CROSSED POLYMODULES AND FUNDAMENTAL RELATIONS

Murat Alp[1], Bijan Davvaz[2]

In this paper, we introduce the notion of crossed polymodule of polygroups and we give some of its properties. Our results extend the classical results of crossed modules to crossed polymodules. One of the main tools in the study of polygroups is the fundamental relations. These relations connect polygroups to groups, and on the other hand, introduce new important classes. So, we consider a crossed polymodule and by using the concept of fundamental relation, we obtain a crossed module. Moreover, we give a crossed polymodule morphism between them.

Keywords: action, crossed module, polygroup, crossed polymodule, fundamental relation.
MSC2010: 13D99, 20N20, 18D35.

1. Introduction

Crossed modules and its applications play very important roles in category theory, homotopy theory, homology and cohomology of groups, Algebra, K-theory etc. The term of crossed module was introduced by J. H. C. Whitehead in his work on combinatorial homotopy theory [24]. Loday explored and gave the new direction to the category of crossed modules by defining equivalent category of cat1-groups in his work [21]. Norrie gave a good example of crossed module such as Actor crossed module in [22] . Improving computer softwares and mathematical tools gave the new directions to the category of crossed modules, category of cat1-groups and actor crossed modules. All of these categories were calculated by GAP (Group, Algorithm and Programming) [17] share package XMod [5].

The polygroup theory is a natural generalization of the group theory. In a group the composition of two elements is an element, while in a polygroup the composition of two elements is a set. Polygroups have been applied in many area, such as geometry, lattices, combinatorics and color scheme. There exists a rich bibliography: publications appeared within 2012 can be found in "Polygroup Theory and Related Systems" by B. Davvaz [13]. This book contains the principal definitions endowed with examples and the basic results of the theory.

In this paper, we give a new application of crossed modules. This application is so important because we use the notion of polygroup to obtain crossed module. Therefore this application can be taught as a generalization of crossed module on groups. In the first two sections of the paper, we review some basic facts about

Professor, Department of Mathematics, Nigde University, Nigde, Turkey
e-mail: muratalp@nigde.edu.tr
Professor, Department of Mathematics, Yazd University, Yazd, Iran
e-mail: davvaz@yazd.ac.ir

crossed modules and polygroups that underlie the subsequent material. To define croosed polymodule, we need the notion of polygroup action. This is given in Section 4. In Section 5, we introduce the notion of crossed polymodules and give some of their properties. Finally, in Section 6, we consider a crossed polymodule and by using the concept of fundamental relation, we obtain a croosed module. Then, we give a crossed polymodule morphism between them.

2. **Crossed modules**

We mentioned above that crossed modules found too many application areas such as Brown and Mosa replaced algebras by algebroids and defined crossed module of algebroids. Actor crossed module of algebroid was defined by Alp. Pullback crossed module was defined by Brown and Wensley. Pullback crossed module of algebroids was defined by Alp. Pushout crossed module of profinite groups was presented by Korkes and Porter. Pushout cat1-profinite groups, Pushout cat1-Lie algebra and Pushout cat1- commutative algebra were presented by Alp and Gürmen. To get more idea about category of crossed module and category of cat1-groups we refer to read [1, 2, 3, 4, 7, 8, 18, 24] and more about we did not mention. To create a crossed module we need to define an action and a boundary homomorphism. In this section we recall the group action and definition of crossed module.

Definition 2.1. *Let G be a group and Ω be a non-empty set. A (left) group action is a binary operator $\tau : G \times \Omega \to \Omega$ that satisfies the following two axioms:*

(1) $\tau(gh, \omega) = \tau(g, \tau(h, \omega))$, *for all $g, h \in G$ and $\omega \in \Omega$,*
(2) $\tau(e, \omega) = \omega$, *for all $\omega \in \Omega$.*

For $\omega \in \Omega$ and $g \in G$, we write ${}^{g}\omega := \tau(g, \omega)$.

Definition 2.2. *A crossed module $X = (M, G, \partial, \tau)$ consists of groups M and G together with a homomorphism $\partial : M \to G$ and a (left) action $\tau : G \times M \to M$ on M, satisfying the conditions:*

(1) $\partial({}^{g}m) = g\partial(m)g^{-1}$, *for all $m \in M$ and $g \in G$,*
(2) ${}^{\partial(m)}m' = mm'm^{-1}$, *for all $m, m' \in M$.*

The standard examples of crossed modules are inclusion $M \hookrightarrow G$ of a normal subgroup M of G, the zero homomorphism $M \to G$ when M is a G-module, and any surjection $M \to G$ with center central. There is also an important topological example: if $F \to E \to B$ is a fibration sequence of pointed spaces, then the induced homomorphism $\pi_1 F \to \pi_1 E$ of fundamental groups is naturally a crossed module [6].

3. **Polygroups**

Suppose that H is a nonempty set and $\mathcal{P}^*(H)$ is the set of all nonempty subsets of H. Then, we can consider maps of the following type: $f_i : H \times H \to \mathcal{P}^*(H)$, where $i \in \{1, 2, \ldots, n\}$ and n is a positive integer. The maps f_i are called (*binary*) *hyperoperations*. For all x, y of H, $f_i(x, y)$ is called the (*binary*) *hyperproduct* of x and y. An algebraic system $(H, f_1, \ldots, f_n)$ is called a (*binary*) *hyperstructure*. Usually, $n = 1$ or $n = 2$. Under certain conditions, imposed to the maps f_i, we obtain the so-called semihypergroups, hypergroups, hyperrings or hyperfields.

