1.

Remark 3.1. *(i) With the help of* (5), $v_n = P_C[v_n - \alpha_n f(v_n)]$ *implies that* $v_n \in \mathrm{Sol}(C, f)$. *(ii) From* (11), *we can conclude that for all* $n \geq 0$, $\alpha_{n+1} \leq \alpha_n$ *and* $\alpha_n \geq \min\{\alpha_0, \frac{\zeta}{\kappa_1}\}$. *Thus,* $\lim_{n\to\infty} \alpha_n = \alpha^\dagger$ *exists and* $\alpha^\dagger \geq \min\{\alpha_0, \frac{\zeta}{\kappa_1}\} > 0$.

Next, we show the convergence of Algorithm 3.1.

Theorem 3.1. *Suppose that the sequences* $\{\varrho_n\}$, $\{\sigma_n\}$ *and* $\{\varsigma_n\}$ *satisfy the following conditions: (C3):* $0 < \underline{\sigma} < \sigma_n < \overline{\sigma} < \varrho_n \leq \frac{1}{\tau_n + \sqrt{\kappa_2^2 + \tau_n^2}} < \frac{1}{\sqrt{1+\kappa_2^2}+1}$ *for all* $n \geq 0$, *(C4):* $0 < \underline{\lim}_{n\to\infty} \varsigma_n \leq \limsup_{n\to\infty} \varsigma_n < 1$ *and (C5):* $\sum_{n=1}^{\infty} (\tau_n - 1) < +\infty$. *Then the sequence* $\{u_n\}$ *generated by Algorithm 3.1 converges weakly to some point in* Γ.

Proof. Choose any $\tilde{x}$ in Γ. Since $\tilde{x} \in C$, applying inequality (5) and from (9), we have

$$\langle v_n - \alpha_n f(v_n) - w_n, w_n - \tilde{x}\rangle \geq 0.$$

So,

$$\langle w_n - v_n, w_n - \tilde{x}\rangle + \alpha_n \langle f(v_n), w_n - \tilde{x}\rangle \leq 0. \tag{12}$$

As a result of $\tilde{x} \in \mathrm{Sol}^d(C, f)$ and $w_n \in C$, we obtain

$$\langle f(w_n), w_n - \tilde{x}\rangle \geq 0. \tag{13}$$

Combining (12) and (13) to deduce

$$\langle w_n - v_n, w_n - \tilde{x}\rangle + \alpha_n \langle f(v_n) - f(w_n), w_n - \tilde{x}\rangle \leq 0. \tag{14}$$

Note that

$$\langle w_n - v_n, w_n - \tilde{x}\rangle = \frac{1}{2}(\|w_n - v_n\|^2 + \|w_n - \tilde{x}\|^2 - \|v_n - \tilde{x}\|^2),$$

which together with (14) implies that

$$\|w_n - v_n\|^2 + \|w_n - \tilde{x}\|^2 - \|v_n - \tilde{x}\|^2 + 2\alpha_n \langle f(v_n) - f(w_n), w_n - \tilde{x}\rangle \leq 0.$$

Hence,

$$\|w_n - \tilde{x}\|^2 \leq \|v_n - \tilde{x}\|^2 - 2\alpha_n \langle f(v_n) - f(w_n), w_n - \tilde{x}\rangle - \|w_n - v_n\|^2. \tag{15}$$

From (10), we have

$$\begin{aligned}
\|u_{n+1} - \tilde{x}\|^2 &= \|(1-\varsigma_n)(v_n - \tilde{x}) + \varsigma_n(w_n - \tilde{x}) + \varsigma_n \alpha_n [f(v_n) - f(w_n)]\|^2 \\
&= \|(1-\varsigma_n)(v_n - \tilde{x}) + \varsigma_n(w_n - \tilde{x})\|^2 + \varsigma_n^2 \alpha_n^2 \|f(v_n) - f(w_n)\|^2 \\
&\quad + 2\varsigma_n(1-\varsigma_n)\alpha_n \langle v_n - \tilde{x}, f(v_n) - f(w_n)\rangle \\
&\quad + 2\varsigma_n^2 \alpha_n \langle w_n - \tilde{x}, f(v_n) - f(w_n)\rangle.
\end{aligned} \tag{16}$$

By (7), we get

$$\begin{aligned}
\|(1-\varsigma_n)(v_n - \tilde{x}) + \varsigma_n(w_n - \tilde{x})\|^2 &= (1-\varsigma_n)\|v_n - \tilde{x}\|^2 + \varsigma_n \|w_n - \tilde{x}\|^2 \\
&\quad - \varsigma_n(1-\varsigma_n)\|v_n - w_n\|^2
\end{aligned} \tag{17}$$

Substituting (17) into (16), we deduce

$$\begin{aligned}
\|u_{n+1} - \tilde{x}\|^2 &= (1-\varsigma_n)\|v_n - \tilde{x}\|^2 + \varsigma_n \|w_n - \tilde{x}\|^2 - \varsigma_n(1-\varsigma_n)\|v_n - w_n\|^2 \\
&\quad + \varsigma_n^2 \alpha_n^2 \|f(v_n) - f(w_n)\|^2 + 2\varsigma_n^2 \alpha_n \langle w_n - \tilde{x}, f(v_n) - f(w_n)\rangle \\
&\quad + 2\varsigma_n(1-\varsigma_n)\alpha_n \langle v_n - \tilde{x}, f(v_n) - f(w_n)\rangle.
\end{aligned} \tag{18}$$

