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ON PRIME A-IDEALS IN MV-MODULES

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In this paper, we study A-ideals in MV-modules. We introduce the notion of ·-prime ideals in PMV-algebras and study the relations between ·-prime ideals and MVF-algebras. Also we define prime A-ideals in MV-modules and annihilator of A-ideals in MV-modules. We investigate some relations between prime A-ideals and annihilators of A-ideals in MV-modules.

Keywords: (MV, PMV)-algebra, MV-module, A-ideal, Prime A-ideal.
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1. Introduction and Preliminaries

In 2003, Di Nola, et.al. introduced the notion of MV-modules over a PMV-algebra and A-ideals in MV-modules [5]. These are structures that naturally correspond to lu-modules over lu-rings [5]. Recall that an lu-ring is a pair (R,u), where $(R, \oplus, \cdot, 0, \leq)$ is an l-ring and u is a strong unit of R (i.e, u is a strong unit of the underlying l-group) such that $u \cdot u \leq u$ and l-ring is a structure $(R, +, \cdot, 0, \leq)$ that $(R, +, 0, \leq)$ is an l-group such that for any $x, y \in R$, $x \geq 0$ and $y \geq 0$, we have $x \cdot y \geq 0$. They proved that the category of lu-modules over a given lu-ring (R, v) is equivalent to the category of MV-modules over $\Gamma(R, v)$. They also proved there is a natural equivalence between MV-modules and truncated modules [5]. A. Dvurecenskij and A. Di Nola in [6] introduced the notion of PMV-algebras, that is MV-algebras whose product operation $(\cdot)$ is defined on the whole MV-algebra. This operation is associative and left/right distributive with respect to partially defined addition. They showed that the category of product MV-algebras is categorically equivalent to the category of associative unital l-rings. In addition, they introduced and studied MVF-algebras [6]. They also introduced ·-ideals in PMV-algebras. Then they showed that: Any MVF-algebra is a subdirect product of subdirectly irreducible MVF-algebras [6, Corollary 5.6]. Thus they concluded that a product MV-algebra is an MVF-ring if and only if it is a subdirect product of linearly ordered product MV-algebras [6, Theorem 5.8].

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In the present paper, we define $\cdot$-prime ideals in PMV-algebras. Using this notion of ideals we construct the quotient PMV-algebras and investigate the relations between $\cdot$-prime ideals and MVF-algebras. Moreover, we study A-ideals in MV-modules, and introduce the notion of prime A-ideals and annihilators of these ideals in MV-modules.
We investigate the relations between prime A-ideals and annihilators of A-ideals in MV-modules. Finally we prove that if $h : M \to N$ is an A-module homomorphism then all prime A-ideals of N and prime A-ideals of M that contain $kerh$ are in one to one correspondence.

Definition 1.1. [3] An *MV-algebra* is a structure $(M, \oplus, {}^*, 0)$ where $\oplus$ is a binary operation, *, is a unary operation, and 0 is a constant such that the following conditions are satisfied for any $a, b \in M$:
$(MV1)$ $(M, \oplus, 0)$ is an abelian monoid,
$(MV2)$ $(a^*)^* = a$,
$(MV3)$ $0^* \oplus a = 0^*$,
$(MV4)$ $(a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a$.

If we define the constant $1 = 0^*$ and the auxiliary operations $\odot, \vee$ and $\wedge$ by:

$$a \odot b = (a^* \oplus b^*)^*, \qquad a \vee b = a \oplus (b \odot a^*),$$

$$a \wedge b = a \odot (b \oplus a^*) \qquad a \ominus b = a \odot b^*,$$

then $(\mathrm{M}, \odot, 1)$ is an abelian monoid and the structure $(M, \vee, \wedge, 0, 1)$ is a bounded distributive lattice. In an MV-algebra M, the Chang distance function is

$$d : M \times M \longrightarrow M, \qquad d(a, b) := (a \odot b^*) \oplus (b \odot a^*).$$

We recall that an lu-group is an algebra $(G, +, -, 0, \vee, \wedge, u)$, where the following properties hold:
(a) $(G, +, -, 0)$ is a group,
(b) $(G, \vee, \wedge)$ is a lattice,
(c) For any $x, y, a, b \in G$, $x \leq y$ implies $a + x + b \leq a + y + b$,
(d) $u > 0$ is strong unit for G (that is, for all $x \in G$ there is some natural number $n \geq 1$ such that $-nu \leq x \leq nu$) [1].

We will denote by $\mathcal{MV}$ the category whose objects are MV-algebras and whose morphisms are MV-algebra homomorphisms and $\mathcal{UG}$ the category of lu-groups. The elements of this category are pairs (G, u) where G is an Abelian l-group and u is a strong unit of G. The morphisms will be l-group homomorphisms which preserve the strong unit. The functor that establishes the categorical equivalence between $\mathcal{MV}$ and $\mathcal{UG}$ is

$$\Gamma : \mathcal{UG} \longrightarrow \mathcal{MV}.$$

such that $\Gamma(G, u) := [0, u]_G$ for any lu-group (G, u), $\Gamma(h) := h\mid_{[0,u]}$ for any lu-groups homomorphism [9].
The above result allows us to consider an MV-algebra, when necessary, as an interval in the positive cone of an l-group.

Thus, many definitions and properties can be transferred from l-groups to MV-algebras. For example, the group addition becomes a partial operation when it is restricted to an interval so we may define a partial addition on an MV-algebra M as follows:
for any $x, y \in M$, $x + y$ is defined iff $x \leq y^*$
and, in this case, $x + y := x \oplus y$, where $+$ is the partial addition on M [7].
Also, cancellation rule holds in it, That is, if $z + x \leq z + y$ then $x \leq y$ [5].

Lemma 1.1. [3] Let M be an MV-algebra. If $x, y, z, t \in M$ and d is a Chang distance function, then
(1) $x \leq y$ *iff* $y^* \leq x^*$,
(2) If $x \leq y$, *then* $x \oplus z \leq y \oplus z$ *and* $x \odot z \leq y \odot z$,
(3) $(x \vee y)^* = x^* \wedge y^*$, $(x \wedge y)^* = x^* \vee y^*$,
(4) $d(x, y) = 0$ *iff* $x = y$,
(5) $d(x, 0) = x, d(x, 1) = x^*$,
(6) $d(x, z) \leq d(x, y) \oplus d(y, z)$,
(7) $d(x^*, y^*) = d(x, y)$,
(8) If $x \leq y$ *and* $z \leq t$, *then* $x \oplus z \leq y \oplus t$.

Lemma 1.2. [3] Let M be an MV-algebra. For $x, y \in M$, the following conditions are equivalent:
(1) $x^* \oplus y = 1$,
(2) $x \odot y^* = 0$,
(3) There is an element $z \in M$ such that $x \oplus z = y$,
(4) $y = x \oplus (y \ominus x)$.
For any two elements $x, y \in M$, $\quad x \leq y$ iff x and y satisfy the equivalent conditions (1)-(4) in the above lemma.

Definition 1.2. [3] An *ideal* of an MV-algebra M is a nonempty subset I of M satisfying the following conditions:
$(I1)$ If $x \in I$, $y \in M$ and $y \leq x$ then $y \in I$,
$(I2)$ If $x, y \in I$, then $x \oplus y \in I$.
We denote by $Id(M)$ the set of ideals of an MV-algebra M.

Definition 1.3. [3] A proper ideal P is a *prime* ideal of an MV-algebra M, if $x \wedge y \in P$, then $x \in P$ or $y \in P$, for all $x, y \in M$.

Definition 1.4. [6] A *product* MV*-algebra* (or PMV-algebra, for short) is a structure $(A, \oplus, {}^*, \cdot, 0)$, where $(A, \oplus, {}^*, 0)$ is an MV-algebra and $\cdot$ is a binary associative operation on A such that the following property is satisfied:
if $x + y$ is defined, then $x \cdot z + y \cdot z$ and $z \cdot x + z \cdot y$ are defined and

$$(x + y) \cdot z = x \cdot z + y \cdot z, \quad z \cdot (x + y) = z \cdot x + z \cdot y$$

If A is PMV-algebra, then a unity for the product is an element $e \in A$ such that $e \cdot x = x \cdot e = x$ for any $x \in A$. A PMV-algebra that has unity for the product will be called unital.
A $\cdot$-ideal of PMV-algebra A is an ideal I of MV-algebra A such that if $a \in I$ and $b \in A$ entail $a \cdot b \in I$ and $b \cdot a \in I$.

