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PROPER CONFORMAL MOTIONS IN BIANCHI TYPE II SPACE-TIMES

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In this paper, a study of proper conformal motion in non conformally flat Bianchi type II space-times is given by using the direct integration technique. All possibilities for the existence of proper conformal motion are exhausted. Using the above mentioned technique, it is shown that there exist two main cases when the above space-times admit proper conformal motion.

1. Introduction

In this paper we are looking for the existence of proper conformal motion in non conformally flat Bianchi type II space-times using direct integration technique. The conformal symmetry preserving the metric structure up to a conformal function has a great importance in the theory of general relativity formulated by Einstein. It is therefore required to consider such type of symmetry. Different approaches [5-9] were adapted to study conformal motion or conformal vector fields. For studying conformal vector fields in the above space-times we use the direct integration technique.

Throughout M represents a four dimensional, connected, Hausdorff space-time manifold with Lorentz metric g of signature (-, +, +, +). The curvature tensor associated with g_{ab}, through the Levi-Civita connection, is denoted in component form by $R^a{}_{bcd}$ and the Ricci tensor components are $R_{ab} = R^c{}_{acb}$. The usual covariant, partial and Lie derivatives are denoted by a semicolon, a comma and the symbol L, respectively. Round and square brackets denote the usual symmetrization and skew-symmetrization, respectively.

One can decomposed $X_{a;b}$ on M as

$$X_{a;b} = \frac{1}{2} h_{ab} + F_{ab}, \tag{1}$$

where $h_{ab} (= h_{ba}) = L_X g_{ab}$ and $F_{ab} (= -F_{ba})$ are symmetric and skew symmetric tensors on M respectively. Such a vector field X is called conformal if the local diffeomorphisms ϕ_t (for appropriate t) associated with X preserve the metric

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structure up to a conformal factor i.e $\phi_t^{*} g_{ab} = \psi\, g_{ab}$, where ψ is a no where zero smooth function on M, called the conformal function of X and ϕ_t^{*} is a pulback map on M [1-3]. This is equivalent to the condition that h_{ab} satisfies

$$h_{ab} = 2\psi\, g_{ab},$$

which in turn is equievalent to

$$g_{ab,c}\, X^{c} + g_{bc}\, X^{c}_{,a} + g_{ca}\, X^{c}_{,b} = 2\psi\, g_{ab}, \tag{2}$$

for some smooth conformal function ψ on M, X is called a conformal vector field. If ψ is a constant on M, then X is homothetic (proper homothetic if $\psi \neq 0$) while for $\psi = 0$ it is Killing [1]. The vector field X is called proper conformal if it is not homothetic while it is special conformal if the conformal function ψ satisfies the condition $\psi_{a;b} = 0$. A complete study about conformal vector field, conformal bivector and its dimension can be found in [1].

2. Main Results

Consider the Bianchi type II space-times in the usual coordinate system (t, x, y, z) (labeled by (x^0, x^1, x^2, x^3), respectively) with line element [4]

$$ds^2 = -dt^2 + A(t)dx^2 + B(t)dy^2 + \left[B(t)x^2 + C(t)\right]dz^2 + 2\,x\,B(t)\,dy\,dz. \tag{3}$$

where A, B and C are zero functions of t only. The above space-time admits three linearly independent Killing vector fields which are

$$\frac{\partial}{\partial y},\quad \frac{\partial}{\partial z},\quad \frac{\partial}{\partial x} - z\frac{\partial}{\partial y}. \tag{4}$$

A vector field X is called conformal if it satisfies equation (2). Writing (2) explicitly and using equation (3) we get

$$X^{0}_{,0} = \psi, \tag{5}$$

$$A(t)\,X^{1}_{,0} - X^{0}_{,1} = 0, \tag{6}$$

$$B(t)\,X^{2}_{,0} + xB(t)\,X^{3}_{,0} - X^{0}_{,2} = 0, \tag{7}$$

$$xB(t)X^{2}_{,0} + \left[x^2 B(t) + C(t)\right]X^{3}_{,0} - X^{0}_{,3} = 0, \tag{8}$$

$$\dot{A}(t)X^{0} + 2\,A(t)X^{1}_{,1} = 2\,A(t)\psi, \tag{9}$$

$$B(t)\,X^{2}_{,1} + xB(t)\,X^{3}_{,1} + A(t)\,X^{1}_{,2} = 0, \tag{10}$$

$$xB(t)\,X^{2}_{,1} + \left[x^2 B(t) + C(t)\right]X^{3}_{,1} + A(t)\,X^{1}_{,3} = 0, \tag{11}$$

$$\dot{B}(t)X^{0} + 2B(t)X^{2}_{,2} + 2x\,B(t)X^{3}_{,2} = 2B(t)\psi, \tag{12}$$

$$x\,\dot{B}(t)X^{0} + B(t)\,X^{1} + x\,B(t)\,X^{2}_{,2} + \left[x^2 B(t) + C(t)\right]X^{3}_{,2} + B(t)\,X^{2}_{,3} = 2\,x\,B(t)\psi, \tag{13}$$

$$\frac{1}{2}\left[x^2\dot{B}(t)+\dot{C}(t)\right]X^0+xB(t)X^1+xB(t)X^2_{,3}+\left[x^2B(t)+C(t)\right]X^3_{,3}=$$
$$=\psi\left[x^2B(t)+C(t)\right], \tag{14}$$

where $\psi=\psi(t,x,y,z)$. Equations (5) and (6) give

$$X^0=\int\psi\,dt+E^1(x,y,z),\quad X^1=\int\frac{1}{A(t)}\left[\int\psi_x\,dt+E^1_x(x,y,z)\right]dt+E^2(x,y,z).$$