Taking into account (15) and (18), we obtain

$$\begin{aligned}\|u_{n+1}-\tilde{x}\|^2 &\le \|v_n-\tilde{x}\|^2 - \varsigma_n(2-\varsigma_n)\|v_n-w_n\|^2 + \varsigma_n^2\alpha_n^2\|f(v_n)-f(w_n)\|^2 \\ &\quad + 2\varsigma_n(1-\varsigma_n)\alpha_n\langle v_n-w_n, f(v_n)-f(w_n)\rangle \\ &\le \|v_n-\tilde{x}\|^2 - \varsigma_n(2-\varsigma_n)\|v_n-w_n\|^2 + \varsigma_n^2\alpha_n^2\|f(v_n)-f(w_n)\|^2 \\ &\quad + 2\varsigma_n(1-\varsigma_n)\alpha_n\|v_n-w_n\|\|f(v_n)-f(w_n)\|.\end{aligned} \tag{19}$$

By (11), $\|f(w_n)-f(v_n)\| \le \frac{\zeta\|w_n-v_n\|}{\alpha_{n+1}}$. From (19), we achieve

$$\begin{aligned}\|u_{n+1}-\tilde{x}\|^2 &\le \|v_n-\tilde{x}\|^2 - \varsigma_n(2-\varsigma_n)\|v_n-w_n\|^2 + \zeta^2\varsigma_n^2\alpha_n^2\|w_n-v_n\|^2/\alpha_{n+1}^2 \\ &\quad + 2\zeta\varsigma_n(1-\varsigma_n)\alpha_n\|v_n-w_n\|^2/\alpha_{n+1} \\ &= \|v_n-\tilde{x}\|^2 - \varsigma_n[2-\varsigma_n-\zeta^2\varsigma_n\alpha_n^2/\alpha_{n+1}^2 - 2\zeta(1-\varsigma_n)\alpha_n/\alpha_{n+1}]\|v_n-w_n\|^2.\end{aligned} \tag{20}$$

According to Remark 3.1 and condition (C4), we have $\underline{\lim}_{n\to\infty}\varsigma_n\big[2-\varsigma_n-\zeta^2\varsigma_n\alpha_n^2/\alpha_{n+1}^2 - 2\zeta(1-\varsigma_n)\alpha_n/\alpha_{n+1}\big] > 0$. Then, there exists a constant $\varpi > 0$ and an integer $\mathcal{N} > 0$ such that when $n \ge \mathcal{N}$, $\varsigma_n[2-\varsigma_n-\zeta^2\varsigma_n\alpha_n^2/\alpha_{n+1}^2 - 2\zeta(1-\varsigma_n)\alpha_n/\alpha_{n+1}] \ge \varpi$. This together with (20) implies that

$$\|u_{n+1}-\tilde{x}\|^2 \le \|v_n-\tilde{x}\|^2 - \varpi\|v_n-w_n\|^2. \tag{21}$$

Based on (7) and (8), we obtain

$$\begin{aligned}\|v_n-\tilde{x}\|^2 &= \|(1-\sigma_n)(u_n-\tilde{x}) + \sigma_n(T^n(\hat{v}_n)-\tilde{x})\|^2 \\ &= (1-\sigma_n)\|u_n-\tilde{x}\|^2 + \sigma_n\|T^n(\hat{v}_n)-\tilde{x}\|^2 \\ &\quad - \sigma_n(1-\sigma_n)\|u_n-T^n(\hat{v}_n)\|^2.\end{aligned} \tag{22}$$

Using the definition (4) of T, we receive

$$\|T^n(\hat{v}_n)-\tilde{x}\|^2 \le (2\tau_n-1)\|\hat{v}_n-\tilde{x}\|^2 + \|\hat{v}_n-T^n(\hat{v}_n)\|^2, \tag{23}$$

and

$$\|T^n(u_n)-\tilde{x}\|^2 \le (2\tau_n-1)\|u_n-\tilde{x}\|^2 + \|u_n-T^n(u_n)\|^2. \tag{24}$$

Since T is uniformly κ_2-Lipschitz, we obtain

$$\|T^n(u_n)-T^n(\hat{v}_n)\| \le \kappa_2\|u_n-\hat{v}_n\| = \kappa_2\varrho_n\|u_n-T^n(u_n)\|. \tag{25}$$

Applying (7) to (8), we attain

$$\begin{aligned}\|\hat{v}_n-\tilde{x}\|^2 &= \|(1-\varrho_n)(u_n-\tilde{x}) + \varrho_n(T^n(u_n)-\tilde{x})\|^2 \\ &= (1-\varrho_n)\|u_n-\tilde{x}\|^2 + \varrho_n\|T^n(u_n)-\tilde{x}\|^2 - \varrho_n(1-\varrho_n)\|u_n-T^n(u_n)\|^2.\end{aligned}$$

This together with (24) implies that

$$\begin{aligned}\|\hat{v}_n-\tilde{x}\|^2 &\le (1-\varrho_n)\|u_n-\tilde{x}\|^2 + \varrho_n[(2\tau_n-1)\|u_n-\tilde{x}\|^2 + \|u_n-T^n(u_n)\|^2] \\ &\quad - \varrho_n(1-\varrho_n)\|u_n-T^n(u_n)\|^2 \\ &= [1+2(\tau_n-1)\varrho_n]\|u_n-\tilde{x}\|^2 + \varrho_n^2\|u_n-T^n(u_n)\|^2.\end{aligned} \tag{26}$$