Sometimes, external hyperoperations are considered, which are maps of the following type: $h : R \times H \to \mathcal{P}^*(H)$, where $R \neq H$. An example of a hyperstructure, endowed both with an internal hyperoperation and an external hyperoperation is the so-called hypermodule. Applications of hypergroups appear in special subclasses like polygroups that they were studied by Comer [9], also see [13, 14, 15]. Specially, Comer and Davvaz developed the algebraic theory for polygroups. A polygroup is a completely regular, reversible in itself multigroup.

Definition 3.1. [9] *A polygroup is a multi-valued system* $\mathcal{M} =< P, \circ, e, ^{-1} >$, *with* $e \in P$, $^{-1} : P \longrightarrow P$, $\circ : P \times P \longrightarrow \mathcal{P}^*(P)$, *where the following axioms hold for all* x, y, z *in* P*:*

(1) $(x \circ y) \circ z = x \circ (y \circ z)$,
(2) $e \circ x = x \circ e = x$,
(3) $x \in y \circ z$ *implies* $y \in x \circ z^{-1}$ *and* $z \in y^{-1} \circ x$.

In the above definition, $\mathcal{P}^*(P)$ is the set of all the non-empty subsets of P, and if $x \in P$ and A, B are non-empty subsets of P, then $A \circ B = \bigcup\limits_{a \in A, b \in B} a \circ b$, $x \circ B = \{x\} \circ B$ and $A \circ x = A \circ \{x\}$. The following elementary facts about polygroups follow easily from the axioms: $e \in x \circ x^{-1} \cap x^{-1} \circ x$, $e^{-1} = e$ and $(x^{-1})^{-1} = x$. In the rest of this section we present the facts about polygroups that underlie the subsequent material. For further discussion of polygroups, we refer to Davvaz's book [13]. Many important examples of polygroups are collected in [13] such as Double coset algebra, Prenowitz algebras, Conjugacy class polygroups, Character polygroups, Extension of polygroups, and Chromatic polygroups. Here, we recall one of them.

Example 3.1. *Suppose that* H *is a subgroup of a group* G. *Define a system* $G//H =< \{HgH \mid g \in G\}, *, H, ^{-I} >$, *where* $(HgH)^{-I} = Hg^{-1}H$ *and*

$$(Hg_1H) * (Hg_2H) = \{Hg_1hg_2H \mid h \in H\}.$$

The algebra of double cosets $G//H$ *is a polygroup.*

Lemma 3.1. *Every group is a polygroup.*

If K is a non-empty subset of P, then K is called a *subpolygroup* of P if $e \in K$ and $< K, \circ, e, ^{-1} >$ is a polygroup. The subpolygroup N of P is said to be *normal* in P if $a^{-1} \circ N \circ a \subseteq N$, for every $a \in P$. If N is a normal subpolygroup of P, the following elementary facts follows easily from the axioms: (1) $N \circ a = a \circ N$, for all $a \in P$; (2) $(N \circ a) \circ (N \circ b) = N \circ a \circ b$, for all $a, b \in P$; (3) $N \circ a = N \circ b$, for all $b \in N \circ a$. If N is a normal subpolygroup of P, then $< P/N, \bullet, N, ^{-I} >$ is a polygroup, where $N \circ a \bullet B \circ b = \{N \circ c \mid c \in N \circ a \circ b\}$ and $(N \circ a)^{-I} = N \circ a^{-1}$ [13]. There are several kinds of homomorphisms between polygroups [13]. In this paper, we apply only the notion of strong homomorphisms. Let $< P, \circ, e, ^{-1} >$ and $< P', \star, e, ^{-1} >$ be two polygroups. A mapping ϕ from P into P' is said to be a *strong homomorphism* if $\phi(e) = e$ and for all $a, b \in P$, $\phi(a \circ b) = \phi(a) \star \phi(b)$, for all $a, b \in P$. A strong homomorphism ϕ is said to be an *isomorphism* if ϕ is one to one and onto. Two polygroups P and P' are said to be *isomorphic* if there is an isomorphism from P onto P'. The defining condition for a strong homomorphism is also valid for sets, i.e., if A, B are nonempty subsets of P, then it follows that $f(A \circ B) = f(A) \star f(B)$.

If ϕ is a strong homomorphism from P into P', then the *kernel* of ϕ is defined as usual, i.e., $ker\phi = \{x \in P \mid \phi(x) = e\}$. It is easy to see that $ker\phi$ is a subpolygroup of P but in general is not normal. We say that ϕ is a *kernel-closed homomorphism* if $x \circ x^{-1} \subseteq ker\phi$, for all $x \in P$. If ϕ is a kernel-closed homomorphism, then $ker\phi$ is normal and $P/ker\phi \cong Im\phi$ [13].

4. **Polygroup action**

By using the concept of generalized permutation, in [11], Davvaz defined permutation polygroups and action of a polygroup on a set. For the definition of crossed polymodule, we need the notion of polygroup action.

