

## UPPER SETS IN RESIDUATED LATTICES

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*In this paper, we investigate several properties of upper sets in residuated lattices and study the connection between filters and upper sets in residuated lattices. At last, the notion of Krull dimension of a residuated lattice is introduced.*

**Keywords:** Residuated lattice,  $G(RL)$ -algebra, Boolean algebra,  $G$ -filter, Implicative filter, Upper set

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### 1. Introduction

Residuated lattices, introduced by Ward and Dilworth in [12], include important classes of algebras such as  $BL$ -algebras, introduced by Hájek as the algebraic counterpart of Basic Logic [3], and  $MV$ -algebras, the algebraic setting for Łukasiewicz propositional logic[2].

The structure of the paper is as follows: In section 2 of the article we recall some definitions and facts about residuated lattices that we use in the sequel. In section 3, some properties of upper sets in connections with filters over residuated lattices is studied. This fact helps us to study the notion of Krull dimension of a residuated lattices is introduced in the last section.

### 2. Preliminaries

A residuated lattices ([9], [12]) is an algebra  $A = (A, \wedge, \vee, *, \rightarrow, 0, 1)$  with four binary operations  $\wedge, \vee, *, \rightarrow$  and two constants  $0, 1$  such that:

( $LR_1$ )  $A = (A, \vee, \wedge, 0, 1)$  is a bounded lattice,

( $LR_2$ )  $A = (A, *, 1)$  is a commutative monoid,

( $LR_3$ )  $*$  and  $\rightarrow$  form a adjoint pair, i.e,

$a * c \leq b$  if and only if  $c \leq a \rightarrow b$ , for all  $a, b, c \in A$ ,

A residuated lattices  $A$ , is called a  $G(RL)$ -algebra if  $x^2 = x$ , for all  $x \in A$  where  $x^2 = x * x$  [13].

**Lemma 2.1.**[8,11] In any residuated lattices  $A$ , the following relations hold for all  $x, y, z \in A$ :

(1)  $x * y \leq x, y$ ,

(2)  $1 \rightarrow x = x$ ,

(3)  $x \rightarrow x = 1$ ,

(4)  $x \leq y \rightarrow x$ ,

(5)  $x * (x \rightarrow y) \leq y$ ,

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- (6)  $x \leq (y \rightarrow (x * y))$ ,
- (7)  $x \leq y$  if and only if  $x \rightarrow y = 1$ ,
- (8)  $x \rightarrow (y \rightarrow z) = (x * y) \rightarrow z = y \rightarrow (x \rightarrow z)$ ,
- (9) If  $x \leq y$  then  $z \rightarrow x \leq z \rightarrow y$  and  $y \rightarrow z \leq x \rightarrow z$ ,
- (10)  $y \leq (y \rightarrow x) \rightarrow x$ ,
- (11)  $y \rightarrow x \leq (z \rightarrow y) \rightarrow (z \rightarrow x)$ ,
- (12)  $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$ ,
- (13)  $(x \rightarrow y) * (y \rightarrow z) \leq x \rightarrow z$ .
- (14)  $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$ .

In [9] there has been defined a filter of a residuated lattice to be a nonempty subset  $F$  of  $A$  such that (i)  $a * b \in F$ , for all  $a, b \in F$  and (ii)  $a \leq b$  and  $a \in F$  imply  $b \in F$ . A deductive system of a residuated lattice  $A$  is a nonempty subset  $D$  of  $A$  such that (i)  $1 \in D$  and (ii) If  $x \in D$  and  $x \rightarrow y \in D$ , then  $y \in D$  [7]. Note that a subset  $F$  of a  $BL$ -algebra  $A$  is a deductive system of  $A$  if and only if  $F$  is a filter of  $A$  [9].

**Definition 2.2.** [13] A subset  $F$  of residuated lattice  $A$  is called

- (1) a positive implicative filter of  $A$  if it  $1 \in F$  and  $x \rightarrow (y \rightarrow z) \in F$  and  $x \rightarrow y \in F$  imply  $x \rightarrow z \in F$ ,
- (2) a  $G$ -filter if it is a filter of  $A$  that satisfies the condition  $x^2 \rightarrow y \in F$  implies  $x \rightarrow y \in F$ ,
- (3) a Boolean filter of  $A$  if it is a filter of  $A$  that satisfies the condition  $x \vee x' \in F$ , for all  $x, y \in A$ , where  $x' = x \rightarrow 0$ .

**Theorem 2.3.** [13] In any residuated lattice  $A$ , the following assertions hold:

- (1) Let  $F$  be a subset of  $A$ . Then  $F$  is a positive implicative filter of  $A$  if and only if  $F$  is a  $G$ -filter of  $A$ ,
- (2)  $A$  is a  $G(RL)$ -algebra if and only if  $\{1\}$  is a  $G$ -filter of  $A$ ,
- (3)  $A$  is a Boolean algebra if and only if every filter of  $A$  is a Boolean filter of  $A$  if and only if  $x = (x \rightarrow y) \rightarrow x$ , for all  $x, y \in A$ .
- (4) Let  $F$  be a subset of  $A$ . Then  $F$  is a Boolean filter of  $A$  if and only if  $(x \rightarrow y) \rightarrow x \in F$  implies  $x \in F$ , for all  $x, y \in A$ .

