U.P.B. Sci. Bull., Series A, Vol. 87, Iss. 4, 2025 ISSN 1223-7027

ON CO-r-SUBMODULES AND CO-r-NOETHERIAN MODULES

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In this paper, we investigate some properties and characterizations of co-r-submodules which is the dual notion of r-submodules. We prove that every nonzero submodule of a finitely generated module is a co-r-submodule. We investigate when a submodule N of an R-module M contains a co-r-submodule. Also, we study the further properties of co-r-Noetherian modules as a generalization of Noetherian modules.

Keywords: *co*-r-submodule, *co*-r-Noetherian module, r-submodule, r-Noetherian module.

2020 Mathematics Subject Classification: 13C13, 13C99.

1. Introduction

Throughout this paper, we focus only on commutative rings with nonzero identity and nonzero unital modules. Let R always denote such a ring and M denote such an R-module. In recent years, some new classes of ideals and submodules have been defined and investigated by various authors (see [1], [9], [13], [19]).

The set of zero-divisors of an R-module M is defined as the set

$$Z(M) := \{a \in R : am = 0 \text{ for some } 0 \neq m \in M\} \text{ [8].}$$

For every submodule N of M, the annihilator of N is denoted by $ann_R(N) := \{r \in R : rN = 0\}$ [8].

Koç and Tekir [15] introduced the following concept: a proper submodule N of M is said to be an *r-submodule* if $Z(M/N) \subseteq Z(M)$ [15]. Also, they proved various properties of r-submodules, which are similar to those of prime submodules and gave a new characterization of torsion free modules in terms of r-submodules.

In recent years, there have been various studies about r-submodules. For instance, Anebri et al. investigated ascending and descending chain conditions on r-submodules in [3] and [4].

Recall that an R-module M is said to satisfy Property (A) if for each finitely generated ideal I of R contained in $Z(M)$ there exists $0 \neq m \in M$ such that $Im = 0$ [17]. Mahdou et al. gave a characterization of modules satisfying Property (A) in terms of r-submodules in [17].

The dual notion of prime submodules was firstly introduced and studied by S. Yassemi in [21].

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Recall from [21] that a nonzero submodule N of an R-module M is said to be a *second submodule* if, for all $r \in R$, $rN = 0$ or $rN = N$. If N is a second submodule of M, then $p = ann_R(N)$ is a prime ideal of R and N is called a p-second submodule of M [21].

In recent years, second submodules have been studied by various authors in a number of papers (see for example [6], [7], [10], [11]).

In 2023, F. Farshadifar introduced the dual notion of r-submodules, which is called co-r-submodule, and studied ascending chain condition on co-r-submodules.

Recall from [20] that the dual notion of zero-divisors of a submodule N of M is defined as the set $W(N) := \{r \in R : rN \neq N\}$. A nonzero submodule N of M is said to be a *co-r-submodule* if $W(N) \subseteq W(M)$ [13].

Also M is said to be a *co-r-Noetherian module* if it satisfies the ascending chain condition on co-r-submodules [13].

Our aim in this paper is to study further properties of co-r-submodules and co-r-Noetherian modules. Among the other results in this paper, we prove that every nonzero submodule of a finitely generated module is an *co*-r-submodule (see Proposition 2.9). We investigate when a submodule N of an R-module M contains a *co*-r-submodule (see Theorem 2.18).

We give a characterization of reduced *co*-r-Noetherian modules via localization (see Theorem 2.21). We also investigate *co*-r-Noetherian property for multiplication and comultiplication modules (see Proposition 2.25 and Theorem 2.27).

2. Main Results

Definition 2.1 [13] We say that a non-zero submodule N of an R-module M is a *co-r-submodule* of M if for $a \in R$ and a submodule K of M, whenever $aN \subseteq K$ and $aM = M$, then $N \subseteq K$.

Remark 2.2. [13, Remark 2.3] Let M be an R-module and N be a non-zero submodule of M. It is easily seen that N is a *co*-r-submodule of M if and only if $W(N) \subseteq W(M)$.

Remark 2.3.

(1) It is clear from the definition that every non-zero R-module M is a *co*-r-submodule of itself.
(2) If R is an integral domain and N is a non-zero divisible submodule of an R-module M, then N is a *co*-r-submodule of M.
(3) Let M be a semisimple R-module. Then every non-zero submodule of M is a *co*-r-submodule of M.