Definition 4.1. [11] *Let $\mathcal{P} =< P, \circ, e, ^{-1} >$ be a polygroup and Ω be a non-empty set. A map $\alpha : P \times \Omega \to \mathcal{P}^*(\Omega)$ is called a (left) polygroup action on Ω if the following axioms hold:*

1. $\alpha(e, \omega) = \{\omega\} = \omega$, *for all* $\omega \in \Omega$,
2. $\alpha(h, \alpha(g, \omega)) = \bigcup\limits_{x \in h \circ g} \alpha(x, \omega)$, *for all* $g, h \in P$ *and* $\omega \in \Omega$,
3. $\bigcup\limits_{\omega \in \Omega} \alpha(g, \omega) = \Omega$, *for all* $g \in P$,
4. *for all* $g \in P$, $x \in \alpha(g, y) \Rightarrow y \in \alpha(g^{-1}, x)$.

From the second condition, we get $\bigcup\limits_{\omega_0 \in \alpha(g,\omega)} \alpha(h, \omega_0) = \bigcup\limits_{x \in h \circ g} \alpha(x, \omega)$. *For* $\omega \in \Omega$, *we write* ${}^{g}\omega := \alpha(g, \omega)$. *Therefore, we have*

1. ${}^{e}\omega = \omega$,
2. ${}^{h}({}^{g}\omega) = {}^{h \circ g}\omega$, *where* ${}^{g}A = \bigcup\limits_{a \in A} {}^{g}a$ *and* ${}^{B}\omega = \bigcup\limits_{b \in B} {}^{b}\omega$, *for all* $A \subseteq \Omega$ *and* $B \subseteq P$,
3. $\bigcup\limits_{\omega \in \Omega} {}^{g}\omega = \Omega$,
4. *for all* $g \in P$, $a \in {}^{g}b \Rightarrow b \in {}^{g^{-1}}a$.

Example 4.1. *Suppose that $< P, \circ, e, ^{-1} >$ is a polygroup. Then, P acts on itself if we define ${}^{g}x := x \circ g^{-1}$ or ${}^{g}x := g \circ x$, for all $x, g \in P$.*

Example 4.2. *Suppose that $< P, \circ, e, ^{-1} >$ is a polygroup. Then, P acts on itself by conjugation. Indeed, if we consider the map $\alpha : P \times P \to \mathcal{P}^*(P)$ by $\alpha(g, x) = {}^{g}x := g \circ x \circ g^{-1}$, then*

1. ${}^{e}x = x$,
2. ${}^{h}({}^{g}x) = {}^{h}(g \circ x \circ g^{-1}) = h \circ g \circ x \circ g^{-1} \circ h^{-1} = (h \circ g) \circ x \circ (h \circ g)^{-1} = \bigcup\limits_{b \in h \circ g} (b \circ x \circ b^{-1}) = \bigcup\limits_{b \in h \circ g} {}^{b}x = {}^{h \circ g}x$,
3. $\bigcup\limits_{x \in P} {}^{g}x = \bigcup\limits_{x \in P} g \circ x \circ g^{-1} = P$,
4. *if* $a \in {}^{g}b = g \circ b \circ g^{-1}$, *then* $g \in a \circ g \circ b^{-1}$ *and hence* $b^{-1} \in g^{-1} \circ a^{-1} \circ g$. *This implies that* $b \in g^{-1} \circ a \circ g$.

Example 4.3. *Suppose that $< P, \circ, e, ^{-1} >$ is a polygroup and N is a normal subpolygroup of P. Let Ω denote the set of all right cosets $N \circ x$ $(x \in P)$. We define ${}^{g}(N \circ x) = \{N \circ z \mid z \in N \circ x \circ g^{-1}\}$. Then, we have a (left) polygroup action on Ω. Indeed, we have*

(1) ${}^{e}(N \circ x) = N \circ x$,

(2) ${}^{h}\left({}^{g}(N \circ x)\right) = {}^{h}\left(\{N \circ z \mid z \in N \circ x \circ g^{-1}\}\right) = \bigcup_{z \in N \circ x \circ g^{-1}} \{N \circ t \mid t \in N \circ z \circ h^{-1}\} = \{N \circ t \mid t \in N \circ x \circ g^{-1} \circ h^{-1}\} = \bigcup_{a \in h \circ g} {}^{a}(N \circ x) = {}^{h \circ g}(N \circ x)$,

(3) *suppose that* $N \circ y \in \Omega$, *where* $y \in P$. *For every* $g \in P$, *there exists* $a \in P$ *such that* $y \in a \circ g^{-1}$. *This implies that* $y \in N \circ a \circ g^{-1}$ *and so* $N \circ y \in {}^{g}(N \circ a)$. *Therefore,* $N \circ y \in \bigcup_{N \circ x \in \Omega} {}^{g}(N \circ x)$,

(4) *we show that* $N \circ x \in {}^{g}(N \circ y) \Rightarrow N \circ y \in {}^{g^{-1}}(N \circ x)$. *Since* $N \circ x \in \{N \circ z \mid z \in N \circ y \circ g^{-1}\}$, *there exists* $z_0 \in N \circ y \circ g^{-1}$ *such that* $N \circ x = N \circ z_0$. *From* $z_0 \in N \circ y \circ g^{-1}$, *we obtain* $g^{-1} \in y^{-1} \circ N \circ z_0$. *Hence,* $y^{-1} \in g^{-1} \circ z_0^{-1} \circ N$ *and so* $y \in N \circ z_0 \circ g$. *Therefore,* $y \in N \circ x \circ g$ *which implies that* $N \circ y \in {}^{g^{-1}}(N \circ x)$.