Taking into account (7), (8) and (25), we derive

$$\begin{aligned}\|\hat{v}_n-T^n(\hat{v}_n)\|^2 &= \|(1-\varrho_n)(u_n-T^n(\hat{v}_n)) + \varrho_n(T^n(u_n)-T^n(\hat{v}_n))\|^2 \\ &= (1-\varrho_n)\|u_n-T^n(\hat{v}_n)\|^2 + \varrho_n\|T^n(u_n)-T^n(\hat{v}_n)\|^2 \\ &\quad - \varrho_n(1-\varrho_n)\|u_n-T^n(u_n)\|^2 \\ &\le (1-\varrho_n)\|u_n-T^n(\hat{v}_n)\|^2 - \varrho_n(1-\varrho_n-\kappa_2^2\varrho_n^2)\|u_n-T^n(u_n)\|^2.\end{aligned} \tag{27}$$

In the light of (23), (26) and (27), we have

$$\begin{aligned}\|T^n(\hat{v}_n)-\tilde{x}\|^2 &\le (2\tau_n-1)[1+2(\tau_n-1)\varrho_n]\|u_n-\tilde{x}\|^2+(2\tau_n-1)\varrho_n^2\|u_n-T^n(u_n)\|^2\\&\quad+(1-\varrho_n)\|u_n-T^n(\hat{v}_n)\|^2-\varrho_n(1-\varrho_n-\kappa_2^2\varrho_n^2)\|u_n-T^n(u_n)\|^2\\&=(2\tau_n-1)[1+2(\tau_n-1)\varrho_n]\|u_n-\tilde{x}\|^2+(1-\varrho_n)\|u_n-T^n(\hat{v}_n)\|^2\\&\quad-\varrho_n(1-2\tau_n\varrho_n-\kappa_2^2\varrho_n^2)\|u_n-T^n(u_n)\|^2.\end{aligned}\tag{28}$$

Based on condition (C3), $\varrho_n \le \frac{1}{\tau_n+\sqrt{\kappa_2^2+\tau_n^2}}$. Then, $1-2\tau_n\varrho_n-\kappa_2^2\varrho_n^2>0$. By (28), we have

$$\|T^n(\hat{v}_n)-\tilde{x}\|^2 \le (2\tau_n-1)[1+2(\tau_n-1)\varrho_n]\|u_n-\tilde{x}\|^2+(1-\varrho_n)\|u_n-T^n(\hat{v}_n)\|^2. \tag{29}$$

Combining (22) and (29), we receive

$$\begin{aligned}\|v_n-\tilde{x}\|^2 &\le \sigma_n(2\tau_n-1)[1+2(\tau_n-1)\varrho_n]\|u_n-\tilde{x}\|^2+(1-\sigma_n)\|u_n-\tilde{x}\|^2\\&\quad+\sigma_n(1-\varrho_n)\|u_n-T^n(\hat{v}_n)\|^2-\sigma_n(1-\sigma_n)\|u_n-T^n(\hat{v}_n)\|^2\\&=[1+2(\tau_n-1)\sigma_n+2(\tau_n-1)(2\tau_n-1)\varrho_n\sigma_n]\|u_n-\tilde{x}\|^2\\&\quad+\sigma_n(\sigma_n-\varrho_n)\|u_n-T^n(\hat{v}_n)\|^2.\end{aligned}$$

which results, together with (21), that

$$\begin{aligned}\|u_{n+1}-\tilde{x}\|^2 &\le [1+2(\tau_n-1)\sigma_n+2(\tau_n-1)(2\tau_n-1)\varrho_n\sigma_n]\|u_n-\tilde{x}\|^2\\&\quad-(\varrho_n-\sigma_n)\sigma_n\|u_n-T^n(\hat{v}_n)\|^2-\varpi\|v_n-w_n\|^2, n\ge \mathcal{N}.\end{aligned}\tag{30}$$

By the assumptions, we have $2(\tau_n-1)\sigma_n+2(\tau_n-1)(2\tau_n-1)\varrho_n\sigma_n \le 8(\tau_n-1)$. It follows from (30) that

$$\|u_{n+1}-\tilde{x}\|^2 \le [1+8(\tau_n-1)]\|u_n-\tilde{x}\|^2. \tag{31}$$

Using condition (C5) and (31), we conclude tat $\lim_{n\to\infty}\|u_n-\tilde{x}\|$ exists. Therefore, $\{u_n\}$ is bounded. According to (30) and (31), we obtain

$$(\varrho_n-\sigma_n)\sigma_n\|u_n-T^n(\hat{v}_n)\|^2+\varpi\|v_n-w_n\|^2 \le [1+8(\tau_n-1)]\|u_n-\tilde{x}\|^2-\|u_{n+1}-\tilde{x}\|^2,$$

which implies that

$$\lim_{n\to\infty}\|u_n-T^n(\hat{v}_n)\|=0. \tag{32}$$

and

$$\lim_{n\to\infty}\|v_n-w_n\|=0. \tag{33}$$

In combination with the Lipschitz continuity of f, we have

$$\lim_{n\to\infty}\|f(v_n)-f(w_n)\|=0. \tag{34}$$

From (8), $v_n-u_n=\sigma_n(T^n(\hat{v}_n)-u_n)$. Then, by (32), we have

$$\lim_{n\to\infty}\|v_n-u_n\|=0. \tag{35}$$

By virtue of (10), (32), (33) and (34), we deduce

$$\lim_{n\to\infty}\|u_{n+1}-u_n\|=0. \tag{36}$$

It is obvious that the sequences $\{v_n\}$ and $\{w_n\}$ are bounded. Since T is uniformly κ_2-Lipschitz continuous, we have

$$\begin{aligned}\|u_n-T^n(u_n)\| &\le \|u_n-T^n(\hat{v}_n)\|+\|T^n(\hat{v}_n)-T^n(u_n)\|\\&\le \|u_n-T^n(\hat{v}_n)\|+\kappa_2\varrho_n\|u_n-T^n(u_n)\|.\end{aligned}$$