Definition 2.4. [18] A proper submodule N of an R-module M is called a *pure submodule* of M if $rN = rM \cap N$ for every $r \in R$.

Remark 2.5. It is clear that if N is a non-zero pure submodule of M, then N is a *co*-r-submodule of M.

Definition 2.6. [12] An R-module M is said to be a *multiplication module* if each submodule N of M has the form $N = IM$ for some ideal I of R .

Remark 2.7. It is well-known that M is a multiplication module if and only if $N = (N : M)M$ for every submodule N of M (see [12]).

Proposition 2.8. [13, Theorem 2.4] *Every non-zero submodule of a multiplication module is a co-r-submodule.*

Proposition 2.9. *Every nonzero submodule N of a finitely generated R-module M is a co-r-submodule.*

Proof. Let M be a finitely generated R-module and N be a nonzero submodule of M. Assume that there exists $a \in W(N)\backslash W(M)$. Then it is clear that $aM = M$. Since M is finitely generated, by [8, Corollary 2.5], there exists $x \in R$ such that $(1 + xa)M = 0$ and so $(1 + xa)N = 0$.

This implies that $N = xaN \subseteq aN$ and so $N = aN$, a contradiction. Hence, we have $W(N) \subseteq W(M)$. □

Proposition 2.10. *Let M be an R-module. Then the following hold.*
(1) [13, Remark 2.3] *If N is a co-r-submodule of M, then $ann_R(N) \subseteq W(M)$.*
(2) [13, Proposition 2.6] *The sum of an arbitrary non-empty set of co-r-submodules of an R-module M is a co-r-submodule.*

Proposition 2.11. [13, Proposition 2.11] *Let N be a non-zero submodule of an R-module M. Then the following are equivalent.*
(1) *N is a co-r-submodule.*
(2) *$(0 :_M a) + N = (N :_M a)$ for every $a \in R\backslash W(M)$.*
(3) *$aN = N$ for every $a \in R\backslash W(M)$.*

Recall from [16], an R-module M is said to be an *α-reduced module* where α is an endomorphism of R with $\alpha(1) = 1$, if for any $a \in R$ and $m \in M$,
(1) $a^2m = 0$ implies $Rm \cap aM = 0$.
(2) $am = 0$ if and only if $\alpha(a)m = 0$.
If α is the identity map on R, then M is called a *reduced module* [16]. By [16, Lemma 1.2], M is a reduced module if and only if for any $a \in R$ and $m \in M$, $a^2m = 0$ implies that $am = 0$.

Proposition 2.12. *Let M be a reduced module and N be a nonzero submodule of M. If N is a co-r-submodule, then $N = (N :_M a)$ for each $a \in R\backslash W(M)$.*

Proof. Assume that M is a reduced module and N is a *co*-r-submodule of M. Then, by Proposition 2.12, we have $(0 :_M a) + N = (N :_M a)$ for each $a \in R\backslash W(M)$. In order to complete the proof, it is enough to show that $(0 :_M a) = 0$ for each $a \in R\backslash W(M)$. Let $m^* \in (0 :_M a)$. Then we have $am^* = 0$. Since $a \notin W(M)$, $aM = M$ and so $m^* = am'$ for some $m' \in M$. This implies that $am^* = a^2m' = 0$. As M is reduced, we conclude that $m^* = am' = 0$ and thus $(0 :_M a) = 0$. Therefore, $N = (N :_M a)$, as required. □

Recall that a proper submodule N of M is said to be a *prime submodule* if $am \in N$, where $a \in R$ and $m \in M$, then either $a \in (N : M)$ or $m \in N$. Note that a submodule N of M is a prime submodule if and only if $N = (N :_M a)$ for each $a \in R\backslash(N : M)$. As an immediate consequence of Proposition 2.12 we give the following explicit result.

Corollary 2.13. *Assume that M is a reduced module and N is a nonzero submodule of M with $W(M) \subseteq (N : M)$. If N is a co-r-submodule, then N is prime.*

Definition 2.14. Let S be a non-empty subset of R. We say that S is a *co*-r-multiplicatively closed subset of R if $R\backslash W(M) \subseteq S$ and $ab \in S$ for every $a \in R\backslash W(M)$ and $b \in S$.