5. **Crossed polymodules**

Now, in this section, we give the notion of crossed polymodule. To define a crossed module, we need the notion of polygroup action and boundary strong homomorphism.

Definition 5.1. *A crossed polymodule* $\mathcal{X} = (C, P, \partial, \alpha)$ *consists of polygroups* $< C, \star, e, ^{-1} >$ *and* $< P, \circ, e, ^{-1} >$ *together with a strong homomorphism* $\partial : C \to P$ *and a (left) action* $\alpha : P \times C \to \mathcal{P}^*(C)$ *on* C, *satisfying the conditions:*

(1) $\partial({}^{p}c) = p \circ \partial(c) \circ p^{-1}$, *for all* $c \in C$ *and* $p \in P$,

(2) ${}^{\partial(c)}c' = c \star c' \star c^{-1}$, *for all* $c, c' \in C$.

When we wish to emphasize the codomain P, we call $\mathcal{X}$ a *crossed* P*-polymodule.* The strong homomorphism $\partial : C \to P$ is called the *boundary*, while the polygroups C and P are referred to as, respectively, the *top polygroup* and the *base* of the crossed polymodule. A structure with the same data as a crossed polymodule and satisfying the first condition of Definition 5.1 but not the second condition is called a *precrossed polymodule.*

Example 5.1. *A conjugation crossed polymodule is an inclusion of a normal subpolygroup* N *of* P, *with action given by conjugation. In particular, for any polygroup* P *the identity map* $Id_P : P \to P$ *is a crossed polymodule with the action of* P *on itself by conjugation. Indeed, there are two canonical ways in which a polygroup* P *may be regarded as a crossed polymodule: via the identity map or via the inclusion of the trivial subpolygroup.*

Example 5.2. *If* C *is a* P*-polymodule, then there is a well defined action* α *of* P *on* C. *This together with the zero homomorphism yields a crossed polymodule* $(C, P, 0, \alpha)$.

Theorem 5.1. *Every crossed module is a crossed polymodule.*

Proof. By using Lemma 3.1, the proof is straightforward. □

The crossed polymodule axioms impose some restriction on the kernel and image of boundary strong homomorphism.

Proposition 5.1. *Let* $\mathcal{X} = (C, P, \partial, \alpha)$ *be a crossed polymodule. Then,* $\partial(C)$ *is a normal subpolygroup of* P.

Proof. Clearly, $\partial(C) = \{\partial(c) \mid c \in C\}$ is a subpolygroup of P. Suppose that $x \in \partial(C)$ and $p \in P$. Then, $x = \partial(c)$ for some $c \in C$ and $p \circ \partial(c) \circ p^{-1} = \partial({}^{p}c)$. Now, ${}^{p}c \subseteq C$, and so $p \circ \partial(c) \circ p^{-1} \subseteq \partial(C)$. □

Note that Proposition 5.1 depends only on the first condition of Definition 5.1. So, it is true for any precrossed polymodule.

The centralizer $\mathcal{C}(A)$ of a subset A of a polygroup P is the set of elements P which commute with all elements of P. In particular, $\mathcal{C}(P)$ is written $Z(P)$ and called the *center* of P and is abelian. Any subset of $Z(P)$ is called *central* in P.

Proposition 5.2. *Let $\mathcal{X} = (C, P, \partial, \alpha)$ be a crossed polymodule. Then, $ker\partial$ is central in C.*

Proof. It is easy to see that $ker\partial$ is a subpolygroup of C. Suppose that $c \in C$ and $k \in ker\partial$. We have ${}^{\partial(k)}c = k \star c \star k^{-1}$, but $\partial(k) = e$, hence ${}^{\partial(k)}c = {}^{e}c = c$. Therefore, $k \star c \star k^{-1}$ is singleton. Thus, $c \star k = k \star c \star k^{-1} \star k$. Since $e \in k^{-1} \star k$, we conclude that $k \star c \subseteq c \star k$. Similarly, $k^{-1} \star c = k^{-1} \star k \star c \star k^{-1}$. Since $e \in k^{-1} \star k$, we conclude that $c \star k^{-1} \subseteq k^{-1} \star c$, for all $k \in ker\partial$. Hence, $c \star k \subseteq k \star c$. □