It yields

$$\|u_n - T^n(u_n)\| \le \frac{1}{1-\kappa\varrho_n}\|u_n - T^n(\hat{v}_n)\|.$$

This together with (32) implies that

$$\lim_{n\to\infty} \|u_n - T^n(u_n)\| = 0. \tag{37}$$

Observe that

$$\begin{aligned} \|u_{n+1} - T(u_{n+1})\| &\le \|u_{n+1} - T^{n+1}u_{n+1}\| + \|T^{n+1}u_{n+1} - T^{n+1}u_n\| \\ &\quad + \|T^{n+1}u_n - Tu_{n+1}\| \\ &\le \|u_{n+1} - T^{n+1}u_{n+1}\| + 2\kappa_2\|u_{n+1} - u_n\| + \kappa_2\|T^n u_n - u_n\|. \end{aligned} \tag{38}$$

By virtue of (36), (37) and (38), we get

$$\lim_{n\to\infty} \|u_n - T(u_n)\| = 0. \tag{39}$$

Next, we show that $\omega_w(u_n) \subset \Gamma$. Choose any $x^* \in \omega_w(u_n)$ and $\{u_{n_i}\}$ is a subsequence of $\{u_n\}$ such that $u_{n_i} \rightharpoonup x^*$ as $i \to \infty$. Thanks yo (33) and (35), we have $v_{n_i} \rightharpoonup x^*$ and $w_{n_i} \rightharpoonup x^*$. Take into account of (39) and Lemma 2.1, we obtain that $x^* \in Fix(T)$. Now, we prove that $x^* \in \mathrm{Sol}(C, f)$. Noting that $w_{n_i} = P_C[v_{n_i} - \alpha_{n_i} f(v_{n_i})]$ and applying (5), we receive

$$\langle w_{n_i} - v_{n_i} + \alpha_{n_i} f(v_{n_i}), w_{n_i} - \hat{u}\rangle \le 0,\ \forall \hat{u} \in C,$$

which yields

$$\frac{1}{\alpha_{n_i}}\langle v_{n_i} - w_{n_i}, \hat{u} - w_{n_i}\rangle + \langle f(v_{n_i}), w_{n_i} - v_{n_i}\rangle \le \langle f(v_{n_i}), \hat{u} - v_{n_i}\rangle,\ \forall \hat{u} \in C. \tag{40}$$

Owing to (33), $\lim_{i\to\infty} \|v_{n_i} - w_{n_i}\| = 0$. It follows from (40) that

$$\underline{\lim}_{i\to\infty}\langle f(v_{n_i}), \hat{u} - v_{n_i}\rangle \ge 0,\ \forall \hat{u} \in C \tag{41}$$

Now, we consider two cases: $\underline{\lim}_{i\to+\infty}\|f(v_{n_i})\| = 0$ and $\underline{\lim}_{i\to+\infty}\|f(v_{n_i})\| > 0$.

Suppose that $\underline{\lim}_{i\to+\infty}\|f(v_{n_i})\| = 0$. Since $v_{n_i} \rightharpoonup x^*$ and f satisfies condition (C2), we obtain that $f(x^*) = 0$. Hence, $x^* \in \mathrm{Sol}(C, f)$. Suppose that $\underline{\lim}_{i\to+\infty}\|f(v_{n_i})\| > 0$. Then there exists an integer $\mathfrak{I} > 0$ verifying $f(v_{n_i}) \neq 0$ for all $i \ge \mathfrak{I}$. In view of (41), we achieve

$$\underline{\lim}_{i\to+\infty}\langle f(v_{n_i})/\|f(v_{n_i})\|, \hat{u} - v_{n_i}\rangle \ge 0,\ \forall \hat{u} \in C. \tag{42}$$

Let $\{\xi_j\}$ be a sequence satisfying (i) $\xi_j > 0$ for all $j > 0$; (ii) $\xi_{j+1} < \xi_j, \forall j > 0$ and (iii) $\lim_{j\to\infty} \xi_j = 0$. By virtue of (42), there exists a strictly increasing subsequence $\{n_{i_j}\}$ satisfying $n_{i_j} \ge \mathfrak{I}$ and $\forall j \ge 0$,

$$\langle f(v_{n_{i_j}})/\|f(v_{n_{i_j}})\|, \hat{u} - v_{n_{i_j}}\rangle + \xi_j > 0,\ \forall \hat{u} \in C,$$

which results in that

$$\langle f(v_{n_{i_j}}), \hat{u} - v_{n_{i_j}}\rangle + \xi_j\|f(v_{n_{i_j}})\| > 0,\ \forall \hat{u} \in C, \forall j \ge 0. \tag{43}$$

Write $\tilde{v}_j = f(v_{n_{i_j}})/\|f(v_{n_{i_j}})\|^2, \forall j \ge 0$. It is clear that $\langle f(v_{n_{i_j}}), \tilde{v}_j\rangle = 1$ for each $j \ge 0$. Owing to (43), we have

$$\langle f(v_{n_{i_j}}), \hat{u} + \xi_j\|f(v_{n_{i_j}})\|\tilde{v}_j - v_{n_{i_j}}\rangle > 0,\ \forall \hat{u} \in C, \forall j \ge 0. \tag{44}$$

Since f is quasimonotone on H, from (44), we have

$$\langle f(\hat{u} + \xi_j\|f(v_{n_{i_j}})\|\tilde{v}_j), \hat{u} + \xi_j\|f(v_{n_{i_j}})\|\tilde{v}_j - v_{n_{i_j}}\rangle \ge 0,\ \forall \hat{u} \in C, \forall j \ge 0. \tag{45}$$