Proposition 2.15. *If N is a co-r-submodule of M, then $R\backslash ann_R(N)$ is a co-r-multiplicatively closed subset of R.*

Proof. Since N is a *co*-r-submodule of M, by Proposition 2.10 (1), we have $R\backslash W(M) \subseteq R\backslash ann_R(N)$. Let $a \in R\backslash W(M)$ and $b \in R\backslash ann_R(N)$. $a \in R\backslash W(M)$ and $W(N) \subseteq W(M)$ (since N is a co-r-submodule of M) implies $aN = N$. Assume that $ab \in ann_R(N)$. Then $abN = 0$ and $aN = N$.

It follows that $abN = bN = 0$ and so $b \in ann_R(N)$, a contradiction. Hence, $ab \in R\backslash ann_R(N)$, as required. □

Proposition 2.16. *Suppose that N is a nonzero submodule of M. Then N is a co-r-submodule if and only if $R\backslash W(N)$ is a co-r-multiplicatively closed subset of R.*

Proof. Assume that $R\backslash W(N)$ is a *co-r*-multiplicatively closed subset of R. Then $R\backslash W(M) \subseteq R\backslash W(N)$ and so $W(N) \subseteq W(M)$.

Conversely, assume that N is a *co-r*-submodule of M. Then $W(N) \subseteq W(M)$ and so $R\backslash W(M) \subseteq R\backslash W(N)$. Let $a \in R\backslash W(M)$ and $b \in R\backslash W(N)$. Now, we will show that $ab \in R\backslash W(N)$.

If $ab \in W(N)$, then $abN \neq N$. Since $b \in R\backslash W(N)$, we have $bN = N$ and thus $abN = aN \neq N$ and this yields that $a \in W(N)$. As N is a *co-r*-submodule, $W(N) \subseteq W(M)$ and so $a \in W(M)$, a contradiction.

Hence $ab \in R\backslash W(N)$, that is, $R\backslash W(N)$ is a *co-r*-multiplicatively closed subset of R. □

Definition 2.17. Let T be a *co-r*-multiplicatively closed subset of R and T^* be a non-empty subset of M. We say that T^* is a T-closed subset of M if $ax \in T^*$ for each $a \in T$ and $x \in T^*$.

The following theorem is the dual result of [15, Theorem 4].

Theorem 2.18. *Let T be a co-r-multiplicatively closed subset of R and T^* be a T-closed subset of M. Suppose that N is a submodule of M with $N \cup T^* = M$. Then there exists a co-r-submodule L of M such that $L \subseteq N$ and $L \cup T^* = M$.*

Proof. Let $\Psi := \{L' : L'$ is a submodule of M with $L' \subseteq N$ and $L' \cup T^* = M\}$. Then $\Psi \neq \emptyset$ as $N \in \Psi$. By Zorn's Lemma, Ψ has a minimal element, say L. Suppose that L is not a *co-r*-submodule of M. Then there exists an $a \in R$ such that $aL \neq L$ and $aM = M$. By the minimality of L, we have $aL \cup T^* \neq M$. Therefore, there exists $m \in M$ such that $m \notin aL$ and $m \notin T^*$.

Since $aM = M$, we have $m = ax$ for some $x \in M$. $m \notin aL$ implies $x \notin L$. It follows that $x \in T^*$. Since T^* is a T-closed subset of M, we have $m = ax \in T^*$, a contradiction. Therefore, L is a *co-r*-submodule of M. □

Definition 2.19. Let M be an R-module. If M satisfies ascending chain condition on *co-r*-submodules, then M is called *co-r*-Noetherian module.

Let M be an R-module and S a multiplicatively closed subset of R. Then $S^{-1}M$ denotes the quotient module of M. Note that $S^{-1}M$ is both an R-module and $S^{-1}R$-module. $0_{S^{-1}M}$ denotes the zero element of $S^{-1}M$.

The natural R-homomorphism $\pi : M \to S^{-1}M$ is defined as $\pi(m) = \frac{m}{1}$ for each $m \in M$. We use the notation K^c to denote $\pi^{-1}(K)$ for a submodule K of $S^{-1}M$ and N^e to denote the submodule generated by $\pi(N)$. It is well-known that $K^{ce} = K$ for a submodule K of $S^{-1}M$ [8].

