U.P.B. Sci. Bull., Series A, Vol. 72, Iss. 1, 2010 ISSN 1223-7027

SOME NEW TYPES OF VERTICAL 2-JETS ON THE TANGENT BUNDLE OF A FINSLER MANIFOLD

Adelina MANEA *

În fibratul vertical a unei varietăţi Finsler definim o bază adaptată foliaţiei Liouville $F_{L'}$. Definim fibratul 2-jeturilor verticale $J^{v,2}(TM^0)$, precum şi fibratele 2-jeturilor verticale leafwise, transversale şi mixte în raport cu $F_{L'}$. Principalul rezultat al lucrării este existenţa unui difeomorfism între spaţiul total al fibratului $J^{v,2}(TM^0)$ şi cel al produsului fibrat al fibratelor 2-jeturilor verticale leafwise, transversale şi mixte.

We provide a basis of the vertical bundle of a Finsler manifold, adapted to the Liouville foliation $F_{L'}$. We define the vertical 2-jet bundle $J^{v,2}(TM^0)$ and the leafwise, transversal and mixed vertical 2-jet bundles with respect to $F_{L'}$. The main result is the existence of a diffeomorphism between the total space $J^{v,2}(TM^0)$ and the total space of the fiber product of the bundles of the leafwise, transversal and mixed vertical 2-jet bundles.

Keywords: Finsler manifold, foliation, leafwise, jets.
AMS Subject Classification: 53C12, 58A20, 53C60.

1. Introduction

The language of jet bundles, [1], [2], is an interest point for the geometries and the mathematical physicists. The foliations on manifolds, the Finsler and Lagrange manifolds are important in theoretical physics, too. We used [3] to learn about Finsler manifolds and [4], [5] for initiating in domain of foliated manifolds.

*Lect., Dept. of Algebra, Geometry and Equations of *Transilvania* Univ. of Brasov, e-mail: amanea28@yahoo.com

In [6] there are introduced two foliations on the tangent bundle of a Finsler manifold (M, F). These foliations are: F_V the foliation by the fibers of the bundle TM^0 and $F_{L'}$, the foliation by the c-indicatrices of (M, F), which is a subfoliation of the first one. The structural bundle with respect to F_V is the vertical tangent bundle VTM^0. We remind some notions about Finsler manifolds and we present these foliations in the first section of the paper, following [6], [3]. We provide a basis of $\Gamma(VTM^0)$, adapted to the foliation $F_{L'}$ and present some useful properties of the vectors of this basis, in the second section of this paper. The last section is dedicated to vertical 2-jets. The leafwise and transversal 2-jets on a foliated manifolds were introduced in [7], [8]. In a recent work, [9], we consider the leafwise 2-jets of (TM^0, F_V) and we related them to the cohomology of TM^0. In the case of the Berwald-Moore metric F, an one-dimensional cohomology group of TM^0 is expressed by the vertical 2-jets. Another types of 2-jets, namely the transversal and mixed 2-jets, were introduced in [10]. They could be related with some gravitational fields, [11]. We define here the vertical 2-jet bundle $J^{v,2}(TM^0)$ being the leafwise 2-jet bundle with respect to F_V. We also define the leafwise, transversal and mixed vertical 2-jet bundles with respect to the second foliation, $F_{L'}$: $J^{l,v,}(TM^0)$, $J^{t,v}(TM^0)$, $J^{t,l,v}(TM^0)$. The main result is the existence of a diffeomorphism between the total space $J^{v,2}(TM^0)$ and the total space of the fiber product $J^{l,v}(TM^0) \times_{TM^0} J^{t,v}(TM^0) \times_{TM^0} J^{t,l,v}(TM^0)$.

2. The vertical and the fundamental foliations on the tangent bundle of a Finsler manifold

In this section we follow [3], [6] to present some facto about Finsler manifolds. Let M be an n-dimensional paracompact manifold and TM its tangent bundle. If E is a bundle over M, we denote by $\Gamma(E)$ the module of its smooth sections. If $(x^i)_{i=\overline{1,n}}$ are the local coordinates on M and $(y^i)_{i=\overline{1,n}}$ are the fiber coordinates, then $(x^i, y^i)_{i=\overline{1,n}}$ are the local coordinates on TM. It is well-known that the transformations of local coordinates on TM are

$$\widetilde{x}^{i_1} = \widetilde{x}^{i_1}(x^1, \ldots, x^n), \quad \widetilde{y}^{i_1} = \frac{\partial \widetilde{x}^{i_1}}{\partial x^i} y^i. \tag{1}$$

In this paper the indices take the values $i, j, k, i_1, j_1, ... = \overline{1, n}$. We use the Einstein convention for summation when an index is repeated up and down.

Now, we assume that M is a Finsler manifold, so there is a function $F : TM \to [0, \infty)$ which vanishes only on the zero section on TM and is smooth on $TM^0 = TM - \{0\}$, such that it is 1-homogeneous on y,

$$F(x, ky) = |k|\, F(x, y), \quad \forall k \in \mathbf{R},$$

and the matrix

$$[g_{ij}(x, y)]_{i,j} = \left[\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}\right]_{i,j}, \tag{2}$$

is positive definite at any point of the domain of the local chart.

Let F_V be the foliation on TM^0 determinated by the fibers of $\pi : TM^0 \to M$, called the *vertical foliation*. The leaves are exactly $\{T_x M^0\}_{x \in M}$. The local coordinates $(x^i, y^i)_{i=\overline{1,n}}$ are adapted to this foliation, which means that the leaves are locally defined by $x^i = constant$. The sections of the *structural* (also called vertical) bundle VTM^0 are locally spanned by $\{\frac{\partial}{\partial y^i}\}_i$. We consider the functions

$$G^i = \frac{1}{4} g^{ik}\left(\frac{\partial^2 F^2}{\partial y^k \partial x^h} y^h - \frac{\partial F^2}{\partial x^k}\right), \quad G_i^j = \frac{\partial G^j}{\partial y^i}, \tag{3}$$

where the matrix $(g^{ik})_{i,k=\overline{1,n}}$ is the inverse of the matrix (2).