Since $\lim_{j\to+\infty}\xi_j\|f(v_{n_{i_j}})\|\|\tilde{v}_j\| = \lim_{j\to+\infty}\xi_j = 0$ and f is Lipschitz continuous, we conclude that $\lim_{j\to\infty} f(x+\xi_j\|f(v_{n_{i_j}})\|\tilde{v}_j) = f(x)$. Letting $j\to+\infty$ in (45), we deduce

$$\langle f(\hat{u}), \hat{u}-x^*\rangle \geq 0,\ \forall \hat{u}\in C,$$

which indicates $x^*\in \mathrm{Sol}^d(C,f)$. Next, we show that x^* is the unique weak cluster point of $\{u_n\}$ in $\mathrm{Sol}^d(C,f)$. Let $\bar{x}\in \mathrm{Sol}^d(C,f)$ be another weak cluster point of $\{u_n\}$. Then, there exists a sequence $\{u_{n_j}\}$ of $\{u_n\}$ satisfying $u_{n_j}\rightharpoonup \bar{x}$ as $j\to+\infty$. Note that for all $k\geq 0$,

$$2\langle u_n, x^*-\bar{x}\rangle = \|u_n-\bar{x}\|^2 - \|u_n-x^*\|^2 + \|x^*\|^2 - \|\bar{x}\|^2. \tag{46}$$

Note that $\lim_{n\to+\infty}\|u_n-x^*\|$ and $\lim_{n\to+\infty}\|u_n-\bar{x}\|$ exist. From (46), $\lim_{n\to+\infty}\langle u_n, x^*-\bar{x}\rangle$ exists. Hence,

$$\lim_{i\to+\infty}\langle u_{n_i}, x^*-\bar{x}\rangle = \lim_{j\to+\infty}\langle u_{n_j}, x^*-\bar{x}\rangle \tag{47}$$

Since $u_{n_i}\rightharpoonup x^*$ and $u_{n_j}\rightharpoonup \bar{x}$, from (47), we have

$$\langle x^*, x^*-\bar{x}\rangle = \langle \bar{x}, x^*-\bar{x}\rangle$$

which implies that $\|x^*-\bar{x}\|^2 = 0$ and hence $x^*=\bar{x}$. Therefore, $\{u_n\}$ has the unique weak cluster point in $\mathrm{Sol}^d(C,f)$. By the condition (C1), $\{x\in C, f(x)=0\}\setminus \mathrm{Sol}^d(C,f)$ is a finite set. Therefore, $\{u_n\}$ has finite weak cluster points in $\mathrm{Sol}(C,f)$ denoted by $a_1, a_2, \cdots, a_m$. Set $N_0=\{1,2,\cdots,m\}$ and $\nu = \min\{\frac{\|a_j-a_k\|}{3}, j,k\in N_0, j\neq k\}$. Let $a_j, j\in N_0$ be any weak cluster point in $\mathrm{Sol}(C,f)$ and $\{u^j_{n_i}\}$ be a subsequence of $\{u_n\}$ satisfying $u^j_{n_i}\rightharpoonup a_j$ as $i\to+\infty$. Then, we have

$$\lim_{i\to+\infty}\langle u^j_{n_i}, \frac{a_j-a_k}{\|a_j-a_k\|}\rangle = \langle a_j, \frac{a_j-a_k}{\|a_j-a_k\|}\rangle,\ \forall k\in N_0 \text{ and } k\neq j. \tag{48}$$

By the definition of ν, we have $\forall k\neq j$,

$$\begin{aligned}\langle a_j, \frac{a_j-a_k}{\|a_j-a_k\|}\rangle &= \frac{\|a_j-a_k\|}{2} + \frac{\|a_j\|^2-\|a_k\|^2}{2\|a_j-a_k\|}\\ &> \nu + \frac{\|a_j\|^2-\|a_k\|^2}{2\|a_j-a_k\|}.\end{aligned} \tag{49}$$

In the light of (48) and (49), there exists an integer q^j_i such that when $i\geq q^j_i$,

$$u^j_{n_i}\in\{x:\langle x, \frac{a_j-a_k}{\|a_j-a_k\|}\rangle > \nu + \frac{\|a_j\|^2-\|a_k\|^2}{2\|a_j-a_k\|}\},\ k\in N_0, k\neq j. \tag{50}$$

Set

$$Sb_j = \bigcap_{k=1,k\neq j}^{m}\{x:\langle x, \frac{a_j-a_k}{\|a_j-a_k\|}\rangle > \nu + \frac{\|a_j\|^2-\|a_k\|^2}{2\|a_j-a_k\|}\}. \tag{51}$$

Take into account of (50) and (51), we have $u^j_{n_i}\in Sb_j$ when $i\geq \max\{q^j_i, j\in N_0\}$.

Now we show that $u_n\in\bigcup_{j=1}^m Sb_j$ for a large enough n. If not, there exists a subsequence $\{u_{n_l}\}$ of $\{u_n\}$ such that $u_{n_l}\notin\bigcup_{j=1}^m Sb_j$. By the boundedness of $\{u_{n_l}\}$, there exists a subsequence of $\{u_{n_l}\}$ convergent weakly to x^*. Without loss of generality, we still denote the subsequence as $\{u_{n_l}\}$. According to assumptions $u_{n_l}\notin\bigcup_{j=1}^m Sb_j$, so $u_{n_l}\notin Sb_j$ for any $j\in N_0$. Therefore, there exists a subsequence $\{u_{n_{l_s}}\}$ of $\{u_{n_l}\}$ such that $\forall s\geq 0$,

$$u_{n_{l_s}}\notin\{x:\langle x, \frac{a_j-a_k}{\|a_j-a_k\|}\rangle > \nu + \frac{\|a_j\|^2-\|a_k\|^2}{2\|a_j-a_k\|}\}, k\in N_0, k\neq j. \tag{52}$$

Thus,

$$x^*\notin\{x:\langle x, \frac{a_j-a_k}{\|a_j-a_k\|}\rangle > \nu + \frac{\|a_j\|^2-\|a_k\|^2}{2\|a_j-a_k\|}\}, k\in N_0, k\neq j,$$

which implies that $x^* \neq a_j, j \in N_0$. This is impossible. So, for a large enough positive integer N_1, $u_n \in \bigcup_{j=1}^m Sb_j$ when $n \geq N_1$.