Let $< P, \circ, e, ^{-1} >$ be a polygroup. We define the relation β^*_P as the smallest equivalence relation on P such that the quotient P/β^*_P, the set of all equivalence classes, is a group. In this case β^*_P is called the *fundamental equivalence relation* on P and P/β^*_P is called the *fundamental group*. The product $\odot$ in P/β^*_P is defined as follows: $\beta^*_P(x) \odot \beta^*_P(y) = \beta^*_P(z)$, for all $z \in \beta^*_P(x) \circ \beta^*(y)$. This relation is introduced by Koskas [19] and studied mainly by Corsini [10], Leoreanu-Fotea [20] and Freni [16] concerning hypergroups, Vougiouklis [23] concerning H_v-groups, Davvaz concerning polygroups [12], and many others. We consider the relation β_P as follows:

$$x \ \beta_P \ y \ \Leftrightarrow \ \text{there exist } z_1, \dots z_n \text{ such that } \{x, y\} \subseteq \circ \prod_{i=1}^{n} z_i.$$

Freni in [16] proved that for hypergroups $\beta = \beta^*$. Since polygroups are certain subclass of hypergroups, we have $\beta^*_P = \beta_P$. The kernel of the *canonical map* $\varphi_P : P \longrightarrow P/\beta^*_P$ is called the *core* of P and is denoted by ω_P. Here we also denote by ω_P the unit of P/β^*_P. It is easy to prove that the following statements: $\omega_P = \beta^*_P(e)$ and $\beta^*_P(x)^{-1} = \beta^*_P(x^{-1})$, for all $x \in P$.

Lemma 5.1. *ω_P is a subpolygroup of P.*

Proof. Suppose that $x, y \in \omega_P$ are arbitrary. Then, $\beta^*_P(x) = \beta^*_P(y) = \omega_P$. So, $\beta^*_P(x \circ y) = \beta^*_P(x) \odot \beta^*_P(y) = \omega_P$ and $\beta^*_P(x^{-1}) = \omega_P$. Therefore, we obtain $x \circ y \subseteq \omega_P$ and $x^{-1} \in \omega_P$. □

Lemma 5.2. *For every $p \in P$, $p \circ p^{-1} \subseteq \omega_p$.*

Proof. Suppose that $x \in p \circ p^{-1}$ is arbitrary. Since $e \in p \circ p^{-1}$, $\beta^*_P(e) = \beta^*_P(x)$ and so $\omega_P = \beta^*_P(x)$ which implies that $x \in \omega_P$. □

Throughout the paper, we denote the binary operations of the fundamental groups P/β^*_P and C/β^*_C by $\odot$ and $\otimes$, respectively.

Now, we can consider another notion of the kernel of a strong homomorphism of polygroups. Let $< P, \circ, e, ^{-1} >$ and $< C, \star, e, ^{-1} >$ be two polygroups and $\partial : C \rightarrow$

P be a strong homomorphism. The *core-kernel* of ∂ is defined by

$$ker^*\partial = \{x \in C \mid \partial(x) \in \omega_P\}.$$

Lemma 5.3. *$ker^*\partial$ is a normal subpolygroup of C.*

Proof. Suppose that $x, y \in ker^*\partial$ are arbitrary. Then, $\partial(x), \partial(y) \in \omega_P$. By Lemma 5.1, ω_P is a subpolygroup of P, so $\partial(x \circ y) = \partial(x) \circ \partial(y) \subseteq \omega_P$ and $\partial(x^{-1}) \in \omega_P$. Thus, $x \star y \subseteq ker^*\partial$ and $x^{-1} \in ker^*\partial$. Now, assume that $c \in C$ and $x \in ker^*\partial$ are arbitrary. We show that $\partial(c \star x \star c^{-1}) = \partial(c) \circ \partial(x) \circ \partial(c^{-1}) \subseteq \omega_P$. In order to show this, we have

$$\begin{aligned}
\beta_P^*(\partial(c) \circ \partial(x) \circ \partial(c^{-1})) &= \beta_P^*(\partial(c)) \odot \beta_P^*(\partial(x)) \odot \beta_P^*(\partial(c^{-1})) \\
&= \beta_P^*(\partial(c)) \odot \omega_P \odot \beta_P^*(\partial(c^{-1})) \\
&= \beta_P^*(\partial(c)) \odot \beta_P^*(\partial(c^{-1})) \\
&= \beta_P^*(\partial(c) \circ \partial(c^{-1})) \\
&= \beta_P^*(\partial(c \star c^{-1})) \\
&= \beta_P^*(\partial(e)) \text{ (since } e \in c \star c^{-1}) \\
&= \beta_P^*(e) = \omega_P,
\end{aligned}$$

which implies that $\partial(c) \circ \partial(x) \circ \partial(c^{-1}) \subseteq \omega_P$. □

Theorem 5.2. *Let $\mathcal{X} = (C, P, \partial, \alpha)$ be a crossed polymodule. Then, $ker^*\partial$ is a $P/\partial(C)$-polymodule.*