Lemma 1.3. [6] If A is a unital PMV-algebra, then:
(a) The unity for the product is $e = 1$,
(b) $x \cdot y \leq x \wedge y$ for any $x, y \in A$.

Definition 1.5. [5] Let $(A, \oplus, ^*, \cdot, 0)$ be a PMV-algebra and $(M, \oplus, ^*, 0)$ an MV-algebra. We say that M is a *(left) MV-module over A (or, simply, A-module)* if there is an external operation:

$$\varphi : A \times M \longrightarrow M, \quad \varphi(\alpha, x) = \alpha x,$$

such that the following properties hold for any $x, y \in M$ and $\alpha, \beta \in A$:
(1) If $x + y$ is defined in M, then $\alpha x + \alpha y$ is defined and

$$\alpha(x + y) = \alpha x + \alpha y,$$

(2) If $\alpha + \beta$ is defined in A then $\alpha x + \beta x$ is defined in M and

$$(\alpha + \beta)x = \alpha x + \beta x,$$

(3) $(\alpha \cdot \beta)x = \alpha(\beta x)$.
We say that M is a unital MV-module if A is a unital PMV-algebra and M is an MV-module over A such that $1_A x = x$ for any $x \in M$.

Example 1.1. [5] Let $M_2(\mathbb{R})$ be the ring of square matrices of order 2 with real elements and 0 be the matrix with all element 0. If we define the order relation on components $A = (a_{ij})_{i,j=1,2} \geq 0$ iff $a_{ij} \geq 0$ for any i, j, such that $v = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$, then $A = \Gamma(M_2(\mathbb{R}), v)$ is a PMV-algebra. Let $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ be the direct product with the order relation defined on components. If $M = \Gamma(\mathbb{R}^2, u)$ is an MV-algebra, where $u = (1, 1)$, then M is an A-module, where the external operation is the usual matrix multiplication $(A, (x, y)) \mapsto A \begin{pmatrix} x \\ y \end{pmatrix}$. The above construction can be generalized for any order $n \geq 2$.
(1) If $(x, y) + (z, t)$ is defined in M, so $(x, y) \leq (z, t)^* = (1, 1) - (z, t)$ or $(x, y) + (z, t) \leq (1, 1)$, suppose that $A = (a_{ij})_{i,j=1,2}$ such that $a_{ij} \leq 1/2$ for $i, j = 1, 2$. Hence $A \begin{pmatrix} x \\ y \end{pmatrix} + A \begin{pmatrix} z \\ t \end{pmatrix} \leq A \begin{pmatrix} 1 \\ 1 \end{pmatrix} \leq (1, 1)$. Then $A \begin{pmatrix} x \\ y \end{pmatrix} + A \begin{pmatrix} z \\ t \end{pmatrix}$ is defined in M.
(2) If $A + B$ is defined in A, so $A \leq B^* = v - B$ or $A + B \leq v$. Let $X = (x, y) \in M$ such that $(x, y) \leq (1, 1)$ or $x, y \leq 1$. Hence $A \begin{pmatrix} x \\ y \end{pmatrix} + B \begin{pmatrix} x \\ y \end{pmatrix} \leq v \begin{pmatrix} x \\ y \end{pmatrix} \leq (1, 1)$.

Then $A\begin{pmatrix} x \\ y \end{pmatrix} + B\begin{pmatrix} x \\ y \end{pmatrix}$ is defined in M.

(3) $(A.B)\begin{pmatrix} x \\ y \end{pmatrix} = A(B\begin{pmatrix} x \\ y \end{pmatrix})$.

Definition 1.6. [5] Let M and N be two MV-modules over a PMV-algebra A. An *A-module homomorphism* is an MV-algebra homomorphism $h : M \to N$ such that $h(\alpha x) = \alpha h(x)$, for any $\alpha \in A$ and $x \in M$.

Definition 1.7. [5] Let M be an A-module. Then ideal $I \subseteq M$ is called an *A-ideal* if it satisfies the following condition*:*
if $x \in I$ and $\alpha \in A$, then $\alpha x \in I$.

Lemma 1.4. [5] If M is an A-module, then the following properties hold for any $x, y \in M$ and α,$\beta \in A$*:*
(a) $0x = 0$*,*
(b) $\alpha 0 = 0$*,*
(c) $(n\alpha)x = \alpha(nx)$ *for any* $n \in \mathbb{N}$*,*
(d) $\alpha x^* \leq (\alpha x)^*$*,*
(e) $\alpha^* x \leq (\alpha x)^*$*,*
(f) $(\alpha x)^* = \alpha^* x + (1x)^*$*, if $+$ is defined,*
(g) $x \leq y$ *implies* $\alpha x \leq \alpha y$*,*
(h) $\alpha \leq \beta$ *implies* $\alpha x \leq \beta x$*,*
(i) $(\alpha x) \odot (\alpha y)^* \leq \alpha(x \odot y^*)$*,*
(j) $\alpha(x \oplus y) \leq \alpha x \oplus \alpha y$*,*
(k) $d(\alpha x, \alpha y) \leq \alpha d(x, y)$*.*

Proposition 1.1. [5] If A is a unital PMV-algebra and M is a unital A-module, then any ideal of M is an A-ideal. Thus, the ideals and the A-ideals of M coincide.

Remark 1.1. [5] Let M be an A-module and $I \subseteq M$ an A-ideal of M. We recall that the relation $\sim_I$ defined by:

$$x \sim_I y \quad \text{if and only if} \quad d(x, y) \in I,$$

for any $x, y \in M$, is a congruence with respect to the MV-algebra operations. We notice that $x \sim_I y$ implies $\alpha x \sim_I \alpha y$, for any $\alpha \in A$. Thus, the quotient MV-algebra M/I has a canonical structure of A-module

$$\alpha[x]_I := [\alpha x]_I \quad or \quad \alpha(x/I) := (\alpha x)/I,$$

where $[x]_I$ is the congruence class of x. $x/I = y/I$ if and only if $d(x, y) \in I$ and if $x, y \in M$, then $x/I \leq y/I$ if and only if $x \odot y^* \in I$.

Definition 1.8. [4] A *residuated lattice* is an algebra $(A, \wedge, \vee, \odot, \to, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ equipped with an order $\leq$ satisfying the following:
(LR_1) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice,

(LR_2) $(A, \odot, 1)$ is a commutative monoid,
(LR_3) $\odot$ and $\to$ are form an adjoint pair, i.e., $c \leq a \to b$ if and only if $a \odot c \leq b$, for all $a, b, c \in A$.

Remark 1.2. [2] Any Boolean algebra can be regarded as a residuated lattice where the operations $\odot$ and $\wedge$ coincide and $x \to y = x^* \vee y$.

2. Some results on A-ideals in MV-modules

In the sequel A is a PMV-algebra and M is an A-module.

Remark 2.1. In general, the union of any family of A-ideals of M is not an A-ideal of M.

Example 2.1. Let M be $\Gamma(\mathbb{R}^2, u)$ such that $u = (1, 1)$, $A = \Gamma(M_2(\mathbb{R}), v)$ and $v = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$. By Example 1.1, M is an A-module but $M = \Gamma(\mathbb{R}^2, u) = [(0, 0), (1, 1)]$ and $A = \Gamma(M_2(\mathbb{R}), v) = [0, \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}]$. Then $Id_A(M) = \{(0, 0), M\}$.

We denote by $Id_A(M)$ the set of A-ideals of an MV-module over a PMV-algebra A.
We recall that for a nonempty subset $N \subseteq M$, the smallest A-ideal of M which contains N, i.e., $\bigcap\{I \in Id_A(M) : N \subseteq I\}$, is said to be the A-ideal of M generated by N and will be denoted by $(N]$.