Next, we show that $\{u_n\}$ has the unique weak cluster point in $\mathrm{Sol}(C, f)$. First, there exists a positive integer $N_2 \geq N_1$ such that $\|u_{n+1} - u_n\| < \nu$ for all $n \geq N_2$. Assume that $\{u_n\}$ has at least two weak cluster points in $\mathrm{Sol}(C, f)$. Then, there exists $\hat{n} \geq N_2$ such that $u_{\hat{n}} \in Sb_j$ and $u_{\hat{n}+1} \in Sb_k$, where $j, k \in N_0$ and $m \geq 2$, that is, $u_{\hat{n}} \in Sb_j = \bigcap_{k=1,k\neq j}^m \{x : \langle x, \frac{a_j - a_k}{\|a_j - a_k\|}\rangle > \nu + \frac{\|a_j\|^2 - \|a_k\|^2}{2\|a_j - a_k\|}\}$ and $u_{\hat{n}+1} \in Sb_k = \bigcap_{j=1,j\neq k}^m \{x : \langle x, \frac{a_k - a_j}{\|a_k - a_j\|}\rangle > \nu + \frac{\|a_k\|^2 - \|a_j\|^2}{2\|a_k - a_j\|}\}$. Therefore,

$$\langle u_{\hat{n}}, \frac{a_j - a_k}{\|a_j - a_k\|}\rangle > \nu + \frac{\|a_j\|^2 - \|a_k\|^2}{2\|a_j - a_k\|} \tag{53}$$

and

$$\langle u_{\hat{n}+1}, \frac{a_k - a_j}{\|a_k - a_j\|}\rangle > \nu + \frac{\|a_k\|^2 - \|a_j\|^2}{2\|a_k - a_j\|}. \tag{54}$$

Combining (53) and (54), we achieve

$$\langle u_{\hat{n}} - u_{\hat{n}+1}, \frac{a_j - a_k}{\|a_j - a_k\|}\rangle > 2\nu. \tag{55}$$

At the same time, we have

$$\|u_{\hat{n}+1} - u_{\hat{n}}\| < \nu. \tag{56}$$

Based on (55) and (56), we deduce $2\nu < \langle u_{\hat{n}} - u_{\hat{n}+1}, \frac{a_j - a_k}{\|a_j - a_k\|}\rangle \leq \|u_{\hat{n}} - u_{\hat{n}+1}\| < \nu$. This leads to a contradiction. Then, $\{u_n\}$ has the unique weak cluster point in $\mathrm{Sol}(C, f)$. So, the sequence $\{u_n\}$ has the unique weak cluster point $x^* \in \Gamma$. Therefore, the sequence $\{u_n\}$ converges weakly to $x^* \in \Gamma$. This completes the proof. □

Based on Algorithm 3.1 and Theorem 3.1, we can obtain the following algorithms and the corresponding corollaries.

Algorithm 3.2. *Let $u_0 \in H$ be an initial guess. Set $n = 0$.*
Step 1. Let the n-th iterate u_n and the n-th step-size α_n be given. Compute $w_n = P_C[u_n - \alpha_n f(u_n)]$, and $u_{n+1} = (1 - \varsigma_n)u_n + \varsigma_n w_n + \varsigma_n \alpha_n[f(u_n) - f(w_n)]$.
Step 2. Update the $n+1$-th step-size by the following form

$$\alpha_{n+1} = \begin{cases} \min\left\{\alpha_n, \frac{\zeta\|w_n - u_n\|}{\|f(w_n) - f(u_n)\|}\right\}, & \text{if } f(w_n) \neq f(u_n), \\ \alpha_n, & \text{else.} \end{cases}$$

Set $n := n + 1$ and return to step 1.

Corollary 3.1. *Assume that the operator $f : H \to H$ is quasimonotone, κ_1-Lipschitz continuous and satisfies condition (C2). Suppose that $\mathrm{Sol}^d(C, f) \neq \emptyset$, $\{x \in C : f(x) = 0\} \setminus \mathrm{Sol}^d(C, f)$ is a finite set and condition (C4) holds. Then the sequence $\{u_n\}$ generated by Algorithm 3.2 converges weakly to some point in $\mathrm{Sol}(C, f)$.*

Algorithm 3.3. *Take $u_0 \in C$ and $\alpha_0 > 0$. Set $n = 0$.*
Step 1. For known u_n, compute $u_{n+1} = (1 - \sigma_n)u_n + \sigma_n T^n[(1 - \varrho_n)u_n + \varrho_n T^n(u_n)]$.
Step 2. Set $n := n + 1$ and return to step 1.

Corollary 3.2. *Assume that T is a uniformly κ_2-Lipschitz and τ_n-asymptotically pseudocontractive operator on H. Suppose that $Fix(T) \neq \emptyset$ and conditions (C3) and (C5) hold. Then the sequence $\{u_n\}$ generated by Algorithm 3.3 converges weakly to some point in $Fix(T)$.*

4. **Conclusions**

In this paper, we investigate a fixed point problem and a quasimonotone variational inequality in a Hilbert space. We construct an iterative algorithm (Algorithm 3.1) for finding a fixed point of an asymptotically pseudocontractive operator T and a solution of quasimonotone variational inequality. Our algorithm consists of Tseng-type method and self-adaptive rule. With the help of conditions (C1) and (C2), we show that the proposed iterative sequence (10) converges weakly to a fixed point of an asymptotically pseudocontractive operator T and a solution of quasimonotone variational inequality.