Lemma 2.20. *Let M be an R-module and $S = R\backslash W(M)$. If L is a non-zero submodule of $S^{-1}M$, then $\pi^{-1}(L)$ is a co-r-submodule of M.*

Proof. We have $\pi^{-1}(L) = L^c \neq 0$ because if $L^c = 0$, then we would have $L^{ce} = L = (0_{S^{-1}M})$, a contradiction. Put $\pi^{-1}(L) := N$. Let $a \in R\backslash W(M)$. We will show that $aN = N$. Let $x \in N$. Then $x = ay$ for some $y \in M$. We have $\pi(x) = \frac{x}{1} \in L$. Then $\frac{y}{1} = \frac{1}{a}\frac{ay}{1} \in L$.

This shows that $y \in \pi^{-1}(L) = N$. Thus $x = ay \in aN$ and so $aN = N$. Therefore, N is a *co-r*-submodule of M. □

Theorem 2.21. *Let M be an R-module. Let us consider the following two assertions.*

(1) *M is a co-r-Noetherian R-module.*

(2) $S^{-1}M$ *is a Noetherian* $S^{-1}R$*-module, where* $S = R\backslash W(M)$.
Then (1) *implies* (2). *If* M *is a reduced* R*-module, then* (2) *implies* (1).

Proof. (1) $\Rightarrow$ (2) Suppose that M is a *co*-r-Noetherian module and let

$$L_1 \subseteq L_2 \subseteq \cdots \subseteq L_n \subseteq \cdots$$

be an ascending chain of $S^{-1}R$-submodules of $S^{-1}M$. Consider the natural R-homomorphism π previously defined. By Lemma 2.20 we have the following ascending chain of *co*-r-submodules of M :

$$\pi^{-1}(L_1) \subseteq \pi^{-1}(L_2) \subseteq \cdots \subseteq \pi^{-1}(L_n) \subseteq \cdots$$

As M is a *co*-r-Noetherian module, then there exists $k \in \mathbb{Z}^+$ such that $\pi^{-1}(L_k) = \pi^{-1}(L_n)$ for all $n \geq k$. We fix an integer $n \geq k$ and we show that $L_n = L_k$. By the above chain, we have $L_k \subseteq L_n$.

For the converse, take $\frac{m}{s} \in L_n$, where $m \in M$, $s \in R\backslash W(M)$. Since $sM = M$, $m = sm'$ for some $m' \in M$. It follows that $\frac{m}{s} = \frac{sm'}{s} = \frac{m'}{1} = \pi(m') \in L_n$ and so $m' \in \pi^{-1}(L_n) = \pi^{-1}(L_k)$. Thus $\frac{m'}{1} = \frac{m}{s} \in L_k$. Hence, $L_n = L_k$. Thus $S^{-1}M$ is a Noetherian $S^{-1}R$-module.

(2) $\Rightarrow$ (1) Suppose that $S^{-1}M$ is a Noetherian $S^{-1}R$-module. Take any ascending chain of *co*-r-submodules $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n \subseteq \cdots$ of M. Then, by hypothesis, there is a $k \in \mathbb{Z}^+$ such that $S^{-1}N_k = S^{-1}N_n$ for all $n \geq k$.

We fix an integer $n \geq k$. Note that $N_k \subseteq N_n$. Let $m \in N_n$. Then there exists an element $s \in R\backslash W(M)$ such that $m \in (N_k :_M s)$. Since M is reduced, by Proposition 2.12, we have $N_k = (N_k :_M s)$. Hence $m \in N_k$ and, thus $N_n = N_k$. This shows that M is a *co*-r-Noetherian module. □

In the previous theorem, if we remove the condition "M is a reduced module", then (2) $\Rightarrow$ (1) may be wrong. See the following example derived from [13, Example 3.7].

Example 2.22. Let p be a prime number and consider $\mathbb{Z}$-module

$M = \mathbb{Z}(p^\infty) \times \mathbb{Q}$, where $\mathbb{Z}(p^\infty) = \{x \in \mathbb{Q}/\mathbb{Z} : x = (r/p^t) + \mathbb{Z}$ for some $r \in \mathbb{Z}$ and $t \in \mathbb{N} \cup \{0\}\}$ is the Prüfer group.

Then note that M is a divisible $\mathbb{Z}$-module, so by Example 2.28, every nonzero submodule of M is a *co*-r-submodule of M. Also note that

$$\left\langle \frac{1}{p} + \mathbb{Z} \right\rangle \times \mathbb{Q} \subsetneq \left\langle \frac{1}{p^2} + \mathbb{Z} \right\rangle \times \mathbb{Q} \subsetneq \cdots \subsetneq \left\langle \frac{1}{p^n} + \mathbb{Z} \right\rangle \times \mathbb{Q} \subsetneq \cdots$$

is a strictly ascending chain of *co*-r-submodules of M.