There is a complementary bundle HTM^0 to VTM^0 in TTM^0, called the *transversal* (or horizontal) bundle of the foliation, whose sections are locally spanned by

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - G_i^j \frac{\partial}{\partial y^j}, \quad i = \overline{1, n}. \tag{4}$$

We have the decomposition

$$TTM^0 = HTM^0 \oplus VTM^0, \tag{5}$$

so every vector field X on TM^0 has a vertical part VX and an horizontal part HX. The relations

$$G\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = G\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = g_{ij}, \quad G\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right) = 0, \tag{6}$$

define a Riemannian metric on TM^0, called the *Sasaki-Finsler metric*.

In the following we consider the globally defined vertical Liouville vector field on TM^0:

$$Z = y^i \frac{\partial}{\partial y^i}.$$

From the Euler theorem on positively homogeneous functions we have, [6],

$$F^2(x,y) = y^i y^j g_{ij}(x,y), \quad \frac{\partial F}{\partial y^k} = \frac{1}{F} y^i g_{ki}, \quad y^i \frac{\partial g_{ij}}{\partial y^k} = 0, \quad \forall k = \overline{1,n}. \quad (7)$$

From the above equalities we have

$$G(Z,Z) = F^2. \quad (8)$$

The line distribution $L = span\{Z\}$ is called *the vertical Liouville distribution* on TM^0. Let L' and $L^\perp$ be the complementary orthogonal distributions to L in VTM^0 and TTM^0, respectively. It is proved, [6] that both the distributions L', $L^\perp$ are integrable. Moreover, the foliations determined by these integrable distributions can be defined by means of the fundamental function F, [6], (p.140-141):

Theorem 1 *a) The fundamental foliation F_F determined by the level hypersurfaces of the fundamental function F of the Finsler manifold (M,F) is exactly the foliation determined by the integrable distribution $L^\perp$.*

b) The vertical Liouville vector field Z is orthogonal to F_F.

c) The leaves of the foliation $F_{L'}$ determined by the integrable distribution L' are the c-indicatrices of (M,F):

$$I_{x_0}M(c) : x = x_0, \quad F(x_0,y) = c, \forall y \in T_{x_0}M^0,$$

where $x_0 = (x_0^i)$ is a fixed point on M. The foliation $F_{L'}$ is called the Liouville foliation on TM^0.

d) The foliation $F_{L'}$ is a subfoliation of the vertical foliation F_V.

From the above considerations, we have the decomposition

$$TTM^0 = HTM^0 \oplus L' \oplus L.$$

3. An adapted basis on the vertical vector fields space

We know that the basis $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$ is a local basis on TTM^0, adapted to the vertical foliation F_V. Searching for a basis on $\Gamma(VTM^0)$, adapted to the foliation $F_{L'}$, we consider the following vertical vector fields:

$$X_k = \frac{\partial}{\partial y^k} - t_k Z, \quad k = \overline{1,n}, \quad (9)$$

where the functions t_k are defined by the conditions

$$G(X_k, Z) = 0, \forall k = \overline{1, n}. \tag{10}$$

The above conditions become

$$G\left(\frac{\partial}{\partial y^k}, y^i \frac{\partial}{\partial y^i}\right) - t_k G(Z, Z) = 0,$$

which give by relations (6), (7) and (8),

$$t_k = \frac{1}{F^2} y^i g_{ki} = \frac{1}{F} \frac{\partial F}{\partial y^k}, \quad \forall k = \overline{1, n}. \tag{11}$$

If $(\tilde{U}, (\tilde{x}^{i_1}, \tilde{y}^{i_1}))$ is another local chart on TM^0, in $U \cap \tilde{U} \neq \oslash$ we have:

$$\tilde{t}_{k_1} = \frac{1}{F^2} \tilde{y}^{i_1} \tilde{g}_{i_1 k_1} = \frac{1}{F^2} \frac{\partial \tilde{x}^{i_1}}{\partial x^i} y^i \frac{\partial x^k}{\partial \tilde{x}^{k_1}} \frac{\partial x^i}{\partial \tilde{x}^{i_1}} g_{ki} = \frac{\partial x^k}{\partial \tilde{x}^{k_1}} t_k,$$

so we obtain from (9) the following changing rule for the vector fields:

$$\tilde{X}_{k_1} = \frac{\partial x^k}{\partial \tilde{x}^{k_1}} X_k, \quad \forall k = \overline{1, n}. \tag{12}$$

Proposition 1 *The functions $\{t_k\}_{k=\overline{1,n}}$ defined by (11) are satisfying:*

$$a) \quad y^i t_i = 1; \tag{13}$$

$$b) \quad y^i X_i = 0; \tag{14}$$

$$c) \quad \frac{\partial t_r}{\partial y^k} = -2 t_k t_r + \frac{1}{F^2} g_{kr}, \quad Z t_k = -t_k, \quad \forall k, r = \overline{1, n}; \tag{15}$$

$$d) \quad y^j \frac{\partial t_j}{\partial y^i} = -t_i, \quad \forall i = \overline{1, n}; \tag{16}$$

$$e) \quad y^i (Z t_i) = -1, \quad y^i (Z X_i) = 0. \tag{17}$$

Proof: a) Taking into account relations (7) and (11), we have:

$$y^i t_i = \frac{1}{F^2} y^i y^j g_{ij} = \frac{1}{F^2} F^2 = 1.$$

b) We can calculate using the definition of the vector field Z and the relation (13):

$$y^i X_i = y^i \left(\frac{\partial}{\partial y^i} - t_i Z\right) = y^i \frac{\partial}{\partial y^i} - y^i t_i Z = Z - Z = 0.$$

c) From the relation (11) we obtain

$$\frac{\partial t_r}{\partial y^k} = \frac{\partial}{\partial y^k}\left(\frac{1}{F^2}y^i g_{ri}\right) = -\frac{2}{F^3}\frac{\partial F}{\partial y^k}y^i g_{ri} + \frac{1}{F^2}g_{kr} + \frac{1}{F^2}y^i\frac{\partial g_{ri}}{\partial y^k},$$