Proposition 2.1. Let M be an A-module.
(i) If $N \subseteq M$ is a nonempty set, then we have $(N] = \{x \in M : x \leq x_1 \oplus \ldots \oplus x_n \oplus \alpha_1 y_1 \oplus \ldots \oplus \alpha_m y_m \quad \text{for some} \quad x_1, \ldots, x_n, y_1, \ldots y_m \in N, \alpha_1, \ldots \alpha_m \in A\}$, where by $(N]$, we mean the ideal generated by N.
In particular, for $a \in M$,

$$(a] = \{x \in M : x \leq na \oplus m(\alpha a) \quad \text{for some integer} \quad n, m \geq 0\},$$

(ii) If $I_1, I_2 \in Id_A(M)$, then
$I_1 \vee I_2 = (I_1 \cup I_2] = \{a \in M : a \leq x_1 \oplus x_2 \quad \text{for some} \quad x_1 \in I_1 \quad \text{and} \quad x_2 \in I_2\}$,

(iii) If $x, y \in A$, then $(x \wedge y] \subseteq (x] \cap (y]$.

Proof. (i) We denote $I = \{x \in M : x \leq x_1 \oplus \ldots \oplus x_n \oplus \alpha_1 y_1 \oplus \ldots \oplus \alpha_m y_m \quad \text{for some} \quad x_1, \ldots, x_n, y_1, \ldots y_m \in N, \alpha_1, \ldots \alpha_m \in A\}$ and prove that I is the smallest A-ideal containing N. It is clear that $N \subseteq I$, if $x \in N$, then $x \in M$, $x \leq x \oplus 0 \quad \text{for some} \quad x \in N, 0 \in A$, hence, $x \in I$. Let $a \leq b$ and $b \in I$. So there exist $n \geq 1$ and $x_1, \ldots x_n \in N$ such that $a \leq b \leq x_1 \oplus \ldots \oplus x_n \oplus \alpha_1 y_1 \oplus \ldots \oplus \alpha_m y_m$. It follows that $a \in I$. Now, let $a, b \in I$. Then $a \leq x_1 \oplus \ldots \oplus x_n \oplus \alpha_1 y_1 \oplus \ldots \oplus \alpha_m y_m \quad \text{for some} \quad x_1, \ldots, x_n, y_1, \ldots y_m \in N, \alpha_1, \ldots \alpha_m \in A$, and $b \leq t_1 \oplus \ldots \oplus t_k \oplus \beta_1 z_1 \oplus \ldots \oplus \beta_s z_s \quad \text{for some} \quad t_1, \ldots, t_k, z_1, \ldots z_s \in N$ and $\beta_1, \ldots \beta_s \in A$, by Lemma 1.1 (8), we have $a \oplus b \leq x_1 \oplus \ldots \oplus x_n \oplus t_1 \oplus \ldots \oplus t_k \oplus \alpha_1 y_1 \oplus \ldots \oplus \alpha_m y_m \oplus \beta_1 z_1 \oplus \ldots \oplus \beta_s z_s$, so $a \oplus b \in I$. Let $\alpha \in A$, $x \in I$. Then $x \leq$

$x_1\oplus\ldots\oplus x_n\oplus\alpha_1 y_1\oplus\ldots\oplus\alpha_m y_m$ for some $x_1,\ldots,x_n,y_1,\ldots y_m\in N,\alpha_1,\ldots\alpha_m\in A$, by Lemma 1.4 ($h$), ($j$), we have $\alpha x\leq\alpha x_1\oplus\ldots\oplus\alpha x_n\oplus(\alpha\cdot\alpha_1)y_1\oplus\ldots\oplus(\alpha\cdot\alpha_m)y_m$ for some $x_1,\ldots,x_n,y_1,\ldots y_m\in N$, $\gamma_1,\ldots\gamma_m\in A$ such that $\gamma_i=\alpha\cdot\alpha_i, i=1,\ldots m$. Hence, $\alpha x\in I$. Thus, I is an A-ideal containing N. Let K be another A-ideal of M that contains N and $a\in I$. Hence, $a\leq x_1\oplus\ldots\oplus x_n\oplus\alpha_1 y_1\oplus\ldots\oplus\alpha_m y_m$ for some $x_1,\ldots,x_n,y_1,\ldots y_m\in N,\alpha_1,\ldots\alpha_m\in A$. Since K is an A-ideal, it follows that $x_1\oplus\ldots\oplus x_n\in K$ and $\alpha_i y_i\in K$ for $i=1\ldots m$. Hence, $x_1\oplus\ldots\oplus x_n\oplus\alpha_1 y_1\oplus\ldots\oplus\alpha_m y_m\in K$, so $a\in K$, we deduce that $I\subseteq K$. Therefore $(N]=I$.
Clearly, for $a\in M$

$$(a]=\{x\in M: x\leq na\oplus m(\alpha a)\quad\text{for some integers}\quad n,m\geq 0\}.$$

(ii) Follows by (i).
(iii) Obviously, $x\in(x]$ and $y\in(y]$. Since $x\wedge y\leq x,y$, we get that $x\wedge y\in(x]$ and $x\wedge y\in(y]$. It follows that $x\wedge y\in(x]\cap(y]$.
Now, let $t\in(x\wedge y]$. Then, $t\leq n(x\wedge y)\oplus m(\alpha(x\wedge y))$ for some integers $n,m\geq 0$, we deduce that $t\in(x]\cap(y]$, so $(x\wedge y]\subseteq(x]\cap(y]$.

□

If in the above theorem, we consider M unitary A-module, then we have:

Corollary 2.1. Let M be a unitary A-module. If $N\subseteq M$ is a nonempty set, then we have: (i) $(N]=\{x\in M: x\leq\alpha_1 x_1\oplus\ldots\oplus\alpha_n x_n\quad\text{for some}\quad x_1,\ldots,x_n\in N\text{and}\quad\alpha_1,\ldots\alpha_n\in A\}$, In particular, for $a\in M$,

$$(a]=\{x\in M: x\leq n(\alpha a)\quad\text{for some integer}\quad n\geq 0\},$$

(ii) If $I_1,I_2\in Id_A(M)$, then $I_1\vee I_2=(I_1\cup I_2]=\{a\in M: a\leq x_1\oplus x_2\quad\text{for some}\quad x_1\in I_1,x_2\in I_2\}$,
(iii) If $x,y\in A$, then $(x\wedge y]=(x]\cap(y]$.

For $I\in Id_A(M)$ and $a\in A-I$, we denote by $I(a)=(a]\vee I=(I\cup\{a\}]$.

Remark 2.2. Let M be an A-module. Then
$I(a)=\{x\in M: x\leq y\oplus ma\oplus n(\alpha a),\quad\text{for some}\quad y\in I,\quad integers\quad n,m\geq 0,\alpha\in A\}$.

Proof. Let $T=\{x\in M: x\leq y\oplus ma\oplus n(\alpha a),\quad\text{for some } y\in I,\quad integers\quad n,m\geq 0,\alpha\in A\}$. We suppose that, $x\in I(a)=(a]\vee I=\{x\in M: x\leq x_1\oplus y\quad\text{for some}\quad x_1\in(a]\quad and\quad y\in I\}$. Since $x_1\in(a]$, then $x_1\leq ma\oplus n(\alpha a)$, for some integer $m,n\geq 0$ and $\alpha\in A$, we have $x\leq x_1\oplus y\leq ma\oplus n(\alpha a)\oplus y$, it follows that $x\in T$.
Conversely, if $x\in T$, then we get that $x\leq y\oplus ma\oplus n(\alpha a)$, for some $y\in I$ and integer $n\geq 0, x_1=ma\oplus n(\alpha a)\in(a]$, so $x\leq y\oplus x_1$ such that $x_1\in(a]$ and $y\in I$. It follows that $x\in(a]\vee I=I(a)$. □

Remark 2.3. *Let M be a unitary A-module. We have:* $I(a)=\{x\in M: x\leq y\oplus n(\alpha a),\quad\textit{for some}\quad y\in I\quad\textit{and}\quad\textit{integer}\quad n\geq 0\}$.

Corollary 2.2. Let $I \in Id_A M$ and $a, b \in A - I$. Then $I(a \wedge b) \subseteq I(a) \cap I(b)$.