REFERENCES

[1] F. Alvarez, *Weak convergence of a relaxed and inertial hybrid projection proximal point algorithm for maximal monotone operators in Hilbert spaces*, SIAM J. Optim., **14**(2004), 773–782.

[2] T.V. Anh, *Linesearch methods for bilevel split pseudomonotone variational inequality problems*, Numerical Algorithms, **81**(2019), 1067–1087.

[3] S. Banert and R.I. Bot, *A forward-backward-forward differential equation and its asymptotic properties*, J. Convex Anal., **25**(2018), 371–388.

[4] T.Q. Bao and P.Q. Khanh, *A projction-type algorithm for pseudomonotone nonlipschitzian multivalued variational inequalities*, Nonconvex Optim. Appl., **77**(2005), 113–129.

[5] H.H. Bauschke and P.L. Combettes, *Convex analysis and monotone operator theory in Hilbert spaces*, CMS books in mathematics. New York: Springer, 2011.

[6] R.L. Bot, E.R. Csetnek and P.T. Vuong, *The forward-backward-forward method from continuous and discrete perspective for pseudo-monotone variational inequalities in Hilbert spaces*, Eur. J. Oper. Res., **287**(2020), 49–60.

[7] L.C. Ceng, A. Petrusel, J.C. Yao and Y. Yao, *Systems of variational inequalities with hierarchical variational inequality constraints for Lipschitzian pseudocontractions*, Fixed Point Theory, **20**(2019), 113–133.

[8] Y. Censor, A. Gibali and S. Reich, *The subgradient extragradient method for solving variational inequalities in Hilbert space*, J Optim. Theory Appl., **148**(2011), 318–335.

[9] Y. Censor, A. Gibali and S. Reich, *Extensions of Korpelevichs extragradient method for the variational inequality problem in Euclidean space*, Optim., **61**(2012), 1119–1132.

[10] C. Chen, S. Ma and J. Yang, *A general inertial proximal point algorithm for mixed variational inequality problem*, SIAM J. Optim., **25**(2014), 2120–2142.

[11] V. Dadashi and M. Postolache, *Forward-backward splitting algorithm for fixed point problems and zeros of the sum of monotone operators*, Arab. J. Math., **9** (2020), 89–99.

[12] Q.L. Dong, L. Liu and Y. Yao, *Self-adaptive projection and contraction methods with alternated inertial terms for solving the split feasibility problem*, J. Nonlinear Convex Anal., in press.

[13] Q.L. Dong, Y. Peng and Y. Yao, *Alternated inertial projection methods for the split equality problem*, J. Nonlinear Convex Anal., **22**(2021), 53–67.

[14] F. Facchinei and J.S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*, Volume I. Springer Series in Operations Research. Springer, New York, 2003.

[15] M. Fukushima, *A relaxed projection method for variational inequalities*, Math. Program., **35**(1986), 58–70.

[16] R. Glowinski, *Numerical methods for nonlinear variational problems*, Springer, New York, 1984.

[17] A.A. Goldstein, *Convex programming in Hilbert space*, Bull. Am. Math. Soc., **70**(1964), 709–710.

[18] H. Iiduka, *Iterative algorithm for triple-hierarchical constrained nonconvex optimization problem and its application to network bandwidth allocation*, SIAM J. Optim., **22**(2012), 862–878.

[19] P.D. Khanh and P.T. Vuong, *Modified projection method for strongly pseudomonotone variational inequalities*, J. Global Optim., **58**(2014), 341–350.

[20] G.M. Korpelevich, *An extragradient method for finding saddle points and for other problems*, Ekon. Matorsz. Metod., **12**(1976), 747–756.

[21] P.E. Maingé, *Strong convergence of projected reflected gradient methods for variational inequalities*, Fixed Point Theory, **19**(2018), 659–680.

[22] Y. Malitsky, *Proximal extrapolated gradient methods for variational inequalities*, Optim. Meth. Soft., **33**(2018), 140–164.

[23] Y. Mudasir, S. Deepak, R. Stojan and I. Mohammad, *Convergence theorems for generalized contractions and applications*, Filomat, **34**(2020), 945–964.

[24] D.R. Sahu, A. Pitea, M. Verma, *A new iteration technique for nonlinear operators as concerns convex programming and feasibility problems*, Numer. Algorithms, **83(2)**(2020), 421–449.

[25] W. Sintunavarat and A. Pitea, *On a new iteration scheme for numerical reckoning fixed points of Berinde mappings with convergence analysis*, J. Nonlinear Sci. Appl., **9** (2016), 2553–2562.

[26] M.V. Solodov and B.F. Svaiter, *A new projection method for variational inequality problems*, SIAM J. Control Optim., **37**(1999), 765–776.

[27] G. Stampacchi, *Formes bilineaires coercivites surles ensembles convexes*, C. R. Acad. Sciences, **258**(1964), 4413–4416.

[28] D. Sun, *A projction and contraction method for the nonlinear complementarity problems and its extensions*, Math. Numer. Sin., **16**(1994), 183–194.

[29] B.S. Thakur, D. Thakur and M. Postolache, *A new iterative scheme for numerical reckoning fixed points of Suzuki's generalized nonexpansive mappings*, Appl. Math. Comput., **275**(2016), 147–155.

[30] B.S. Thakur, D. Thakur and M. Postolache, *A new iteration scheme for approximating fixed points of nonexpansive mappings*, Filomat, **30**(2016), 2711–2720.