and using again (11) and the last equality from (7), it results

$$\frac{\partial t_r}{\partial y^k} = -2t_k t_r + \frac{1}{F^2}g_{kr}.$$

Now, from the above relation and using also (11) and (13),

$$Zt_r = y^i\frac{\partial t_r}{\partial y^i} = y^i(-2t_i t_r + \frac{1}{F^2}g_{ir}) = -2t_r y^i t_i + \frac{1}{F^2}y^i g_{ri} = -2t_r + t_r = -t_r.$$

d) Taking into account the definition (11) of the functions $\{t_k\}$ and the relations (13), (15), we have

$$y^j\frac{\partial t_j}{\partial y^i} = y^j(-2t_i t_j + \frac{1}{F^2}g_{ij}) = -2t_i y^j t_j + \frac{1}{F^2}y^j g_{ij} = -2t_i + t_i = -t_i.$$

e) Using the second relation from (15) and (13), we obtain

$$y^i(Zt_i) = y^i(-t_i) = -1.$$

Now, using the definition (9) of the vector fields X_i, the expression of Z and (13), it results:

$$y^i(ZX_i) = y^i(Z\frac{\partial}{\partial y^i} - Zt_i Z - t_i Z^2) = y^i y^j\frac{\partial^2}{\partial y^i \partial y^j} + y^i t_i Z - y^i t_i Z^2 =$$

$$= y^i y^j\frac{\partial^2}{\partial y^i \partial y^j} + y^i\frac{\partial}{\partial y^i} - (y^i\frac{\partial}{\partial y^i})(y^j\frac{\partial}{\partial y^j}) = 0.$$

Remark 1 *The vector fields $\{X_1, X_2, ..., X_n\}$ are $n-$vectors from VTM^0, orthogonal on the Liouville vector also from VTM^0, so they are linear dependent. The relation (14) proves also that a linear combination of these vectors, with non-zero coefficients, vanishes. Moreover, the rank of the matrix*

$$A = \begin{pmatrix} 1-t_1y^1 & -t_2y^1 & ... & -t_ny^1 \\ -t_1y^2 & 1-t_2y^2 & ... & -t_ny^2 \\ ... & ... & ... & ... \\ -t_1y^n & -t_2y^n & ... & 1-t_ny^n \end{pmatrix}$$

is $n-1$, because its $(n-1)\times(n-1)$ left corner minor is non-zero, since the local coordinate function y^n doesn't vanish. We can suppose this fact. It is also known that the functions from (11) are non-zero. This minor is equal to $y^n t_n$. For calculation we used the Proposition 1. So, the first $n-1$ vectors $\{X_k\}$ are linear independent and, from (15),

$$X_n = -\frac{1}{y^n}\sum_{k=1}^{n-1} y^k X_k. \tag{18}$$

Returning now to (12), we can say that $\tilde{X}_{k_1}$ depends only on $\{X_1, X_2, ..., X_{n-1}\}$. For the simplicity of calculations, we shall use (12) in the following, keeping in mind (18).

Theorem 2 *The set $\{X_1, X_2, ..., X_{n-1}, Z\}$ of vector fields is a basis of $\Gamma(VTM^0)$, adapted to the Liouville foliation.*

Proof: The matrix of change from the natural basis $\{\frac{\partial}{\partial y^i}\}_{i=\overline{1,n}}$ to $\{X_1, X_2, ..., X_{n-1}, Z\}$ is

$$\begin{pmatrix} 1-t_1y^1 & -t_2y^1 & ... & -t_{n-1}y^1 & y^1 \\ -t_1y^2 & 1-t_2y^2 & ... & -t_{n-1}y^2 & y^2 \\ ... & ... & ... & ... & ... \\ -t_1y^{n-1} & -t_2y^{n-1} & ... & 1-t_{n-1}y^{n-1} & y^{n-1} \\ -t_1y^n & -t_2y^n- & ... & -t_{n-1}y^n & y^n \end{pmatrix}$$

We multiply the last column with t_k and add to the k-column, for $k=\overline{1,n}$ and, using Proposition 1, we obtain the determinant of the above matrix equal to y^n, which is supposed to be non-zero everywhere. So, the system of vectors $\{X_1, X_2, ..., X_{n-1}, Z\}$ is linear independent, hence it is a basis. Moreover, it is an adapted basis to the foliation $F_{L'}$, because $L = span\{Z\}$ and $L' = span\{X_1, X_2, ..., X_{n-1}\}$.

Proposition 2 *For every $k, r = \overline{1,n}$, we have*

$$X_kX_r - X_rX_k = t_kX_r - t_rX_k.$$

Proof: We can compute using Proposition 1

$$X_kX_r = \left(\frac{\partial}{\partial y^k} - t_kZ\right)\left(\frac{\partial}{\partial y^r} - t_rZ\right) = \tag{19}$$

$$= \frac{\partial^2}{\partial y^k \partial y^r} - t_k y^i \frac{\partial^2}{\partial y^i \partial y^r} - t_r y^i \frac{\partial^2}{\partial y^i \partial y^k} + t_k t_r (Z + Z^2) - \frac{1}{F^2} g_{kr} Z - t_r \frac{\partial}{\partial y^k}.$$

Then, the Poisson bracket of the vector fields X_k, X_r is

$$X_k X_r - X_r X_k = t_k \frac{\partial}{\partial y^r} - t_r \frac{\partial}{\partial y^k},$$

or, equivalently,

$$X_k X_r - X_r X_k = t_k \left(\frac{\partial}{\partial y^r} - t_r Z \right) - t_r \left(\frac{\partial}{\partial y^k} - t_k Z \right) = t_k X_r - t_r X_k,$$

which ends the proof.

4. Vertical 2-jets on (TM^0, G)

The notion of *vertical 2-jets* correspond to the leafwise 2-jets on the foliated manifold (TM^0, G, F_V). We denote by $\Omega^0(TM^0)$ the ring of real differentiable functions on TM^0.