Proof. We have $a\wedge b \leq y \oplus m(a\wedge b) \oplus n(\alpha(a\wedge b))$ for some $y \in I$ and integers $m, n \geq 0$. Then $a \wedge b \in I(a \wedge b)$. Since $a \wedge b \leq a, b$ and $a \in (a]$, $b \in (b]$, so $a \wedge b \in (a] \subseteq I(a)$ and $a \wedge b \in (b] \subseteq I(b)$, also $I \subseteq I(a)$, $I \subseteq I(b)$, we deduce that $a \wedge b \in I(a) \cap I(b)$, if $x \in I(a \wedge b)$, then $x \leq y \oplus m(a \wedge b) \oplus n(\alpha(a \wedge b))$. It follows that $x \in I(a) \cap I(b)$, Thus $I(a \wedge b) \subseteq I(a) \cap I(b)$. □

Corollary 2.3. Let M be a unitary A-module, $I \in Id_A M$ and $a, b \in A - I$. Then $I(a \wedge b) = I(a) \cap I(b)$.

Proof. Since M is a unitary A-module, by Proposition 1.1, it is clear that $I \in Id(M)$, so $I(a \wedge b) = I(a) \cap I(b)$ [11]. □

We recall that if $h : M_1 \to M_2$ is an A-module homomorphism, then $ker(h) = \{x \in M_1 : h(x) = 0\}$ is an A-ideal of M_1 [5].

Lemma 2.1. Let M, N be MV-modules over a PMV-algebra A and $f : M \longrightarrow N$ be an A-module homomorphism. Then the following properties hold:
(i) For each ideal $J \in Id_A(N)$, the set $f^{-1}(J) = \{x \in M : f(x) \in J\}$ is an ideal of A. Thus, in particular, $ker(f) \in Id_A(M)$,
(ii) $f(x) \leq f(y)$ if and only if $x \ominus y \in ker(f)$,
(iii) f is injective if and only if $ker(f) = \{0\}$,
(iv) $ker(f) \neq M$ if and only if N is nontrivial.

The well-known isomorphism theorems have corresponding versions for MV-modules. We mention only the first and the second isomorphism theorem.

Theorem 2.1. *(The first isomorphism theorem)* If M and N are two MV-modules and $f : M \longrightarrow N$ is an A-module homomorphism, then $M/ker(f)$ and $Im(f)$ are isomorphic MV-modules.

Theorem 2.2. *(The second isomorphism theorem)* If M is an MV-module and I, J are two A-ideals such that $I \subseteq J$, then $(M/I)/P_I(J)$ and M/J are isomorphic MV-module, such that $P_I : M \to M/I$ is the quotient module of M.

Proposition 2.2. If $\sim$ is a congruence relation on M, then $I_\sim = \{x \in M : \quad x \sim 0\} \in Id_A(M)$ and $x \sim y$ if and only if $d(x, y) \sim 0$ [11].

Proposition 2.3. Let I be an A-ideal of M and $\sim$ be a congruence relation on M. The assignment $I \rightsquigarrow \sim_I$ is a bijection from the set $Id_A(M)$ of A-ideals of M onto the set of congruences on M; more precisely, the function $\alpha : Id_A(M) \to Con(M)$ defined by $\alpha(I) = \sim_I$ is an isomorphism of partially ordered sets [11].

3. Prime A-ideals in an MV-module

In the sequel A is a PMV-algebra and M is an MV-module.

Definition 3.1. Let N be an A-ideal of M.

$$(N : M) = \{r \in A : rM \subseteq N\}$$

such that $rM = \{rm : m \in M\}$.

Definition 3.2. Let N be an A-ideal. We denote *annihilator* of N by $Ann_A(N)$, which is defined as $Ann_A(N) = \{r \in A : rN = 0\}$.

Using Lemma 3.11 from [5], we obtain more properties of MV-modules.

Lemma 3.1. The following properties *hold for any* $x, m \in M$ *and* $\alpha, \beta \in A$:
(a) $\alpha m + (\beta m)^* \geq (\alpha + \beta^*)m$,
(b) $(\alpha \odot \beta)m \geq \alpha m \odot \beta m$,
(c) $(\alpha x)^* \odot (\beta x) \leq (\alpha^* \odot \beta)x$,
(d) $d(\alpha x, \beta x) \leq d(\alpha, \beta)x$,
(e) $(\alpha \oplus \beta)x \leq \alpha x \oplus \beta x$.

Proof. (a) by Lemma 1.4 (e), we have $\beta^* m \leq (\beta m)^*$, hence

$$\alpha m + (\beta m)^* \geq \alpha m + \beta^* m = (\alpha + \beta^*)m.$$

(b) Since $\alpha \odot \beta \leq \alpha, \beta$, $(\alpha \odot \beta)m \leq \alpha m, \beta m$. It follows that $(\alpha \odot \beta)m = (\alpha \odot \beta)m \wedge \alpha m = [(\alpha \odot \beta)m + (\alpha m)^*] \odot (\alpha m)$, using (a), we get $(\alpha \odot \beta)m \geq ((\alpha \odot \beta) + \alpha^*)m \odot (\alpha m) = (\alpha^* \vee \beta)m \odot \alpha m \geq \beta m \odot \alpha m$.

(c) Since $\alpha, \beta \leq \alpha \vee \beta$, by Lemma 1.4 (h), we get that $\alpha x \vee \beta x \leq (\alpha \vee \beta)x$. Thus, we have

$$((\alpha x) \odot (\beta x)^*) + \beta x = \alpha x \vee \beta x \leq (\alpha \vee \beta)x = ((\alpha \odot \beta^*) + \beta)x = (\alpha \odot \beta^*)x + \beta x.$$

Since cancellation rule holds in it [5], the desired inequality is straightforward.

(d) $d(\alpha x, \beta x) = [\alpha x \odot (\beta x)^*] \oplus [(\alpha x)^* \odot \beta x]$ by using (c), we get that

$$d(\alpha x, \beta x) \leq x(\alpha \odot \beta^*) + x(\alpha^* \odot \beta) = ((\alpha \odot \beta^*) + (\alpha^* \odot \beta))x = d(\alpha, \beta)x.$$

(e) By using (c) and Lemma 1.4 (h), we get that

$$(\alpha \oplus \beta)x \odot (\alpha x)^* \leq ((\alpha \oplus \beta) \odot \alpha^*)x = (\alpha^* \wedge \beta)x \leq \beta x.$$

It follows that $(\alpha \oplus \beta)x = (\alpha \oplus \beta)x \vee \alpha x \leq [(\alpha \oplus \beta)x \odot (\alpha x)^*] \oplus \alpha x \leq \beta x \oplus \alpha x$. $\square$

Proposition 3.1. Let N be an A-ideal of M. Then $Ann_A(N)$ is a $\cdot$-ideal of a PMV-algebra A.

Proof. Suppose that $a, b \in A$ such that $a \leq b$, and $b \in Ann_A(N)$, then $a \leq b$ and $bx = 0$ for every $x \in N$, it follows from Lemma 1.4 (h), $ax \leq bx$ and $bx = 0$, then $ax = 0$, for every $x \in N$. Hence $aN = 0$. Therefore $a \in Ann_A(N)$.

If $a, b \in Ann_A(N)$, then $aN = 0$ and $bN = 0$. By Lemma 3.1 (e), for every $x \in N$, we have

$$(a \oplus b)x \leq ax \oplus bx = 0.$$

So $(a \oplus b)N = 0$, hence $a \oplus b \in Ann_A(N)$.

Let $\alpha \in A$, $r \in Ann_A(N)$. We show that $\alpha \cdot r \in Ann_A(N)$. Since $r \in Ann_A(N)$, it follows that $rN = 0$ or for every $x \in N$, $\quad rx = 0$. Now, we have

$$(\alpha \cdot r)x = \alpha(rx) = \alpha 0 = 0;$$

for every $x \in N$, then $\alpha \cdot r \in Ann_A(N)$. Therefore $Ann_A(N)$ is a $\cdot$-ideal of A. □

Remark 3.1. If N is an A-ideal of a MV-module M, then $(N : M) = Ann_A(M/N)$. Hence $(N : M)$ is a $\cdot$-ideal of A.

Definition 3.3. Let N be an A-ideal of M and $T(N) = \{n \in N : \exists 0 \neq a \in A; an = 0\}$. Then $T(N)$ is called *torsion A-ideal* of N.

Definition 3.4. Let P be a $\cdot$-ideal of A. P is called a *$\cdot$-prime* if (i) $P \neq A$, (ii) for every $a, b \in A$, if $a \cdot b \in P$, then $a \in P$ or $b \in P$.

Remark 3.2. Let N be an A-ideal of M and $\{0\}$ be a $\cdot$-prime ideal of A. Then $T(N)$ is an A-ideal of M.