Thus, M is not a *co*-r-Noetherian R-module.

On the other hand, note that $\mathbb{Z}\backslash W(M) = \mathbb{Z}\backslash\{0\}$ and $S^{-1}\mathbb{Z}$-module $S^{-1}M$ is isomorphic to $\mathbb{Q}$-module $\mathbb{Q}$.

Thus, $S^{-1}\mathbb{Z}$-module $S^{-1}M$ is a Noetherian module.

Let S be a multiplicatively closed subset of R and M be an R-module. An increasing sequence $(N_n)_{n\in\mathbb{Z}^+}$ of submodules of M is called S-stationary if there exists a positive integer k and $s \in S$ such that for each $n \geq k$, $sN_n \subseteq N_k$ [14].

Proposition 2.23. *Let* M *be an* R*-module and* S *be a multiplicatively closed subset of* R *such that* $S \cap W(M) = \emptyset$. *If every ascending chain of co-*r*-submodules of* M *is* S*-stationary, then* M *is a co-*r*-Noetherian module.*

Proof. Let $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n \subseteq \cdots$ be an ascending chain of *co*-r-submodules of M. Then, by hypothesis, there exists an $s \in S$ and $k \in \mathbb{Z}^+$ such that $sN_n \subseteq N_k$ for all $n \geq k$. Since $S \subseteq R\backslash W(M)$, $sN_n = N_n$ by Proposition 2.11. Thus $N_n = N_k$ for all $n \geq k$ and this shows that M is a *co*-r-Noetherian module. □

Recall from [2] that a multiplicatively closed subset S of R is said to *satisfy the maximal multiple condition* if there exists an $s \in S$ such that $t|s$ for all $t \in S$.

Note that every finite multiplicatively closed set S of R and the set of all units in R are examples of multiplicatively closed sets satisfying the maximal multiple condition.

Theorem 2.24. *Let M be an R-module such that $S = R\backslash W(M)$ satisfies the maximal multiple condition. Then the following are equivalent:*
(1) M is a co-r-Noetherian R-module.
(2) $S^{-1}M$ is a Noetherian $S^{-1}R$-module.

Proof. (1) $\Rightarrow$ (2) It follows from Theorem 2.21.

(2) $\Rightarrow$ (1) Suppose that $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n \subseteq \cdots$ is an ascending chain of *co*-r-submodules of M. Then by assumption, there exists $k \in \mathbb{Z}^+$ such that $S^{-1}N_k = S^{-1}N_n$ for all $n \geq k$. Let $m \in N_n$. Then we have $\frac{m}{1} \in S^{-1}N_n = S^{-1}N_k$ and this yields $tm \in N_k$ for some $t \in R\backslash W(M)$.

Since $R\backslash W(M)$ satisfies maximal multiple condition, there exists $s \in S$ such that $t|s$ for each $t \in S$. Then we have $sm \in N_k$ and so $sN_n \subseteq N_k$. Thus every ascending chain of *co*-r-submodules of M is S-staionary. Then by Proposition 2.23, M is a *co*-r-Noetherian module. □

Note that in Example 2.22, $S = R\backslash W(M) = \mathbb{Z}\backslash\{0\}$ does not satisfy the maximal multiple condition. In Example 2.22, although $S^{-1}\mathbb{Z}$-module $S^{-1}M$ is a Noetherian module, M is not a co-r-Noetherian $\mathbb{Z}$-module. This shows that the condition "$S = R\backslash W(M)$ satisfies the maximal multiple condition" in the previous theorem is necessary.

Recall from [15] that a proper ideal I of R is called an r-ideal if $ab \in I$ and $ann_R(a) = 0$, then $b \in I$ for all $a, b \in R$. Recall from [3] that a ring R is said to be *r-Artinian* if it satisfies descending chain condition on r-ideals.

An R-module M is said to be a *comultiplication module* if each submodule N of M has the form $N = (0 :_M I)$ for some ideal I of R.

According to [5, Lemma 3.7], an R-module M is a comultiplication module if and only if for each submodule N of M, $N = (0 :_M ann_R(N))$.