Definition 1 *We say that two functions $f, g \in \Omega^0(TM^0)$ determine the same **vertical 2-jet** (or **v,2-jet**) at $(x, y) \in TM^0$ if they determine the same 2-jet at (x, y) in $T_x M^0$, which means $f(x, y) = g(x, y) = 0$ and in a local chart $(U, (x^i, y^i))$ at (x, y)*

$$\frac{\partial f}{\partial y^i}(x, y) = \frac{\partial g}{\partial y^i}(x, y), \quad \frac{\partial^2 f}{\partial y^i \partial y^j}(x, y) = \frac{\partial^2 g}{\partial y^i \partial y^j}(x, y), \quad \forall i, j = \overline{1, n}. \quad (20)$$

The relation "to determine the same *v,2*-jet at (x, y)" is an equivalence one and the equivalence class containing f is called the *v,2-jet of f at* (x, y) and it is denoted by $j^{v,2}_{(x,y)} f$. By a straightforward calculation, the relations (20) have geometrical meaning. The space

$$J^{v,2}(TM^0) = \{ j^{v,2}_{(x,y)} f, \quad f \in \Omega^0(TM^0), (x, y) \in TM^0 \},$$

is a $\frac{n(n+7)}{2}$ - dimensional differentiable manifold. Indeed, given an atlas $\{(U, (x^i, y^i))\}$ on TM^0, the collection of charts (U^v, u^v) is an C^∞-atlas on $J^{v,2}(TM^0)$, where

$$U^v = \{ j^{v,2}_{(x,y)} f, \quad f \in \Omega^0(U), (x, y) \in U \}, \quad u^v = (x^i, y^i, \omega_i, \omega_{ij})_{1 \le i \le j \le n},$$

$$x^i(j^{v,2}_{(x,y)}f) = x^i(x), \quad y^i(j^{v,2}_{(x,y)}f) = y^i(x,y), \quad \omega_i(j^{v,2}_{(x,y)}f) = \frac{\partial f}{\partial y^i}(x,y),$$

$$\omega_{ij}(j^{v,2}_{(x,y)}f) = \frac{\partial^2 f}{\partial y^i \partial y^j}(x,y).$$

Moreover, the map

$$\pi^v : J^{v,2}(TM^0) \to TM^0, \quad \pi^v(j^{v,2}_{(x,y)}f) = (x,y),$$

is a surjective submersion, so $(J^{v,2}(TM^0), \pi^v, TM^0)$ is a fiber bundle over TM^0.

In the following we introduce some new types of vertical 2-jets on TM^0, namely the leafwise, transversal and mixed vertical 2-jets with respect to the foliation $F_{L'}$.

Definition 2 *We say that two functions $f, g \in \Omega^0(TM^0)$ determine the same **leafwise vertical 2-jet** (or **l,v,2-jet**) at $(x,y) \in TM^0$ if $f(x,y) = g(x,y) = 0$ and if in a local chart $(U,(x^i,y^i))$ at (x,y) we have*

$$(X_k f)(x,y) = (X_k g)(x,y), \quad (X_k X_r f)(x,y) = (X_k X_r g)(x,y), \tag{21}$$

$\forall k, r = \overline{1, n-1}$, where $\{X_1, ..., X_{n-1}\}$ is the adapted basis on L' from Theorem 2.

The relation "to determine the same *l,v,2*-jet at (x,y)" is an equivalence one and the equivalence class containing f is called the *l,v,2-jet of f at* (x,y) and it is denoted by $j^{l,v}_{(x,y)}f$.

Remark 2 *The conditions (21) imply also*

$$(X_n f)(x,y) = (X_n g)(x,y), \quad (X_k X_n f)(x,y) = (X_k X_n g)(x,y),$$

$$(X_n X_k f)(x,y) = (X_n X_k g)(x,y), \quad (X_n X_n f)(x,y) = (X_n X_n g)(x,y),$$

$\forall k = \overline{1, n-1}$. Indeed, taking into account relation (18), the above equalities hold.

Proposition 3 *The notion introduced in Definition 2 has geometrical meaning.*

Proof: From the transformation rules (12) of X_k and taking into account the above remark, we have

$$(\tilde{X}_{k_1}f)(x,y)=\frac{\partial x^k}{\partial \tilde{x}^{k_1}}(x,y)(X_kf)(x,y)=\frac{\partial x^k}{\partial \tilde{x}^{k_1}}(x,y)(X_kg)(x,y)=(\tilde{X}_{k_1}g)(x,y),$$

where the repeated indices show summation from 1 to n. On the other hand, by (12) it results

$$\tilde{X}_{k_1}\tilde{X}_{r_1}=\left(\frac{\partial x^k}{\partial \tilde{x}^{k_1}}X_k\right)\left(\frac{\partial x^r}{\partial \tilde{x}^{r_1}}X_l\right)=\frac{\partial x^k}{\partial \tilde{x}^{k_1}}\frac{\partial x^r}{\partial \tilde{x}^{r_1}}X_kX_r+\frac{\partial x^k}{\partial \tilde{x}^{k_1}}X_k\left(\frac{\partial x^r}{\partial \tilde{x}^{r_1}}\right)X_r.$$

But $\{X_k\}_{k=\overline{1,n}}$ are vertical vector fields and $\frac{\partial x^r}{\partial \tilde{x}^{r_1}}$ does not depend on $(y^i)_{i=\overline{1,n}}$, so we have

$$\tilde{X}_{k_1}\tilde{X}_{r_1}=\frac{\partial x^k}{\partial \tilde{x}^{k_1}}\frac{\partial x^r}{\partial \tilde{x}^{r_1}}X_kX_r. \tag{22}$$

Hence,

$$(\tilde{X}_{k_1}\tilde{X}_{r_1}f)(x,y)=\left[\frac{\partial x^k}{\partial \tilde{x}^{k_1}}\frac{\partial x^r}{\partial \tilde{x}^{r_1}}X_kX_rf\right](x,y)=\left[\frac{\partial x^k}{\partial \tilde{x}^{k_1}}\frac{\partial x^r}{\partial \tilde{x}^{r_1}}X_kX_rg\right](x,y)=$$

$$=(\tilde{X}_{k_1}\tilde{X}_{r_1}g)(x,y),$$

from (21) and Remark 2.