If equation (7) is multiplied by x and then subtracted from equation (8), one obtains:

$$X^3=\int\frac{1}{C(t)}\left[\int\psi_z dt-x\int\psi_y\,dt\right]dt+\left[E^1_z(x,y,z)-xE^1_y(x,y,z)\right]\int\frac{dt}{C(t)}+E^3(x,y,z).$$

Equation (7) gives

$$X^2=\int\frac{1}{B(t)}\left[\int\psi_y dt+E^1_y(x,y,z)\right]dt-x\left[E^1_z(x,y,z)-xE^1_y(x,y,z)\right]\int\frac{dt}{C(t)}$$
$$-x\int\frac{1}{C(t)}\left[\int\psi_z dt-x\int\psi_y dt\right]dt+E^4(x,y,z).$$

Thus we have the following system

$$X^0=\int\psi\,dt+E^1(x,y,z);$$
$$X^1=\int\frac{1}{A(t)}\left[\int\psi_x\,dt+E^1_x(x,y,z)\right]dt+E^2(x,y,z),$$
$$X^2=\int\frac{1}{B(t)}\left[\int\psi_y\,dt+E^1_y(x,y,z)\right]dt-x\left[E^1_z(x,y,z)-xE^1_y(x,y,z)\right]\int\frac{dt}{C(t)}$$
$$-x\int\frac{1}{C(t)}\left[\int\psi_z\,dt-x\int\psi_y(t,x,y,z)\,dt\right]dt+E^4(x,y,z),$$
$$X^3=\int\frac{1}{C(t)}\left[\int\psi_z\,dt-x\int\psi_y\,dt\right]dt+\left[E^1_z(x,y,z)-xE^1_y(x,y,z)\right]\int\frac{dt}{C(t)}+E^3(x,y,z), \tag{15}$$

where $E^1(x,y,z)$, $E^2(x,y,z)$, $E^3(x,y,z)$ and $E^4(x,y,z)$ are functions of integration which are determined by solving the remaining six equations. If one proceeds further after some tedious and lengthy calculation we reach the conclusion that there exist two cases when the above space-times admit proper conformal vector fields. These are

Case (1): In this case we have $A(t)=c_6U^2(t)e^{-2c_1V(t)}$, $B(t)=c_7U^2(t)e^{-2c_1V(t)}$ and $C(t)=c_8U^2(t)$, where $c_6,c_7,c_8\in R(c_6\neq0,c_7\neq0,c_8\neq0)$. The space-time (3) in this case becomes

$$ds^2 = -dt^2 + U^2(t)e^{-2c_1V(t)}\left(c_6 dx^2 + c_7 dy^2\right) + U^2(t)\left(c_7 x^2 e^{-2c_1V(t)} + c_8\right)dz^2 + 2xc_7 U^2(t)e^{-2c_1V(t)}dy\,dz \qquad (16)$$

and the conformal vector fields in this case are

$$X^0 = U(t),\ X^1 = c_1 x + c_2,\ X^2 = -c_2 z + c_1 y + c_3,\ X^3 = c_4, \qquad (17)$$

where $U(t) = \int \psi(t)dt + c_5$, $V(t) = \int \frac{dt}{U(t)}$ and $c_1, c_2, c_3, c_4, c_5 \in R.$ After subtracting Killing vector fields from equation (17) becomes:

$$X^0 = U(t),\ X^1 = c_1 x,\ X^2 = c_1 y,\ X^3 = 0. \qquad (18)$$

The space-time (16) admits four linearly independent conformal vector fields in which three are Killing which are given in (4) and one is proper conformal which is given in (18).

Case (2): In this case $A(t) = c_{13}U^2(t)$, $B(t) = c_{14}U^2(t)$ and $C(t) = c_{15}U^2(t)$, where $c_{13}, c_{14}, c_{15} \in R(c_{13} \neq 0, c_{14} \neq 0, c_{15} \neq 0)$. The space-time (3) in this case becomes:

$$ds^2 = -dt^2 + U^2(t)\left[c_{13}\,dx^2 + c_{14}\,dy^2 + \left(c_{14}x^2 + c_{15}\right)dz^2 + 2c_{14}\,xdy\,dz\right]. \qquad (19)$$

and the conformal vector fields in this case are:

$$X^0 = U(t),\ X^1 = c_9,\ X^2 = -c_9 z + c_{10},\ X^3 = c_{11}, \qquad (20)$$

where $U(t) = \int \psi(t)dt + c_{16}$ and $c_9, c_{10}, c_{11}, c_{16} \in R.$ After subtracting Killing vector fields from equation (20) becomes:

$$X = (U(t),\ 0,\ 0,\ 0). \qquad (21)$$

The space-time (19) admits four linearly independent conformal vector fields in which three are Killing which are given in (4) and one is proper conformal which is given in (21).

3. Summary

In this paper a study of conformal motion in Bianchi type II space-times is given by using the direct integration technique. From this study we found that there exist two possible cases when the above space-times admit conformal motion. These space-times are given in equations (16) and (19). The proper conformal vector fields are given in equations (18) and (21).

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