[31] D. Thakur, B.S. Thakur and M. Postolache, *New iteration scheme for numerical reckoning fixed points of nonexpansive mappings*, J. Inequal. Appl., **2014**(2014), Art. No. 328.

[32] P. Tseng, *A modified forward-backward splitting method for maximal monotone mappings*, SIAM J. Control Optim., **38**(2000), 431–446.

[33] G.I. Usurelu, *Split feasibility handled by a single-projection three-step iteration with comparative analysis*, J. Nonlinear Convex Anal. **22(3)**(2021), 544–558.

[34] G.I. Usurelu, M. Postolache, *Algorithm for generalized hybrid operators with numerical analysis and applications*, J. Nonlinear Variational Anal., in press.

[35] P.T. Vuong, *On the weak convergence of the extragradient method for solving pseudomonotone variational inequalities*, J. Optim. Theory Appl., **176**(2018), 399–409.

[36] Y. Yao, Olaniyi S. Iyiola and Y. Shehu, *Subgradient extragradient method with double inertial steps for variational inequalities*, J. Sci. Comput., **90**(2022), Article number 71.

[37] Y. Yao, L. Leng, M. Postolache and X. Zheng, *Mann-type iteration method for solving the split common fixed point problem*, J. Nonlinear Convex Anal., **18**(2017), 875–882.

[38] Y. Yao, H. Li and M. Postolache, *Iterative algorithms for split equilibrium problems of monotone operators and fixed point problems of pseudo-contractions*, Optim., in press, DOI: 10.1080/02331934.2020.1857757.

[39] Y. Yao, Y.C. Liou and M. Postolache, *Self-adaptive algorithms for the split problem of the demicontractive operators*, Optim., **67**(2018), 1309-1319.

[40] Y. Yao, Y.C. Liou and J.C. Yao, *Split common fixed point problem for two quasi-pseudocontractive operators and its algorithm construction*, Fixed Point Theory Appl., **2015**(2015), Art. No. 127.
[41] Y. Yao, Y.C. Liou and J.C. Yao, *Iterative algorithms for the split variational inequality and fixed point problems under nonlinear transformations*, J. Nonlinear Sci. Appl., **10**(2017), 843–854.
[42] Y. Yao, M. Postolache and Y.C. Liou, *Strong convergence of a self-adaptive method for the split feasibility problem*, Fixed Point Theory Appl., **2013**(2013), Art. No. 201.
[43] Y. Yao, M. Postolache and J.C. Yao, *Iterative algorithms for the generalized variational inequalities*, U.P.B. Sci. Bull., Series A, **81**(2019), 3–16.
[44] Y. Yao, M. Postolache and J.C. Yao, *An iterative algorithm for solving the generalized variational inequalities and fixed points problems*, Mathematics, **7**(2019), Art. No. 61.
[45] Y. Yao, M. Postolache and J.C. Yao, *Strong convergence of an extragradient algorithm for variational inequality and fixed point problems*, U.P.B. Sci. Bull., Series A, **82(1)**(2020), 3–12.
[46] Y. Yao, M. Postolache and Z. Zhu, *Gradient methods with selection technique for the multiple-sets split feasibility problem*, Optim., **69**(2020), 269–281.
[47] Y. Yao, X. Qin and J.C. Yao, *Projection methods for firmly type nonexpansive operators*, J. Nonlinear Convex Anal., **19**(2018), 407–415.
[48] Y. Yao, N. Shahzad, M. Postolache and J.C. Yao, *Convergence of self-adaptive Tseng-type algorithms for split variational inequalities and fixed point problems*, Carpathian J. Math., in press.
[49] Y. Yao, J.C. Yao, Y.C. Liou and M. Postolache, *Iterative algorithms for split common fixed points of demicontractive operators without priori knowledge of operator norms*, Carpathian J. Math., **34**(2018), 459–466.
[50] Y. Yao, J.C. Yao and M. Postolache, *An iterate for solving quasi-variational inclusions and nonmonotone equilibrium problems*, J. Nonlinear Convex Anal., in press.
[51] H. Zegeye, N. Shahzad and Y. Yao, *Minimum-norm solution of variational inequality and fixed point problem in Banach spaces*, Optim., **64**(2015), 453–471.
[52] C. Zhang, Z. Zhu, Y. Yao and Q. Liu, *Homotopy method for solving mathematical programs with bounded box-constrained variational inequalities*, Optim., **68**(2019), 2293–2312.
[53] X. Zhao, M.A. Kobis, Y. Yao and J.C. Yao, *A projected subgradient method for nondifferentiable quasiconvex multiobjective optimization problem*, J. Optim. Theory Appl., **190**(2021), 82–107.
[54] X. Zhao, J.C. Yao and Y. Yao, *A proximal algorithm for solving split monotone variational inclusions*, U.P.B. Sci. Bull., Series A, **82(3)**(2020), 43–52.
[55] X. Zhao and Y. Yao, *Modified extragradient algorithms for solving monotone variational inequalities and fixed point problems*, Optim., **69**(2020), 1987–2002.
[56] H. Zhou, *Demiclosedness principle with applications for asymptotically pseudo-contractions in Hilbert spaces*, Nonlinear Anal., **70** (2009), 3140–3145.
[57] D.L. Zhu and P. Marcotte, *Co-coercivity and its role in the convergence of iterative schemes for solving variational inequalities*, SIAM J. Control. Optim., **6**(1996), 714–726.
[58] L.J. Zhu, Y. Yao and M. Postolache, *Projection methods with linesearch technique for pseudomonotone equilibrium problems and fixed point problems*, U.P.B. Sci. Bull., Series A, **83(1)**(2021), 3–14.