The space

$$J^{l,v}(TM^0)=\{j^{l,v}_{(x,y)}f,\quad f\in\Omega^0(TM^0),(x,y)\in TM^0\},$$

is a $\frac{n(n+5)}{2}$ - dimensional differentiable manifold. Indeed, given an atlas $\{(U,(x^i,y^i))\}$ on TM^0, the collection of charts $(U^{l,v},u^{l,v})$ is an C^∞-atlas on $J^{l,v}(TM^0)$, where

$$U^{l,v}=\{j^{l,v}_{(x,y)}f,\quad f\in\Omega^0(U),(x,y)\in U\},\quad u^{l,v}=(x^i,y^i,\lambda_k,\lambda_{kr})_{1\le k\le r\le n-1},$$

$$x^i(j^{l,v}_{(x,y)}f)=x^i(x),\quad y^i(j^{l,v}_{(x,y)}f)=y^i(x,y),\quad \lambda_k(j^{l,v}_{(x,y)}f)=(X_kf)(x,y),$$

$$\lambda_{kr}(j^{l,v}_{(x,y)}f)=\left[\frac{1}{2}(X_kX_r+X_rX_k)\right]f(x,y),$$

with $i=\overline{1,n}$. Moreover, the map

$$\pi^{l,v}:J^{l,v}(TM^0)\to TM^0,\quad \pi^{l,v}(j^{l,v}_{(x,y)}f)=(x,y),$$

is a surjective submersion, so $(J^{l,v}(TM^0),\pi^{l,v},TM^0)$ is a fiber bundle over TM^0.

Definition 3 *We say that two functions $f, g \in \Omega^0(TM^0)$ determine the same **transversal vertical 2-jet** (or **t,v,2-jet**) at $(x,y) \in TM^0$ if $f(x,y) = g(x,y) = 0$ and if in a local chart $(U,(x^i,y^i))$ at (x,y)*

$$(Zf)(x,y) = (Zg)(x,y), \quad (Z^2 f)(x,y) = (Z^2 g)(x,y), \tag{23}$$

where Z is the vertical Liouville vector field.

The relation "to determine the same *t,v,2*-jet at (x,y)" is an equivalence one and the equivalence class containing f is called the *t,v-jet of f at* (x,y) and it is denoted by $j^{t,v}_{(x,y)} f$.

The above definition has geometrical meaning, Z being a global defined vector field.

We remark that

$$Z^2 = \left(y^i \frac{\partial}{\partial y^i} \right) \left(y^j \frac{\partial}{\partial y^j} \right) = Z + y^i y^j \frac{\partial^2}{\partial y^i \partial y^j}. \tag{24}$$

The space

$$J^{t,v}(TM^0) = \{ j^{t,v}_{(x,y)} f, \quad f \in \Omega^0(TM^0), (x,y) \in TM^0 \},$$

is a $(2n+2)$-dimensional differentiable manifold. Indeed, given an atlas $\{(U,(x^i,y^i))\}$ on TM^0, the collection of charts $(U^{t,v}, u^{t,v})$ is an C^∞-atlas on $J^{t,v}(TM^0)$, where

$$U^{t,v} = \{ j^{t,v}_{(x,y)} f, \quad f \in \Omega^0(U), (x,y) \in U \}, \quad u^{t,v} = (x^i, y^i, z, z^2)_{1 \le i \le n},$$

$$x^i(j^{t,v}_{(x,y)} f) = x^i(x), \quad y^i(j^{t,v}_{(x,y)} f) = y^i(x,y), \quad z(j^{t,v}_{(x,y)} f) = (Zf)(x,y),$$
$$z^2(j^{t,v}_{(x,y)} f) = (Z^2 f)(x,y).$$

Moreover, the map

$$\pi^{t,v} : J^{t,v}(TM^0) \to TM^0, \quad \pi^{t,v}(j^{t,v}_{(x,y)} f) = (x,y),$$

is a surjective submersion, so $(J^{t,v}(TM^0), \pi^{t,v}, TM^0)$ is a fiber bundle over TM^0.

Definition 4 *We say that two functions $f, g \in \Omega^0(TM^0)$ determine the same **leafwise-transversal vertical 2-jet** (or **l,t,v-jet**) at $(x,y) \in TM^0$ if $f(x,y) = g(x,y) = 0$ and if in a local chart $(U,(x^i,y^i))$ at (x,y)*

$$(X_k Z f)(x,y) = (X_k Z g)(x,y), \quad \forall k = \overline{1, n-1}, \tag{25}$$

where $\{X_1, ..., X_{n-1}, Z\}$ is the adapted basis from Proposition 1.

The relation "to determine the same *l,t,v*-jet at (x, y)" is an equivalence one and the equivalence class containing f is called the *l,t,v-jet of f at* (x, y) and it is denoted by $j^{l,t,v}_{(x,y)}f$.

Remark 3 *If* $j^{l,t,v}_{(x,y)}f = j^{l,t,v}_{(x,y)}g$*, then*

$$(X_n Z f)(x, y) = (X_n Z g)(x, y). \tag{26}$$

Indeed, the hypothesis is equivalent to $f(x, y) = g(x, y) = 0$ *and* $(X_k Z f)(x, y) = (X_k Z g)(x, y)$, $\forall k = \overline{1, n-1}$, *in a local chart* $(U, (x^i, y^i))$ *at* (x, y). *From the relation (18) we have*

$$(X_n Z f)(x, y) = \left[\left(\frac{-1}{y^n}\sum_{k=1}^{n-1} y^i X_i\right) Z f\right](x, y) = \frac{-1}{y^n(x, y)}\sum_{k=1}^{n-1} y^i(x, y)\left(X_i Z f\right)(x, y) =$$

$$= \frac{-1}{y^n(x, y)}\sum_{k=1}^{n-1} y^i(x, y)\left(X_i Z g\right)(x, y) = (X_n Z g)(x, y).$$

Proposition 4 *The notion of l,t,v-jet of a function at a fixed point has geometrical meaning.*