Proof. The action of P on C induces an action of P on $ker^*\partial$. It is sufficient to check that ${}^pk \subseteq ker^*\partial$ whenever $k \in ker^*\partial$. In order to show this, we have $\partial({}^pk) = p \circ \partial(k) \circ p^{-1}$. So,

$$\begin{aligned}
\beta_P^*(p \circ \partial(k) \circ p^{-1}) &= \beta_P^*(p) \odot \beta_P^*(\partial(k)) \odot \beta_P^*(p^{-1}) \\
&= \beta_P^*(p) \odot \omega_P \odot \beta_P^*(p^{-1}) \\
&= \beta_P^*(p) \odot \beta_P^*(p^{-1}) \\
&= \omega_P,
\end{aligned}$$

which implies that $p \circ \partial(k) \circ p^{-1} \subseteq \omega_P$. Hence, $\partial({}^pk) \subseteq \omega_P$ and so ${}^pk \subseteq ker^*\partial$. Therefore, $ker^*\partial$ is a P-polymodule. From Proposition 5.1, $\partial(C)$ is a normal subpolygroup of P, so $P/\partial(C)$ is defined. The action of P on $ker^*\partial$ induces an action of $P/\partial(C)$ on $ker^*\partial$ by ${}^{\varphi(p)}k = {}^pk$, for $p \in P$, $k \in ker^*\partial$ and $\varphi : P \to P/\partial(C)$ the cannonical map. This is well defined since if $q \in P$ with $\varphi(p) = \varphi(q)$, then ${}^qk = {}^{\varphi(q)}k = \{ {}^xk \mid x \in \varphi(q)\} = \{ {}^xk \mid x \in \varphi(p)\} = {}^{\varphi(p)}k = {}^pk$. □

Note that Theorem 5.2 uses the second condition of Definition 5.1, so it need not be true of a general precrossed polymodule.

Definition 5.2. *We say that $(A, B, \partial', \alpha')$ is a subcrossed polymodule of the crossed polymodule (C, P, ∂, α) if*

(1) *A is a subpolygroup of C, and B is a subpolygroup of P,*

(2) *∂' is the restriction of ∂ to A,*
(3) *the action of B on A is induced by the action of P on C.*

A subcrossed polymodule $(A, B, \partial', \alpha')$ of (C, P, ∂, α) is normal if

(1) *B is a normal subpolygroup of P,*
(2) *${}^{p}a \subseteq A$, for all $p \in P$ and $a \in A$,*
(3) *${}^{b}c \star c^{-1} \subseteq A$, for all $b \in B$ and $c \in C$.*

Definition 5.3. *Let $\mathcal{X} = (C, P, \partial, \alpha)$ and $\mathcal{X}' = (C', P', \partial', \alpha')$ be two crossed polymodules. A crossed polymodule morphism*

$$< \theta, \phi >: (C, P, \partial, \alpha) \to (C', P', \partial', \alpha')$$

is a commutative diagram of strong homomorphisms of polygroups

$$\begin{array}{ccc} C & \xrightarrow{\ \theta\ } & C' \\ {\scriptstyle \partial}\downarrow & & \downarrow{\scriptstyle \partial'} \\ P & \xrightarrow[\ \phi\ ]{} & P' \end{array}$$

such that for all $p \in P$ and $c \in C$, we have

$$\theta(\,{}^{p}c) = \,{}^{\phi(p)}\theta(c).$$

We say that $< \theta, \phi >$ is an isomorphism if θ and ϕ are both isomorphisms. Similarly, we can define monomorphism, epimorphism and automorphism of crossed polymodules.

6. **Crossed modules derived from crossed polymodules**

In this section, we consider a crossed polymodule and by using the concept of fundamental relation, we obtain a croosed module. Then, we give a crossed polymodule morphism between them.

Proposition 6.1. *Let $< C, \star, e,^{-1} >$ and $< P, \circ, e,^{-1} >$ be two polygroups and let $\partial : C \to P$ be a strong homomorphism. Then, ∂ induces a group homomorphism $\mathcal{D} : C/\beta^*_C \to P/\beta^*_P$ by setting*

$$\mathcal{D}(\beta^*_C(c)) = \beta^*_P(\partial(c)), \text{ forall } c \in C.$$

Proof. First, we prove that $\mathcal{D}$ is well defined. Suppose that $\beta^*_C(c_1) = \beta^*_C(c_2)$. Then, there exist $a_1, \ldots, a_n$ such that $\{c_1, c_2\} \subseteq \star \prod_{i=1}^{n} a_i$. So,

$$\{\partial(c_1), \partial(c_2)\} \subseteq \partial\left(\star \prod_{i=1}^{n} a_i\right) = \circ \prod_{i=1}^{n} \partial(a_i).$$

Hence, $\partial(c_1)\ \beta^*_P\ \partial(c_2)$, which implies that $\mathcal{D}\left(\beta^*_C(c_1)\right) = \mathcal{D}\left(\beta^*_C(c_2)\right)$. Now, we have

$$\begin{aligned} \mathcal{D}(\beta^*_C(c_1) \otimes \beta^*_C(c_2)) &= \mathcal{D}(\beta^*_C(c_1 \star c_2)) = \beta^*_P(\partial(c_1 \star c_2)) \\ &= \beta^*_P(\partial(c_1) \circ \partial(c_2)) = \beta^*_P(\partial(c_1)) \odot \beta^*_P(\partial(c_2)) \\ &= \mathcal{D}(\beta^*_C(c_1)) \odot \mathcal{D}(\beta^*_C(c_2)). \end{aligned}$$

□

We say the action of P on C is *productive*, if for all $c \in C$ and $p \in P$ there exist $c_1, \ldots, c_n$ in C such that ${}^{p}c = c_1 \star \ldots \star c_n$.