Proof. (i) Let $n, m \in T(N)$. Then there exist $0 \neq a, b \in A$ such that $an = 0$, $\quad bm = 0$. We consider $c := a \cdot b \neq 0$, by Lemma 1.4 (j), we have $(a \cdot b)(m \oplus n) \leq (a \cdot b)m \oplus (a \cdot b)n = a(bm) \oplus b(an) = 0$. Then $(a \cdot b)(m \oplus n) = 0$. Hence $m \oplus n \in T(N)$.
(ii) For every $m, n \in M$ such that $m \leq n$, and $n \in T(N)$, we show that $m \in T(N)$. Since $n \in T(N)$, there exists $0 \neq a \in A$; $an = 0$. Since $m \leq n$, by Lemma 1.4 (g), we get that $am \leq an$ and $an = 0$, it follows that $am = 0$, so $m \in T(N)$.
(iii) Let $m \in T(N)$ and $a \in A$. Then there exists $0 \neq b \in A$; $\quad bm = 0$, $a(bm) = a0 = 0$, by Lemma 1.4 (b), $a(bm) = 0$ or $b(am) = (b \cdot a)m = (a \cdot b)m = a(bm) = 0$. Therefore $am \in T(N)$.

□

Example 3.1. Let $\Omega = \{1, 2\}$ and $\mathcal{M} = \mathcal{A} = \mathcal{P}(\Omega) = \{\{1\}, \{2\}, \{1, 2\}, \emptyset\}$. Then $\mathcal{A}$ is a PMV-algebra with $\oplus = \cup$, and $\odot = \cdot = \cap$. Hence $\mathcal{M}$ is an MV-module over $\mathcal{A}$ with the external operation defined by $AX := A \cap X$ for every $A \in \mathcal{A}$ and $X \in \mathcal{M}$ [5]. Clearly, $I = \{\emptyset\}$ is an $\mathcal{A}$-ideal. We have
$T(\mathcal{M}) = \{B \in \mathcal{M} : \exists \emptyset \neq A \in \mathcal{A}, A \cap B = \emptyset\} = \{\emptyset, \{1\}, \{2\}\}$, $\quad T(\emptyset) = \{B = \emptyset : \exists \phi \neq A \in \mathcal{A}; A \cap B = \emptyset\} = \{\emptyset\}$,
$Ann_{\mathcal{A}}\{\emptyset\} = \{A \in \mathcal{A} : A\emptyset = \emptyset\} = \mathcal{A}$, $\quad Ann_{\mathcal{A}}(\mathcal{M}) = \{A \in \mathcal{A} : A\mathcal{M} = \emptyset\} = \{\emptyset\}$,
$(\emptyset : \mathcal{M}) = \{A \in \mathcal{A} : A\mathcal{M} \subseteq \emptyset\} = \{\emptyset\}$, $(\mathcal{M} : \mathcal{M}) = \{A \in \mathcal{A} : A\mathcal{M} \subseteq \mathcal{M}\} = \mathcal{A}$.
Also $I_1 = \{\emptyset, \{1\}\}$ is an $\mathcal{A}$-ideal of $\mathcal{M}$, so $Ann_{\mathcal{A}}(I_1) = \{B \in \mathcal{A} : BI_1 = \emptyset\} = \{\emptyset, \{2\}\}$,
$T(I_1) = \{C \in I_1 : \exists \emptyset \neq A \in \mathcal{A}; CA = \emptyset\} = \{\{1\}, \emptyset\}$ and
$(I_1 : \mathcal{M}) = \{B \in \mathcal{A} : B\mathcal{M} \subseteq I_1\} = \{\emptyset, \{1\}\}$.

Example 3.2. In Example 2.1, $A = \Gamma(M_2(\mathbb{R}), v)$ is a PMV-algebra such that $v = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$. $P = \{\overline{0}\}$ is not a $\cdot$-prime ideal of A, if $C = \begin{pmatrix} 0 & 1/2 \\ 0 & 1/2 \end{pmatrix}$ and $D = \begin{pmatrix} 1/2 & 1/2 \\ 0 & 0 \end{pmatrix}$, then

$$C.D = \begin{pmatrix} 0 & 1/2 \\ 0 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} 1/2 & 1/2 \\ 0 & 0 \end{pmatrix} = \overline{0}.$$

But $C \neq \overline{0}$ and $D \neq \overline{0}$.

It is well known that if A is a unital PMV-algebra, then $x \cdot y \leq x \wedge y$, for any $x, y \in A$. From this, we can prove the following lemma:

Lemma 3.2. if A is a unital PMV-algebra and P is a $\cdot$-prime ideal of A, then P is a prime ideal of A.

Proof. Let P be a $\cdot$-prime ideal of A. Suppose that $x \wedge y \in P$, for any $x, y \in A$. It follows from Lemma 1.3 (b), $x \cdot y \leq x \wedge y \in P$ and P is a $\cdot$-ideal, so $x \cdot y \in P$. Since P is a $\cdot$-prime ideal, hence $x \in P$ or $y \in P$. Thus P is a prime ideal of A. □

We recall that a product MV-algebra A is said to be an MVF-algebra if for all $a, b, c \in A$,

$$a \wedge b = 0 \text{ implies } (a \cdot c) \wedge b = 0 = (c \cdot a) \wedge b.$$

Also, any linearly ordered PMV-algebra is an MVF-algebra [6].

Theorem 3.1. Let A be a unital PMV-algebra and P be a $\cdot$-prime ideal of A. Then A/P is a chain PMV-algebra.

Proof. By Lemma 3.2, we deduce that P is a prime ideal of A. Then $x \odot y^* \in P$ or $y \odot x^* \in P$, for any $x, y \in A$. It follows from Remark 1.1, $x/P \leq y/P$ or $y/P \leq x/P$. Hence A/P is a chain PMV-algebra. □

By the above theorem, we imply that if P is a $\cdot$-prime ideal of unital A, Then A/P is a MVF-algebra.
The following example, we show that the converse of above theorem is not true.

Example 3.3. Let $l_3 = \{0, 1, 2\}$ be a linearly ordered set (chain). l_3 is an MV-algebra with operations $\wedge = min$, $x \oplus y = min\{2, x+y\}$ and $x \odot y = max\{0, x+y-2\}$, for every $x, y \in A$ which is not a Boolean algebra. Also, A is a PMV-algebra by operation $\cdot$ such that $x \cdot y = 0$, for every $x, y \in A$. Clearly, $\cdot$ is associative and if $x+y$ is defined i.e, $x \leq y^* = 2 - y$ or $x + y \leq 2$, then $x \cdot z + y \cdot z \leq 2$, $z \cdot x + z \cdot y \leq 2$ and $(x+y) \cdot z = x \cdot z + y \cdot z$ and $z \cdot (x+y) = z \cdot x + z \cdot y$. Let $P = \{0\}$. Then $A/\{0\} \simeq A$ is an MVF-algebra but $P = \{0\}$ is not $\cdot$-prime ideal of A. Since $2 \cdot 1 \in P$ but $2 \neq 0$ and $1 \neq 0$.

Definition 3.5. Let M be an A-module. Then an A-ideal P of an MV-module M is a *prime A-ideal*, if (i) $P \neq M$ (ii) for every $\alpha \in A$, $x \in M$ if $\alpha x \in P$, then $x \in P$ or $\alpha \in (P : M)$.

Example 3.4. Let $A = \{0, a, b, 1\}$, where $0 < a, b < 1$. Define $\odot$, $\oplus$ and $*$ as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\oplus$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

$*$	0	a	b	1
	1	b	a	0

Then $(A, \oplus, \odot, *, 0, 1)$ is an MV-algebra. If we define $\alpha x := \alpha \cdot x = 0$ for any $\alpha \in A$ and $x \in A$, them A becomes an A-module. It is clear that $P_1 = \{0, a\}$ and $P_2 = \{0, b\}$ are prime A-ideals of A. Let $\alpha x = 0 \in P_1$. If $x \in P$, then the proof is clear. If $x \notin P$, then $\alpha \in (P : M)$. Since $\alpha M = \{0\} \subseteq P_1$. Hence P_1 is a prime A-ideal of A. Similarly P_2 is a prime A-ideal of A.