Proof: We have to prove that the relation (25) does not depend on the local chart. Taking another local chart $(\tilde{U}, (\tilde{x}^{i_1}, \tilde{y}^{i_1}))$ at the fixed point (x, y), we have the relation (12), so

$$\left(\tilde{X}_{k_1} Z f\right)(x, y) = \left(\sum_{k=1}^{n} \frac{\partial x^k}{\partial \tilde{x}^{k_1}} X_k Z f\right)(x, y) = \sum_{k=1}^{n} \frac{\partial x^k}{\partial \tilde{x}^{k_1}}(x, y)\left(X_k Z f\right)(x, y) =$$

and from the relations (25) and (26) we have,

$$\left(\tilde{X}_{k_1} Z f\right)(x, y) = \sum_{k=1}^{n} \frac{\partial x^k}{\partial \tilde{x}^{k_1}}(x, y)\left(X_k Z g\right)(x, y) = \left(\tilde{X}_{k_1} Z g\right)(x, y).$$

Definition 5 *We say that two functions* $f, g \in \Omega^0(TM^0)$ *determine the same* ***transversal-leafwise vertical 2-jet*** *(or* ***t,l,v-jet****) at* $(x, y) \in TM^0$ *if* $f(x, y) = g(x, y) = 0$ *and if in a local chart* $(U, (x^i, y^i))$ *at* (x, y)

$$(Z X_k f)(x, y) = (Z X_k g)(x, y), \quad \forall k = \overline{1, n-1}, \tag{27}$$

where $\{X_1, ..., X_{n-1}, Z\}$ *is the adapted basis from Proposition 1.*

The relation "to determine the same t,l,v-jet at (x,y)" is an equivalence one and the equivalence class containing f is called the *t,l,v-jet of f at (x,y)* and it is denoted by $j^{t,l,v}_{(x,y)}f$.

Remark 4 *By an analogous argument as in Remark 3, we prove that if $j^{t,l,v}_{(x,y)}f = j^{t,l,v}_{(x,y)}g$, then*

$$(ZX_n f)(x,y) = (ZX_n g)(x,y). \tag{28}$$

It is easy to see also that:

Proposition 5 *The notion of t,l,v-jet of a function at a fixed point has geometrical meaning.*

The vertical 2-jets defined in Definitions 4 and 5 will be called *mixed vertical 2-jets* on TM^0.

Theorem 3 *Let be $f, g \in \Omega^0(TM^0)$. We have $j^{v,2}_{(x,y)}f = j^{v,2}_{(x,y)}g$ if and only if $j^{l,v}_{(x,y)}f = j^{l,v}_{(x,y)}g$, $j^{t,v}_{(x,y)}f = j^{t,v}_{(x,y)}g$ and $j^{t,l,v}_{(x,y)}f = j^{t,l,v}_{(x,y)}g$.*

Proof: The vector fields Z, $(X_k)_{k=\overline{1,n}}$ are linear combinations of $\left(\frac{\partial}{\partial y^i}\right)_{i=\overline{1,n}}$, so, taking into account also the relations (19), (24) and

$$ZX_k = y^i \frac{\partial^2}{\partial y^i \partial y^k} + t_k Z - t_k Z^2, \tag{29}$$

the conditions (20) give (21), (23) and (27). Hence, the direct implication is proved.

Then, the hypothesis $j^{l,v}_{(x,y)}f = j^{l,v}_{(x,y)}g$, $j^{t,v}_{(x,y)}f = j^{t,v}_{(x,y)}g$ is equivalent by definitions to a system formed of relations (21) and (23), which goes to

$$\frac{\partial f}{\partial y^k}(x,y) = \frac{\partial g}{\partial y^k}(x,y), \quad \left(y^i y^j \frac{\partial^2 f}{\partial y^i \partial y^j}\right)(x,y) = \left(y^i y^j \frac{\partial^2 f}{\partial y^i \partial y^j}\right)(x,y),$$

$$\left[\frac{\partial^2 f}{\partial y^k \partial y^r} - t_k y^i \frac{\partial^2 f}{\partial y^i \partial y^r} - t_r y^i \frac{\partial^2 f}{\partial y^i \partial y^k}\right](x,y) =$$
$$= \left[\frac{\partial^2 g}{\partial y^k \partial y^r} - t_k y^i \frac{\partial^2 g}{\partial y^i \partial y^r} - t_r y^i \frac{\partial^2 g}{\partial y^i \partial y^k}\right](x,y),$$

for all $k, r = \overline{1, n}$, using also the relations (19), (24), (29) and Remarks 2, 4. Under these hypothesis, the condition $j^{t,l,v}_{(x,y)} f = j^{t,l,v}_{(x,y)} g$ implies

$$\left[y^i \frac{\partial^2 f}{\partial y^i \partial y^k}\right](x, y) = \left[y^i \frac{\partial^2 g}{\partial y^i \partial y^k}\right](x, y).$$

Finally, it results that the conditions (20) are satisfied.

As a consequence of the above theorem we have:

Proposition 6 *Let be $f, g \in \Omega^0(TM^0)$ such that $j^{l,v}_{(x,y)} f = j^{l,v}_{(x,y)} g$ and $j^{t,v}_{(x,y)} f = j^{t,v}_{(x,y)} g$. We have $j^{t,l,v}_{(x,y)} f = j^{t,l,v}_{(x,y)} g$ if and only if $j^{l,t,v}_{(x,y)} f = j^{l,t,v}_{(x,y)} g$.*

The space

$$J^{t,l,v}(TM^0) = \{j^{t,l,v}_{(x,y)} f, \quad f \in \Omega^0(TM^0), (x, y) \in TM^0\},$$

is a $(3n - 1)$-dimensional differentiable manifold. Indeed, given an atlas $\{(U, (x^i, y^i))\}$ on TM^0, the collection of charts $(U^{t,l,v}, u^{t,l,v})$ is an C^∞-atlas on $J^{t,v}(TM^0)$, where

$$U^{t,l,v} = \{j^{t,l,v}_{(x,y)} f, \quad f \in \Omega^0(U), (x, y) \in U\}, \quad u^{t,l,v} = (x^i, y^i, \tau_k)_{1 \le k \le n-1},$$

$$x^i(j^{t,l,v}_{(x,y)} f) = x^i(x), \quad y^i(j^{t,l,v}_{(x,y)} f) = y^i(x, y), \quad \tau_k(j^{t,l,v}_{(x,y)} f) = (ZX_k f)(x, y),$$

for all $i = \overline{1, n}$ and $k = \overline{1, n - 1}$. Moreover, the map

$$\pi^{t,l,v} : J^{t,l,v}(TM^0) \to TM^0, \quad \pi^{t,l,v}(j^{t,l,v}_{(x,y)} f) = (x, y),$$

is a surjective submersion, so $(J^{t,l,v}(TM^0), \pi^{t,v}, TM^0)$ is a fiber bundle over TM^0.