Proposition 2.25. *Let R be an r-Artinian ring and M be a comultiplication R-module such that $W(M) \subseteq Z(R)$. Then M is a co-r-Noetherian module.*

Proof. Let $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n \subseteq \cdots$ be an ascending chain of *co*-r-submodules of M. Since $W(M) \subseteq Z(R)$, one can see that $ann_R(N_i)$ is an r-ideal of R for each i.

Since R is an r-Artinian ring, there exists $k \in \mathbb{Z}^+$ such that $ann_R(N_n) = ann_R(N_k)$ for all $n \geq k$. As M is a comultiplication module, this implies that $N_n = N_k$ for all $n \geq k$. Thus M is a *co*-r-Noetherian module. □

Proposition 2.26. *Let M be a co-r-Noetherian module. Then, for every co-r-submodule N of M and every family of co-r-submodules $\{K_i\}_{i\in\Lambda}$ of N, $\sum_{i\in\Lambda} K_i = N$ implies that $\sum_{i\in\Lambda'} K_i = N$ for some finite subset Λ' of Λ.*

Proof. Let N be a *co*-r-submodule of M and $\{K_i\}_{i\in\Lambda}$ be a family of *co*-r-submodules of N such that $\sum_{i\in\Lambda} K_i = N$. The fact that N is a *co*-r-submodule of M implies that K_i is a *co*-r-submodule of M for all $i \in \Lambda$. Now, set

$\mathfrak{F} = \{\sum_{i\in\Lambda'} K_i : \Lambda' \text{ is a finite subset of } \Lambda\}$.

By Proposition 2.10-(2) and the hypothesis, $\mathfrak{F}$ has a maximal element $N' = \sum_{i\in\Lambda'} K_i$. Let $j \in \Lambda\backslash\Lambda'$. Then $N' \subseteq N' + K_j$. The maximality of N' implies that $N' + K_j = N'$ and so $K_j \subseteq N'$. Thus $N \subseteq N'$ which implies that $N' = \sum_{i\in\Lambda'} K_i = N$. □

The following result can be found in [13, Theorem 3.9]. However, we shall give it with a different proof.

Theorem 2.27. *Let R be a ring satisfying ascending chain condition on r-ideals and M be a multiplication R-module with $W(M) \subseteq Z(R)$. Then M is a Noetherian R-module.*

Proof. Let $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n \subseteq \cdots$ be an ascending chain of *co*-r-submodules of M. Since M is multiplication module, we can write $N_i = (N_i :_R M)M$ for each i. Now, we will show that $(N_i :_R M)$ is an r-ideal of R. Let $ab \in (N_i :_R M)$ with $a \in R\backslash Z(R)$.

Then, by assumption, $aM = M$. It follows that $abM = bM \subseteq N_i$ and so $b \in (N_i :_R M)$. Thus $(N_1 :_R M) \subseteq (N_2 :_R M) \subseteq \cdots \subseteq (N_n :_R M) \subseteq \cdots$ is an ascending chain of r-ideals of R. By the hypothesis, there exists $m \in \mathbb{Z}^+$ such that $(N_i :_R M) = (N_m :_R M)$ for each $i \geq m$.

Since M is a multiplication module, this gives that $N_i = N_m$ for each $i \geq m$. Thus M is a *co*-r-Noetherian module. By Proposition 2.8, M is a Noetherian R-module. □

The condition "M is a multiplication module" can not be removed from Theorem 2.27. See the following example.

Example 2.28. Consider the $\mathbb{Z}$-module $\mathbb{Z}(p^\infty)$ where p is a prime number and $\mathbb{Z}(p^\infty)$ is the Prüfer group. Then clearly $\mathbb{Z}(p^\infty)$ is not a multiplication $\mathbb{Z}$-module and $W(\mathbb{Z}(p^\infty)) = 0 = Z(\mathbb{Z})$.

Also note that $\mathbb{Z}$ satisfies ascending chain condition on r-ideals since it is a domain. However, $\mathbb{Z}(p^\infty)$ is not a Noetherian $\mathbb{Z}$-module.

3. Conclusions

In this paper, we give some new properties and characterizations of co-r-submodules and co-r-Noetherian modules. We prove that every nonzero submodule of a finitely generated module is an *co*-r-submodule. We investigate when a submodule N of an R-module M contains a co-r-submodule. We give a characterization of reduced co-r-Noetherian modules via localization. We also investigate co-r-Noetherian property for multiplication and comultiplication modules.

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