Example 6.1. *The action defined in Examples 4.1 and 4.2 are productive.*

Let $< C, \star, e,^{-1} >$ and $< P, \circ, e,^{-1} >$ be two polygroups and let $\alpha : P \times C \to \mathcal{P}^*(C)$ be a productive action on C. We define the map $\psi : P/\beta^*_P \times P/\beta^*_C \to \mathcal{P}^*(P/\beta^*_C)$ as usual manner:

$$\psi(\beta^*_P(p), \beta^*_C(c)) = \{\beta^*_C(x) \mid x \in \bigcup_{\substack{y \in \beta^*_C(c) \\ z \in \beta^*_P(p)}} {}^{z}y\}.$$

By definition of β^*_C, since the action of P on C is productive, we conclude that $\psi(\beta^*_P(p), \beta^*_C(c)$ is singleton, i.e., we have

$$\psi : P/\beta^*_P \times P/\beta^*_C \to P/\beta^*_C,$$
$$\psi(\beta^*_P(p), \beta^*_C(c)) = \beta^*_C(x), \text{ for all } x \in \bigcup_{\substack{y \in \beta^*_C(c) \\ z \in \beta^*_P(p)}} {}^{z}y.$$

We denote $\psi(\beta^*_P(p), \beta^*_C(c)) = {}^{[\beta^*_P(p)]}[\beta^*_C(c)]$.

Proposition 6.2. *Let $< C, \star, e,^{-1} >$ and $< P, \circ, e,^{-1} >$ be two polygroups and let $\alpha : P \times C \to \mathcal{P}^*(C)$ be a productive action on C. Then, ψ is an action of the group P/β^*_P on the group P/β^*_C.*

Proof. Suppose that $g, h \in P$ and $c \in C$. Then, we have

$$\begin{aligned} \psi(\beta^*_P(h) \odot \beta^*_P(g), \beta^*_C(c)) &= \psi(\beta^*_P(h \circ g), \beta^*_C(c)) \\ &= {}^{[\beta^*_P(h\circ g)]}[\beta^*_C(c)], \end{aligned}$$

and

$$\begin{aligned} \psi(\beta^*_P(h), \psi(\beta^*_P(g), \beta^*_C(c)) &= \psi\left(\beta^*_P(h), {}^{[\beta^*_P(g)]}[\beta^*_C(c)]\right) \\ &= {}^{[\beta^*_P(h)]}\left({}^{[\beta^*_P(g)]}[\beta^*_C(c)]\right). \end{aligned}$$

By condition (2) of Definition 4.1, we have ${}^{h}({}^{g}c) = {}^{h\circ g}c$. Now, it is not difficult to see that

$${}^{[\beta^*_P(h\circ g)]}[\beta^*_C(c)] = {}^{[\beta^*_P(h)]}\left({}^{[\beta^*_P(g)]}[\beta^*_C(c)]\right).$$

□

Theorem 6.1. *Let $\mathcal{X} = (C, P, \partial, \alpha)$ be a crosed polymodule such that the action of P on C is productive. Then, $X = (C/\beta^*_C, P/\beta^*_P, \mathcal{D}, \psi)$ is a crossed module.*

Proof. By Propositions 6.1 and 6.2, it is enough to show that the conditions of Definition 2.2 hold. Suppose that $c \in C$ and $p \in P$ are arbitrary. Then, we have

$$\begin{aligned} {}^{[\beta^*_P(p)]}\mathcal{D}\left([\beta^*_C(c)]\right) &= \mathcal{D}\left([\beta^*_C(z)\right), \text{ for all } z \in {}^{p}c \\ &= \beta^*_P(\partial(z)), \text{ for all } z \in {}^{p}c \\ &= \beta^*_P(\partial({}^{p}c)) \\ &= \beta^*_P(p \circ \partial(c) \circ p^{-1}) \\ &= \beta^*_P(p) \odot \beta^*_P(\partial(c)) \odot \beta^*_P(p^{-1}) \\ &= \beta^*_P(p) \odot \mathcal{D}(\beta^*_C(c)) \odot (\beta^*_P(p))^{-1}. \end{aligned}$$

So, the first condition of Definition 2.2 holds. For the second condition, suppose that $c, c' \in C$ are arbitrary. Then, we have

$$\begin{aligned} {}^{[\mathcal{D}(\beta^*_C(c))]}[\beta^*_C(c')] &= {}^{[\beta^*_P(\partial(c))]}[\beta^*_C(c')] \\ &= \beta^*_C(z), \text{ for all } z \in {}^{\partial(c)}c' \\ &= \beta^*_C(z), \text{ for all } z \in c \star c' \star c \\ &= \beta^*_C(c \star c' \star c) \\ &= \beta^*_C(c) \otimes \beta^*_C(c') \otimes \beta^*_C(c). \end{aligned}$$

□

Theorem 6.2. *Let $\mathcal{X} = (C, P, \partial, \alpha)$ be a crossed polymodule, φ_C and φ_P be canonical maps. Then, $< \varphi_C, \varphi_P >$ is a crossed polymodules morphism.*

Proof. Note that according to Theorem 5.1, we can consider $(C/\beta^*_C, P/\beta^*_P, \mathcal{D}, \psi)$ as a crossed polymodule. We show that the following diagram is commutative.