In the following we shall consider the fiber product of the bundles $\pi^{l,v}$, $\pi^{t,v}$, $\pi^{t,l,v}$, whose total space is

$$J^{l,v}(TM^0) \times_{TM^0} J^{t,v}(TM^0) \times_{TM^0} J^{t,l,v}(TM^0) = \tag{30}$$

$$= \left\{ \left(j^{l,v}_{(x,y)} f, j^{t,v}_{(x,y)} g, j^{t,l,v}_{(x,y)} h \right), \quad f, g, h \in \Omega^0 TM^0, (x, y) \in TM^0 \right\}.$$

The above set is a $\frac{n(n+7)}{2}$-dimensional manifold. Indeed, if $\{(U^{l,v}, u^{l,v})\}$, $\{(U^{t,v}, u^{t,v})\}$, $\{(U^{t,l,v}, u^{t,l,v})\}$ are atlases on the manifolds $J^{l,v}(TM^0)$, $J^{t,v}(TM^0)$, $J^{t,l,v}(TM^0)$, respectively, then

$$\left\{ \left(U^{l,v} \times U^{t,v} \times U^{t,l,v}, u = (x^i, y^i, \lambda_k, z, \lambda_{kr}, z^2, \tau_k)_{1 \le k \le r \le n-1; 1 \le i \le n} \right) \right\},$$

is a differential atlas on $J^{l,v}(TM^0) \times_{TM^0} J^{t,v}(TM^0) \times_{TM^0} J^{t,l,v}(TM^0)$.

Now we can give the main result of this paper:

Theorem 4 *The map*

$$\zeta : J^{v,2}(TM^0) \to J^{l,v}(TM^0) \times_{TM^0} J^{t,v}(TM^0) \times_{TM^0} J^{t,l,v}(TM^0),$$

defined by

$$\zeta\left(j^{v,2}_{(x,y)}f\right) = \left(j^{l,v}_{(x,y)}f, j^{t,v}_{(x,y)}f, j^{t,l,v}_{(x,y)}f\right),$$

is a diffeomorphism.

Proof: The Theorem 3 assures that the map ζ is well-defined and injective. We prove that it is differentiable, too. For every $(\alpha_i, \beta_i, \gamma_i, \delta_{ij})_{1\leq i\leq j\leq n}$ $\in u^v(U^v) \subset \mathbf{R^n}\times\mathbf{R^n}\times\mathbf{R^n}\times\mathbf{R}^{\frac{\mathbf{n(n+1)}}{\mathbf{2}}}$, there is an element $j^{v,2}_{(x,y)}f \in J^{v,2}(TM^0)$ such that $u^v(j^{v,2}_{(x,y)}f) = (\alpha_i, \beta_i, \gamma_i, \delta_{ij})$, or, equivalently, $x^i(j^{v,2}_{(x,y)}f) = \alpha_i$, $y^i(j^{v,2}_{(x,y)}f) = \beta_i$, $\omega_i(j^{v,2}_{(x,y)}f) = \gamma_i$, $\omega_{ij}(j^{v,2}_{(x,y)}f) = \delta_{ij}$. By the expressions of functions of local charts, the above relations are equivalent to

$$x^i(x) = \alpha_i, \quad y^i((x,y) = \beta_i, \quad \frac{\partial f}{\partial y^i} = \gamma_i, \quad \frac{\partial^2 f}{\partial y^i \partial y^j} = \delta_{ij}.$$

Let $(U^v, u^v = (x^i, y^i, \omega_i, \omega_{ij})_{1\leq i\leq j\leq n})$ be a local chart around $j^{v,2}_{(x,y)}f \in J^{v,2}(TM^0)$ and $\left(U^{l,v}\times U^{t,v}\times U^{t,l,v}, u = (x^i, y^i, \lambda_k, z, \lambda_{kr}, z^2, \tau_k)_{1\leq k\leq r\leq n-1; 1\leq i\leq n}\right)$ a local chart around $\zeta(j^{v,2}_{(x,y)}f)$. We obtain

$$\left(u\circ\zeta\circ(u^v)^{-1}\right)(\alpha_i, \beta_i, \gamma_i, \delta_{ij}) = u\left(\zeta(j^{v,2}_{(x,y)}f)\right) = u\left(j^{l,v}_{(x,y)}f, j^{t,v}_{(x,y)}f, j^{t,l,v}_{(x,y)}f\right) =$$

$$= \left(x^i(x), y^i(x,y), \lambda_k(j^{l,v}_{(x,y)}f), z(j^{t,v}_{(x,y)}f), \lambda_{kr}(j^{l,v}_{(x,y)}f), z^2(j^{t,v}_{(x,y)}f), \tau_k(j^{t,l,v}_{(x,y)}f)\right) =$$

$$= \left(\alpha_i, \beta_i, \alpha_k - t_k(x,y)\sum_{j=1}^n \alpha_j\beta_j, \sum_{j=1}^n \alpha_j\beta_j, \xi_{kr}, \theta, \rho_k\right),$$

where

$$\xi_{kr} = \delta_{kr} + t_k(x,y)t_r(x,y)\left(2\sum_{j=1}^n \alpha_j\beta_j + \sum_{i,j=1}^n \beta_i\beta_j\delta_{ij}\right) + \frac{1}{F^2(x,y)}g_{kr}(x,y)\sum_{j=1}^n \alpha_j\beta_j -$$

$$-t_k(x,y)\left(\frac{\alpha_r}{2} + \sum_{j=1}^n \beta_j\delta_{ir}\right) - t_r(x,y)\left(\frac{\alpha_k}{2} + \sum_{j=1}^n \beta_j\delta_{jk}\right),$$

$$\theta = \sum_{j=1}^n \alpha_j\beta_j + \sum_{i,j=1}^n \beta_i\beta_j\delta_{ij}, \quad \rho_k = \sum_{j=1}^n \beta_j\delta_{jk} - t_k(x,y)\sum_{i,j=1}^n \beta_i\beta_j\delta_{ij},$$

for all $k, r = \overline{1, n-1}$. The map $u \circ \zeta \circ (u^v)^{-1}$ is a real differentiable map, so ζ is differentiable.