Example 3.5. Let $A = \{0, 1, 2\}$ be a linearly ordered set (chain). A is an MV-algebra with operations $\wedge = min$, $x \oplus y = min\{2, x+y\}$ and $x \odot y = max\{0, x+y-2\}$, for every $x, y \in A$ [11]. Also, A is PMV-algebra with the following operations:

$\oplus$	0	1	2
0	0	1	2
1	1	2	2
2	2	2	2

$\cdot$	0	1	2
0	0	0	0
1	0	0	0
2	0	0	1

$*$	0	1	2
	2	1	0

Clearly, A ia a PMV-algebra and A becomes an A-module over A with the external operation defined by $\alpha x = \alpha \cdot x$, for any $\alpha \in A$ and $x \in A$. Then $P = \{0\}$ is not a prime A-ideal. Since $2 \cdot 1 \in P$ and $1 \notin P$, also for $\alpha = 2$ and $x = 1$, we have $2M \nsubseteq P$, because $2 \cdot 2 = 1 \notin P$. Hence P is not a prime A-ideal of M.

Example 3.6. Let M be $\Gamma(\mathbb{R}^2, u)$ such that $u = (1, 1)$, $A = \Gamma(M_2(\mathbb{R}), v)$ and $v = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$. By Example 1.1, M is an A-module such that $M = \Gamma(\mathbb{R}^2, u) = [(0, 0), (1, 1)]$ and $A = \Gamma(M_2(\mathbb{R}), v) = [0, \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}]$. Then $Id_A(M) = \{(0, 0), M\}$, but M has not prime A-ideal. If $P = (0, 0)$ is a prime A-ideal, then $B = \begin{pmatrix} 1/2 & 0 \\ 1/2 & 0 \end{pmatrix} \in A$, $(0, 1/2) \in M$, we have $\begin{pmatrix} 1/2 & 0 \\ 1/2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in P$, but $\begin{pmatrix} 1/2 & 0 \\ 1/2 & 0 \end{pmatrix} M \nsubseteq P$ and $(0, 1/2) \notin P$. Let $m = (1/2, 1/2) \in M$. Then $\begin{pmatrix} 1/2 & 0 \\ 1/2 & 0 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} =$

$\begin{pmatrix} 1/4 \\ 1/4 \end{pmatrix} \notin P$ and $(0, 1/2) \notin P$. Therefore, $(0,0)$ is not a prime A-ideal of a MV-module M.

We denoted that $Z_A(M) = \{r \in A : \exists m \in M - \{0\}; rm = 0\}$.

Proposition 3.2. Let M be an A-module. Then $P = \{0\}$ is a prime A-ideal of M if and only if $Ann_A(M) = Z_A(M)$.

Proof. Let $P = \{0\}$ be a prime A-ideal. suppose that $a \in Ann_A(M)$, then $am = 0$ for every $m \in M$. It follows that $a \in Z_A(M)$. Now, let $a \in Z_A(M)$. Then for some $0 \neq m \in M$, $am = 0 \in P$, by hypothesis, we deduce that $m \in P = \{0\}$ or $a \in (P : M)$. Since $m \neq 0$, hence $a \in (P : M)$, it follows that $aM = 0$. Thus, $a \in Ann_A(M)$.
Conversely, let $Ann_A(M) = Z_A(M)$. We show that $P = \{0\}$ is a prime A-ideal. For every $a \in A$, $m \in M$, suppose that $am = 0$, $m \neq 0$, then $a \in Z_A(M) = Ann_A(M)$. It follows that $aM = 0$ or $a \in (\{0\} : M)$.

□

Remark 3.3. Let A be a unital PMV-algebra. Then every $\cdot$-ideal of A is a $\cdot$-prime if and only if it is a prime A-ideal of an A-module A.

Proposition 3.3. Let $h : M \to M'$ be an onto A-module homomorphism. If P is a prime A-ideal of M', then $h^{-1}(P)$ is a prime A-ideal of M.

Theorem 3.2. Let $h : M \to M'$ be an onto A-module homomorphism. Then prime A-ideals of M' and prime A-ideals of M that contain $kerh$ are in one to one correspondence.

Proof. Let ψ: $T \to S$, where $T = \{Q : Q \text{ is prime } A\text{-ideal of } M'\}$ and $S = \{P : P \text{ is a prime } A\text{-ideal of } M \text{ such that } kerh \subseteq P\}$. We define $\psi(Q) := h^{-1}(Q)$. By Proposition 3.3, ψ is well defined. Also ψ is injective. Let $Q \in ker\psi$. Then $\psi(Q) = 0$, hence $h^{-1}(Q) = 0$, it follows that $Q = h(h^{-1}(Q)) = h(0) = 0$. Therefore $Q = 0$, so ψ is injective.

Now, we show that ψ is a surjective. Let $P \in S$ or on the other hand, P be A-ideal of M that contains $kerh$. We claim that there exists a prime A-ideal $Q = h(P)$ of M' such that $\psi(Q) = \psi(h(P)) = P$.
Firstly, $Q = h(P)$ is an ideal of M'.

(i) Suppose that $a, b \in h(P)$, then $a = h(x)$ and $b = h(y)$ for some $x, y \in M$. $a \oplus b = h(x) \oplus h(y) = h(x \oplus y) \in h(P)$
(ii) Suppose that $a \in M'$, $b \in h(P)$ such that, $a \leq b$ and $b \in h(P)$, then $b = h(x)$, for some $x \in P$, and $a \in M'$, h is surjective, there exists $y \in M$ such that $h(y) = a$; but $h(y) \leq h(x)$, hence by Lemma 1.2, $h(y) \odot (h(x))^* = 0$ or $y \odot x^* \in kerh \subseteq P$, then $(y \odot x^*) \oplus x \in P$ or $x \vee y \in P$ and $y \leq x \vee y$, hence, $y \in P$. Therefore, $a = h(y) \in h(P)$.

(iii) Let $x \in h(P)$ and $a \in A$.
Since $x \in h(P)$, then $x = h(b)$ for some $b \in P$ and $a \in A$, hence $ab \in P$ and

$$ax = ah(b) = h(ab) \in h(P).$$

It follows that $h(P)$ is A-ideal of M'.
Second, we show that $Q = h(P)$ is a prime A-ideal of M'.

(i) $h(P)$ is a proper A-ideal of M'. If $h(P) = M' = h(M)$.
Hence for $x \in M$, we have $h(x) \in h(M) = h(P)$, hence $h(x) = h(y)$ for some $y \in P$. Therefore, $h(x) \leq h(y)$ and $h(y) \leq h(x)$. By Lemma 1.2, $h(x) \odot (h(y))^* = 0$, hence $x \odot y^* \in kerh \subseteq P$ and $y \in P$, then $(x \odot y^*) \oplus y \in P$ or $x \vee y \in P$. Since $x \leq x \vee y$ and $x \vee y \in P$, then $x \in P$. Thus, $M \subseteq P$. It follows that $M = P$ which is a contradiction.
(ii) Let $a \in A$, $x \in M'$ such that $ax \in h(P)$, we show that $x \in h(P)$ or $a \in (h(P) : M')$. Since $x \in M'$, there exists $y \in M$ such that $h(y) = x$. Also, since $ax \in h(P)$, $ax = h(t)$ for some $t \in P$, we have $ax = ah(y) = h(ay) = h(t)$, so $h(ay) \leq h(t)$, by Lemma 1.2, $(ay) \odot t^* \in kerh \subseteq P$ and $t \in P$, therefore $((ay) \odot t^*) \oplus t \in P$, then $t \vee (ay) \in P$ and $ay \leq t \vee (ay)$ and P is A-ideal, hence $ay \in P$ for some $a \in A$, $\quad y \in M$; but P is a prime A-ideal of M, then $y \in P$ or $a \in (P : M)$. If $y \in P$, then $h(y) \in h(P)$, if we have $a \in (P : M)$, then $aM \subseteq P$. It follows that $h(aM) \subseteq h(P)$. This implies $ah(M) \subseteq h(P)$, hence $aM' \subseteq h(P)$ or $a \in (h(P) : M')$. Therefore, $h(y) \in h(P)$ or $a \in (h(P) : M')$. Thus, $h(P)$ is a prime A-ideal of M'.
Now, we show that $\psi(Q) = \psi(h(P)) = h^{-1}(h(P))$. Let $x \in h^{-1}(h(P))$. Then $h(x) \in h(P)$, hence $h(x) = h(y)$ for some $y \in P$, $h(x) \leq h(y)$, by Lemma 1.2, $x \odot y^* \in kerh \subseteq P$ and $y \in P$, it follows that $x \vee y = (x \odot y^*) \oplus y \in P$ and $x \leq x \vee y$, so $x \in P$. Thus, $h^{-1}(h(P)) \subseteq P$, so $h^{-1}(h(P)) = P$ or $\psi(h(P)) = P$, therefore ψ is surjective.