$$\begin{array}{ccc} C & \xrightarrow{\varphi_C} & C/\beta^*_C \\ \partial\big\downarrow & & \big\downarrow\mathcal{D} \\ P & \xrightarrow[\varphi_P]{} & P/\beta^*_P \end{array}$$

Indeed, we have $\mathcal{D}\varphi_C(c) = \mathcal{D}\left(\beta^*_C(c)\right) = \beta^*_P(\partial(c)) = \varphi_P\partial(c)$, for all $c \in C$. Moreover,

$$\varphi_C({}^{p}c) = \beta^*_C({}^{p}c) = {}^{[\beta^*_P(p)]}[\beta^*_C(c)] = {}^{\varphi_P(p)}\varphi_C(c),$$

for all $c \in C$ and $p \in P$. Therefore, $< \varphi_C, \varphi_P >$ is a crossed polymodules morphism.

□

The following example gives us another crossed module structure on the fundamental groups.

Example 6.2. *Suppose that $< P, \circ, e, ^{-1} >$ is any polygroup. Then, P/β^*_P is a group. Suppose that $Aut\,(P/\beta^*_P)$ its group of automorphisms. There is an obvious action α of $Aut\,(P/\beta^*_P)$ on P/β^*_P, and a group homomorphism $\partial : P/\beta^*_P \to Aut\,(P/\beta^*_P)$ sending each $\beta^*_P(p) \in P/\beta^*_P$ to the inner automorphism of conjugation by $\beta^*_P(p)$. These together form a crossed module $(P/\beta^*_P, Aut\,(P/\beta^*_P)\,, \partial, \alpha)$.*

REFERENCES

[1] *M. Alp*, Actor of Crossed modules of Algebroids, Proc. 16th Int. Conf. Jangjeon Math. Soc., **16** (2005), 6-15.

[2] *M. Alp and Ö. Gürmen*, Pushouts of profinite crossed modules and cat1-profinite groups, Turkish Journal of Mathematics, **27** (2003), 539-548.

[3] *M. Alp*, Pullback Crossed modules of Algebroids, Iranian J. Sci. & Tech., Transaction A, **32(A3)** (2008), 175-181.

[4] *M. Alp*, Pullbacks of profinite crossed modules and CAT^1-profinite groups, Algebras Groups Geom., **25(2)** (2008), 215-221.

[5] *M. Alp and C. D. Wensley*, XMOD, Crossed modules and cat1-groups in GAP, version 2.19, (2012) 1-49.

[6] *R. Brown and N.D. Gilbert*, Algebraic models of 3-types and authomorphism structures for crossed modules, Proc. London Math. Soc. (3) **59** (1989), 51-73.

[7] *R. Brown and G. H. Mosa*, Double categories, R-categories and crossed modules, U. C. N. W Maths Preprint **88. 11** (1988), 1-18.

[8] *R. Brown and C. D. Wensley*, On finite induced crossed modules, and the homotopy 2-type of mapping cones, Theory Appl. Categ., **1(3)** (1995), 54-71.

[9] *S.D. Comer*, Polygroups derived from cogroups, J. Algebra, **89** (1984), 397-405.

[10] *P. Corsini*, Prolegomena of Hypergroup Theory, Second edition, Aviain editore, 1993.

[11] *B. Davvaz*, On polygroups and permutation polygroups, Math. Balkanica (N.S.), **14(1-2)** (2000), 41-58.

[12] *B. Davvaz*, Isomorphism theorems of polygroups, Bulletin of the Malaysian Mathematical Sciences Society (2), **33(3)** (2010), 385-392.

[13] *B. Davvaz*, Polygroup Theory and Related Systems, World Sci. Publ., 2013.

[14] *B. Davvaz*, Applications of the γ^*-relation to polygroups, Comm. Algebra, **35** (2007), 2698-2706.

[15] *B. Davvaz*, A survey on polygroups and their properties, Proceedings of the International Conference on Algebra 2010, 148–156, World Sci. Publ., Hackensack, NJ, 2012.

[16] *D. Freni*, A note on the core of a hypergroup and the transitive closure β^* of β, Riv. Mat. Pura Appl., **8** (1991), 153-156.

[17] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.7.6, 2014, (`http://www.gap-system.org`).

[18] *F.J. Korkes and T. Porter*, Profinite crossed modules, U.C.N.W. Pure Mathematics preprint, **86(11)** (1986).

[19] *M. Koskas*, Groupoids, demi-groupes et hypergroupes, J. Math. Pures Appl., **49** (1970), 155192.

[20] *V. Leoreanu-Fotea*, The heart of some important classes of hypergroups, Pure Math. Appl., **9** (1998), 351-360.

[21] *J.L. Loday*, Spaces with finitely many non-trivial homotopy groups, J. App. Algebra, **24** (1982), 179-202.

[22] *K.J. Norrie*, Actions and automorphisms of crossed modules, Bull. Soc. Math. France, **118** (1990), 129-146.

[23] *T. Vougiouklis*, Hyperstructures and their Representations, Hadronic Press, Inc, 115, Palm Harber, USA, 1994.

[24] *J.H.C. Whitehead*, Combinatorial homotopy II, Bull. Amer. Math. Soc., **55** (1949), 453-496.