In the following we prove that ζ is a surjection. Let be $\left(j^{l,v}_{(x,y)} f, j^{t,v}_{(x,y)} g, j^{t,l,v}_{(x,y)} h\right)$ an arbitrary element of the codomain of ζ. We search a function $\alpha \in \Omega^0(TM^0)$ such that $j^{l,v}_{(x,y)}\alpha = j^{l,v}_{(x,y)} f$, $j^{t,v}_{(x,y)}\alpha = j^{t,v}_{(x,y)} g$ and $j^{t,l,v}_{(x,y)}\alpha = j^{t,l,v}_{(x,y)} h$. These conditions are equivalent to the following system:

$$\begin{cases} (X_k\alpha)(x,y) = (X_k f)(x,y) \\ (Z\alpha)(x,y) = (Zg)(x,y) \\ (X_k X_r \alpha)(x,y) = (X_k X_r f)(x,y) \quad \forall k, r = \overline{1, n-1}. \\ (Z^2\alpha)(x,y) = (Z^2 g)(x,y) \\ (ZX_k\alpha)(x,y) = (ZX_k h)(x,y) \end{cases} \tag{31}$$

Using the Remarks 2 and 4, these relations hold also for $k, r = n$. Taking into account the relations (9), (19), (24) and (29), from (31) it results that

$$\frac{\partial \alpha}{\partial y^k}(x,y) = (t_k Zg + X_k f)(x,y), \quad \frac{\partial^2 \alpha}{\partial y^k \partial y^l}(x,y) = A_{kr}, \tag{32}$$

$$A_{kr} = \left(X_k X_r f + t_r X_k f + t_k Z X_r h + t_r Z X_k h + + t_k t_r Z^2 g\right)(x,y) -$$
$$- \left(2 t_k t_r Zg + \frac{1}{F^2} g_{kr} Zg\right)(x,y),$$

for all $k, r = \overline{1, n}$. We remark that $A_{kr} = A_{rk}$ by Proposition 2.

Let $\alpha \in \Omega^0(TM^0)$ be the map

$$\alpha = \left[A_k - \left(ZX_k h + t_k Z^2 g - t_k Zg\right)(x,y)\right] y^k + A_{kr} y^k y^r - (Zg)(x,y),$$

where $A_k = \frac{\partial \alpha}{\partial y^k}(x,y)$ from (32). By a direct calculation it results $\alpha(x,y) = 0$ and that the function α is satisfying the system (31). In computation we used also Proposition 1. Hence we obtain that ζ is a surjective map. Being an injection, too, ζ is bijective. Its inverse is also differentiable, as we can see by a straightforward calculation, using (32). So ζ is a diffeomorphism.

As a consequence we have

Proposition 7 *The module of fields of vertical 2-jets on TM^0 admits the following decomposition*

$$\Gamma(J^{v,2}(TM^0)) = \Gamma(J^{l,v}(TM^0)) \oplus \Gamma(J^{t,v}(TM^0)) \oplus \Gamma(J^{t,l,v}(TM^0)).$$

5. Conclusions

This paper introduces some new geometrical objects on the tangent bundle TM^0 of a Finsler manifold endowed with two foliations. These objects are defined with respect to the Liouville foliation and they are related to the vertical 2-jets of TM^0. A decomposition theorem is given for the module of fields of vertical 2-jets.

Acknowledgement

The present research was supported by the Romanian Academy Grant 4/3.06.2009.

R E F E R E N C E S

[1] *I. Kolar, P.W. Michor, J. Slovak*: "Natural Operations in Differential Geometry", Springer-Verlag, Berlin Heidelberg(1993), www.emis.de/monographs.

[2] *D.J. Saunders*: "The Geometry of Jet Bundles", London Math. Soc. Lecture Note Series, 142, Cambridge U.P., 1989.

[3] *R. Miron, M. Anastasiei*: "Vector bundles, Lagrange spaces. Applications to the theory of relativity", Geometry Balkan Press, Bucuresti, 1997.

[4] *Ph. Tondeur*: "Foliations on Riemannian Manifolds", Universitext, Springer-Verlag, New-York, 1988.

[5] *I. Vaisman*: "Cohomology and Differential Forms", Marcel Dekker Inc., New York, 1973.

[6] *A. Bejancu, H.R. Farran*: "Finsler Geometry and Natural Foliations on the Tangent Bundle", Reports on Mathematical Physics, **58**(2006), p. 131-146.

[7] *A. Manea*: "Mastrogiacomo leafwise cohomology on a foliated manifold", Bull. Math. Soc. Sci. Romania, **46**(2003), p. 75-86.

[8] *A. Manea*: "Mastrogiacomo transversal and basic cohomology on a foliated manifold", Ricerche di Matematica, **53**(2004), p. 83-95.

[9] *V.Balan, A.Manea*: "Leafwise 2-jet cohomology on foliated Finsler manifolds", Proceedings of the International Conference of Differential Geometry and Dynamical Systems (DGDS-2008) and The V-th International Colloquium of Mathematics in Engineering and Numerical Physics (MENP-5) - mathematics sections, August 29 - September 2, 2008, Mangalia, Romania, BSG Proceedings **16**, Geometry Balkan Press, Bucharest 2009, 28-41.

[10] *A. Manea*: "Leafwise, transversal and mixed 2-jets of bundles over foliated manifolds", Balkan Journal of Geometry and Its Applications, **12**(2007), p. 87-97.

[11] *V.Balan, S. Ikeda*: "Structural considerations on the theory of gravitational fields in Finsler spaces", U.P.B. Scientific Bulletin, **68**(2006), p. 3-8.