□

Theorem 3.3. *Let M be a unitary A-module and P an A-ideal of M. P is a prime A-ideal of M if and only if P is a prime $A/Ann(M)$-ideal of M.*

Proof. Let P be a prime A-ideal. We show that P is a prime $A/Ann(M)$-ideal of M. Firstly, M is a $A/Ann(M)$-module with operation $A/Ann(M) \times M \to M$ such that $(a/Ann(M), x) \to ax$ or $[a/Ann(M)]x = ax$, for every $a \in A, x \in M$.
(i) It well defined, since $a/Ann(M) = b/Ann(M)$, for every $a, b \in A$, by Remark 1.1, we have $d(a, b) \in Ann(M)$, hence $d(a, b)M = 0$. It follows that $d(a, b)1_M = 0$. Hence, $d(a, b) = 0$, by Lemma 1.1, $a = b$. For $x, y \in M, a, b \in A$:
(1) If $x + y$ is defined in M, then we show that $[a/Ann(M)]x + [a/Ann(M)]y$ is defined in M or $ax + ay$ is defined in M.
Since M is an A-module and $x + y$ is defined in M, so $ax + ay$ is defined in M.
(2) If $a/Ann(M)$,$b/Ann(M) \in A/Ann(M)$ such that $a/Ann(M) + b/Ann(M)$ is defined in $A/Ann(M)$.
We prove that $[a/Ann(M)]x+[b/Ann(M)]x$ is defined M. If $a/Ann(M)+b/Ann(M)$

is defined, then $a/Ann(M) \leq (b/Ann(M))^* = b^*/Ann(M)$, by Remark 1.1, $a \odot (b^*)^* \in Ann(M)$, so $a \odot b \in Ann(M)$ or $(a \odot b)M = 0$ or $(a \odot b)1_M = 0$ or $a \odot b = 0$. Therefore, $a \leq b^*$. i.e., $a + b$ is defined in A, since M is an A-module, for every $x \in M$ $ax + bx$ is defined in M. Thus, $[a/Ann(M)]x + [b/Ann(M)]x$ is defined M.
(3) $(a/Ann(M) \cdot b/Ann(M))x = [(a \cdot b)/Ann(M)]x = (a \cdot b)x = a(bx) = (a/Ann(M))(bx) = (a/Ann(M))[(b/Ann(M))x]$.
Now, let P be a prime A-ideal of M. Then, for every $a/Ann(M) \in A/Ann(M)$ and $x \in M$ such that $[a/Ann(M)]x \in P$, then $ax \in P$, since P is a prime A-ideal of M, then $a \in (P : M)$ or $x \in P$. Consider $a \in (P : M)$, then $aM \subseteq P$ and for every $x \in M$, $ax \in P$ if and only if $(a/Ann(M))x \in P$, for any $x \in M$, if and only if $(a/Ann(M))M \subseteq P$ if and only if $a/Ann(M) \in (P : M)$. Hence, $x \in P$ or $a/Ann(M) \in (P : M)$. We deduce that P is a prime $A/Ann(M)$-ideal.
Conversely, let P be prime $A/Ann(M)$-ideal and for every $a \in A, x \in M$ such that $ax \in P$. We show that $x \in P$ or $a \in (P : M)$.
Let $ax \in P$. Then $(a/Ann(M))x \in P$, by hypothesis, $x \in P$ or $a/Ann(M) \in (P : M)$, so $x \in P$ or $(a/Ann(M))M \subseteq P$, hence $x \in P$ or $aM \subseteq P$. Therefore, $x \in P$ or $a \in (P : M)$. □

Proposition 3.4. Let N be an A-ideal of a MV-module M such that $(N : M)$ is a maximal $\cdot$-ideal of A. Then N is a prime A-ideal of M.

Proof. Let $am \in N$ and $a \notin (N : M)$, for every $a \in A, m \in M$. Since $(N : M)$ is a maximal, then $(N : M) \vee (a] = A$, hence there exist $t \in (N : M)$ and $s \in (a]$, such that $1 = t \oplus s$. Hence by Lemma 3.1 (*e*), we have $m = m(t \oplus s) \leq mt \oplus ms$. Since $t \in (N : M)$, so $tM \subseteq N$, hence for every $m \in M$, $tm \in N$. Also since $s \in (a]$, hence for some integer $n \geq 0$, $s \leq na$, then, by Lemma 1.4 (*c*) $sm \leq (na)m = n(am)$ and by hypothesis, $n(am) \in N$, it follows that $sm \in N$, so $m \leq tm \oplus sm \in N$. Thus, N is a prime A-ideal of M. □

Theorem 3.4. Let M be a unitary A-module. Then A-ideal N of a MV-module M is a prime if and only if $P = (N : M)$ is a $\cdot$-prime ideal of A, and A/P-module M/N is a torsion free.

Proof. Let N be a prime A-ideal of M. We claim that $(N : M)$ is a $\cdot$-prime ideal of A. Firstly $(N : M)$ is a proper ideal. If $(N : M) = A$, then $1 \in (N : M)$, it follows that $1M \subseteq N$, so $M = N$. Which is a contradiction. Now, let $a, b \in A$ such that $a \cdot b \in (N : M)$ and $a \notin (N : M)$. Then $(a \cdot b)M \subseteq N$ and $aM \nsubseteq N$, it follows that for every $m \in M$, $(a \cdot b)m \in N$ and there exists $x \in M$ such that $ax \notin N$, also we have $b(ax) = (a \cdot b)x \in N$. Hence by hypothesis, $b \in (N : M)$, thus $(N : M)$ is a $\cdot$-prime ideal of A.
Now, we show that M/N is A/P-module, by operation: $(a/P, m/N) \to (am)/N$.
We prove that it is well defined, for every $a_1, a_2 \in A$ and $m_1, m_2 \in M$. Suppose that $a_1/P = a_2/P$, $m_1/N = m_2/N$, then by Remark 1.1, we have

$$d(a_1, a_2) \in P \quad and \quad d(m_1, m_2) \in N, \qquad (1)$$

this results $d(a_1, a_2) \in P = (N : M)$, it follows that $d(a_1, a_2)M \subseteq N$, hence $d(a_1, a_2)1 \in N$, then

$$d(a_1, a_2) \in N, \qquad (2)$$

by Lemma 1.4 (k) and Lemma 3.1 (d), we have:

$$d(a_1m_1, a_2m_2) \leq d(a_1m_1, a_1m_2) \oplus d(a_1m_2, a_2m_2)$$

$$\leq a_1d(m_1, m_2) \oplus d(a_1, a_2)m_2$$

and by $(1),(2)$ we deduce that $d(a_1m_1, a_2m_2) \in N$.
(i) If $a_1/P + a_2/P$ is defined in A/P,
we show that $(a_1m)/N+(a_2m)/N$ is defined in M/N, for every $a_1, a_2 \in A, m \in M$. If $a_1/P+a_2/P$ is defined in A/P, then $a_1/P \leq (a_2/P)^*$, it follows that by Remark 1.1, $a_1 \odot a_2 \in P = (N : M)$, then $(a_1 \odot a_2)M \subseteq N$, so $(a_1 \odot a_2)m \in N$, for any $m \in M$ but by Lemma 3.1 (b), we have $a_1m \odot a_2m \leq (a_1 \odot a_2)m$. Thus, $a_1m \odot a_2m \in N$. So by Remark 1.1, we have $(a_1m)/N \leq [(a_2m)/N]^*$, therefore, $(a_1m)/N + (a_2m)/N$ is defined in M/N, for any $m \in M$ and $a_1, a_2 \in A$.
(ii) If $m_1/N + m_2/N$ is defined in M/N, then we show that $(am_1)/N + (am_2)/N$ is defined in M/N.
Let $m_1/N + m_2/N$ be defined in M/N. Then $m_1/N \leq (m_2/N)^*$, it follows from Remark 1.1, $m_1 \odot m_2 \in N$, we have by Lemma 3.1 (b), $am_1 \odot am_2 \leq a(m_1 \odot m_2)$, then $am_1 \odot am_2 \in N$, so by Remark 1.1, we have $(am_1)/N \leq [(am_2)/N]^*$. Thus, $(am_1)/N + (am_2)/N$ is defined in M/N.
(iii) For any $a_1, a_2 \in A$ and $m \in M$, we have: $(a_1/P \cdot a_2/P)(m/N) = [(a_1 \cdot a_2)/P](m/N) = [(a_1 \cdot a_2)m]/N = [a_1(a_2m)]/N = (a_1/P)[(a_2/P)(m/N)]$. Thus, M/N is an A/P-module.

Now, we prove that M/N is torsion free A/P-module. For every $a \in A$, $m \in M$, such that $(a/P)(m/N) = 0/N$, $a/P \neq 0/P$. Then $(am)/N = 0/N$, by Remark 1.1, it follows that $d(am, 0) \in N$, so by Lemma 1.1, we have $am \in N$. Now, let $m/N \neq 0/N$ or $m = d(m, 0) \notin N$. Since P is a prime A-ideal of M, hence $a \in (N : M) = P$, so $a = d(a, 0) \in P$, it follows that $a/P = 0/P$, which is a contradiction. Thus, M/N is a torsion free.
Conversely, we prove that N is a prime A-ideal. Let $am \in N$ and $a \notin (N : M) = P$ for every $m \in M$, $a \in A$. Then $(a/P)(m/N) = (am)/N = 0/N, a/P \neq 0/P$, by hypothesis, since M/N is torsion free A/P-module, it follows that $m/N = 0/N$, then $m \in N$. Also, suppose that $N = M$, thus $P = (N : M) = A$, which is a contradiction. Thus N is a prime A-ideal of M. □

Proposition 3.5. Let N be a proper A-ideal of a unitary MV-module M such that $(N : M) = P$. Then the following are equivalent:
(a) N is a prime A-ideal of M,
(b) M/N is a torsion free A/P-module,
(c) For every $r \in A - P$, $N = \{m \in M : rm \in N\}$,
(d) For every ·-ideal J of A such that $J \nsubseteq P$, $N = \{m \in M : Jm \subseteq N\}$,
(e) For every $m \in M - N$, $P = (N : (m])$,

(f) For every A-ideal L of M such that $L \nsubseteq N$, $P = (N : L)$,
(g) For every $m \in M - N$, $Ann_A(m/N) = P$,
(h) $Z_A(M/N) = P$.

Proof. $(a) \Rightarrow (b)$ is straightforward by Theorem 3.4.

$(b) \Rightarrow (c)$ Let $T = \{m \in M : rm \in N\}$ for every $r \in A - P$. Suppose that $m \in T$, then $rm \in N$, it follows that by Lemma 1.1, $rm = d(rm, 0) \in N$, so by Remark 1.1, we have $(r/P)(m/N) = (rm)/N = 0/N$, hence by hypothesis, since M/N is torsion free, so $m/N = 0/N$, it follows that $m \in N$. Thus, $N = \{m \in M : rm \in N\}$.

$(c) \Rightarrow (d)$ Let J be a $\cdot$-ideal of A such that $J \nsubseteq P$. Then there exists $r \in J - P$. We show that $\{m \in M : Jm \subseteq N\} \subseteq N$. Let $m \in M$ such that $Jm \subseteq N$. Hence $rm \in N$ and $r \notin P$. By (c), we deduce that $m \in N$. Thus $N = \{m \in M : Jm \subseteq N\}$.

$(d) \Rightarrow (e)$ Let $m \in M - N$ and $r \in (N : (m])$. Suppose that $r \notin P$, consider $J = (r]$, then $Jm \subseteq N$ and $J \nsubseteq P$ by hypothesis, we have $m \in N$, which is a contradiction. So $r \in P$, hence $(N : (m]) \subseteq P$. Now, let $r \in P = (N : M)$. Then $rM \subseteq N$, so $rm \in N$ for every $m \in M$, we prove that $r \in (N : (m])$ or $r(m] \subseteq N$. Suppose that $t \in (m]$, hence $t \leq nm$ for some integer $n \geq 0$, so by Lemma 1.4 (c), $rt \leq r(nm) = n(rm)$ and $rm \in N$, it follows that $rt \in N$ or $r(m] \subseteq N$ or $r \in (N : (m])$.

$(e) \Rightarrow (f)$ Let $N \neq L \subseteq M$. Then there exists $m \in L - N$, then by (e), we have $(N : (m]) = P$. Now since $m \in L$ and $(N : M) = P$, hence $(N : L) = P$.

$(f) \Rightarrow (g)$ Let $m \in M - N$. Suppose that $r \in Ann_A(m/N)$, then $r(m/N) = 0/N$, it follows that by Remark 1.1, $rm \in N$, consider that $L = (m]$, by hypothesis, we deduce that $(N : (m]) = P$. Let $r \in (N : (m]) = P$. We show that $r(m] \subseteq N$, suppose that $t \in (m]$, so $t \leq nm$ for some integer $n \geq 0$, hence $rt \leq r(nm)$, by Lemma 1.4 (c), $rt \leq n(rm)$ and we have $rm \in N$, so $rt \in N$ and since $(N : (m]) = P$, hence $r \in P$. Therefore, $Ann(m/N) \subseteq P$.

Conversely, let $r \in P$. Consider $L = (m]$, we deduce by (f), $P = (N : (m])$. It follows that $r \in (N : (m])$, then $r(m] \subseteq N$. Hence $rm \in N$ then by Lemma1.1 and Remark 1.1, we have $d(rm, 0) \in N$ or $(rm)/N = 0/N$ or $r(m/N) = 0/N$, hence $r \in Ann_A(m/N)$. Therefore, $P \subseteq Ann_A(m/N)$. Thus, $Ann_A(m/N) = P$.

$(g) \Rightarrow (h)$ Let

$$Z_A(M/N) = \{r \in A : r(m/N) = 0/N \quad \text{for some} \quad m/N \in M/N \quad \textit{and} \quad m/N \neq 0/N\}$$

$$= \{r \in A : d(rm, 0) \in N \quad \text{for some} \quad m \in M - N\}$$

$$= \{r \in A : rm \in N \quad \text{for some} \quad m \in M - N\}.$$

Now, let $r \in Z_A(M/N)$. Then $r \in Ann_A(m/N)$ but we deduce by (g), $Ann_A(m/N) = P$, hence $r \in P$.

Conversely, let $m \in M - N$ and $r \in P$. This implies by (g), $Ann_A(m/N) = P$, so $r \in Ann_A(m/N)$. It follows that by Remark 1.1, we have $(rm)/N = 0/N$ or $d(rm, 0) \in N$ or $rm \in N$, thus, $r \in Z_A(M/N)$. Therefore, $P \subseteq Z_A(M/N)$.

$(h) \Rightarrow (a)$ Let $Z_A(M/N) = P$. Suppose that $r \in A$, $m \in M$ such that $rm \in N$ and $m \notin N$, by definition of $Z_A(M/N)$ and hypothesis, we deduce that $r \in P = (N : M)$. Thus, N is a prime A-ideal of M. □

4. Conclusion and future research

MV-modules over a PMV-algebra A and A-ideals in MV-modules are introduced by Di Nola, et.al. They proved equivalence between the category of lu-modules over (R, v) and the category of MV-modules over $\Gamma(R, v)$, where (R, v) is an lu-ring [5]. Also A. Dvurecenskij and A. Di Nola in [6] introduced the notions of PMV-algebras, MVF-algebras and $\cdot$-ideals in PMV-algebras. We introduced $\cdot$-prime ideals in PMV-algebras and investigated the relation between $\cdot$-prime ideals and MVF-algebras. We studied A-ideals in MV-modules and introduced the notion of prime A-ideals in an MV-module and annihilator of an A-ideal in an MV-module. We give some conditions on an A-ideal to become prime and proved that if $h : M \to N$ is an A-module homomorphism then all prime A-ideals of N and prime A-ideals of M that contains $kerh$ are in one to one correspondence.

In our future study of MV-modules, we are planning:

(1) to get more results on A-ideals.

(2) to define another types of A-ideals in M.

(3) to get more results on prime A-ideal.

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