**Definition 2.4.** [6,11] Let  $X \subseteq A$ . The filter of  $A$  generated by  $X$  will be denoted by  $\langle X \rangle$ . We have that  $\langle \emptyset \rangle = \{1\}$  and  $\langle X \rangle = \{a \in A \mid x_1 * x_2 * \dots * x_n \leq a, \text{ for some } n \in \mathbb{N}^* \text{ and some } x_1, x_2, \dots, x_n \in X\}$  if  $\emptyset \neq X \subseteq A$ . For any  $a \in A$ ,  $\langle a \rangle$  denotes the principal filter of  $A$  generated by  $\{a\}$ . Then  $\langle a \rangle = \{b \in A \mid a^n \leq b, \text{ for some } n \in \mathbb{N}^*\}$ .

**Definition 2.5.** [5] A Generalized Tarski algebra ( $GT$ -algebra, for short) is an algebra  $(A, \rightarrow, 1)$  with a binary operation  $\rightarrow$ , and a constant 1 such that:

- (T1)  $(\forall a \in A) (1 \rightarrow a = a)$ ,

- (T2)  $(\forall a \in A) (a \rightarrow a = 1)$ ,  
 (T3)  $(\forall a, b, c \in A)(a \rightarrow (b \rightarrow c) = (a \rightarrow b) \rightarrow (a \rightarrow c))$ .

**Definition 2.6.**[4] (1) A relatively pseudocomplemented lattice is an algebra  $(A, \wedge, \vee, \Rightarrow, 1)$  where  $(A, \wedge, \vee)$  is a lattice with greatest element and any element is relatively pseudocomplemented, where the pseudocomplement of  $y$  relative to  $z$ , noted  $y \Rightarrow z$  is  $\max\{x | x \wedge y \leq z\}$ , for all  $x, y, z \in A$ .

(2) A Heyting algebra is a relatively pseudocomplemented lattice with lower bounded (with 0).

### 3. Upper sets in residuated lattices

Let  $A$  be a residuated lattice,  $x \in A$ . We denote the upper set of  $x$  by  $U(x) = \{z \in A \mid z \geq x\}$ . It is easy to see that:

**Proposition 3.1.** Let  $A$  be a residuated lattice, and let  $x, y \in A$ . Then

- (1)  $x \leq y$  if and only if  $U(y) \subseteq U(x)$ ,  
 (2)  $x = y$  if and only if  $U(x) = U(y)$ .

**Proposition 3.2.** In any residuated lattice  $A$ , the following conditions are equivalent:

- (1)  $A$  is  $G(RL)$ -algebra,  
 (2)  $U(x) = U(x^2)$ , for all  $x \in A$ ,  
 (3)  $U(x) = \langle x \rangle$ , for all  $x \in A$ .

**Theorem 3.3.** Let  $A$  be a residuated lattice and  $U_A = \{U(x) \mid x \in A\}$ , then  $(U_A, \subseteq)$  is a lattice and we have  $U(x) \wedge U(y) = U(x) \cap U(y) = U(x \vee y)$  and  $U(x) \vee U(y) = U(x \wedge y)$

**Proof.** As easy check shows that  $U(x) \cap U(y) = U(x \vee y)$ . Hence  $U(x) \cap U(y) \in U_A$  and we get  $U(x) \wedge U(y) = U(x) \cap U(y) = U(x \vee y)$ . Also we have  $x \wedge y \leq x, y$ . By Proposition 3.1,  $U(x), U(y) \subseteq U(x \wedge y)$ . Now let  $U(x), U(y) \subseteq U(z)$ , for some  $z \in A$ . Then  $z \leq x, z \leq y$ . Hence  $z \leq x \wedge y$  and so  $U(x \wedge y) \subseteq U(z)$ . Therefore  $U(x) \vee U(y) = U(x \wedge y)$ .  $\square$

By the above theorem and Proposition 3.1, we get:

**Corollary 3.4.** In any residuated lattice  $A$  and for all  $x, y \in A$ :  $U(x) \cap U(y) = \{1\}$  if and only if  $x \vee y = 1$ .

**Theorem 3.5.**  $U(x) \subseteq U(y) \rightarrow U(z)$  implies  $U(x) \wedge U(y) \subseteq U(z)$ , for all  $x, y, z \in A$ , where  $U(y) \rightarrow U(z) := U(y \rightarrow z)$  and  $A$  is a residuated lattice.

**Proof.** Let  $U(x) \subseteq U(y) \rightarrow U(z)$ , for  $x, y, z \in A$ . Then  $U(x) \subseteq U(y \rightarrow z)$ . By Proposition 3.1 and Lemma 2.1,  $z \leq y \rightarrow z \leq x \leq x \vee y$ . Therefore by Theorem 3.3,  $U(x) \cap U(y) = U(x \vee y) \subseteq U(z)$ .  $\square$

By the following example we get the converse of Theorem 3.5 is not true:

**Example 1.** Let  $A = \{0, a, b, 1\}$ . Define  $*$  and  $\rightarrow$  as follow:

$*$	0	$a$	$b$	1
0	0	0	0	0
$a$	0	0	$a$	$a$
$b$	0	$a$	$b$	$b$
1	0	$a$	$b$	1

$\rightarrow$	0	$a$	$b$	1
0	1	1	1	1
$a$	$a$	1	1	1
$b$	0	$a$	1	1
1	0	$a$	$b$	1

Then  $(A, \wedge, \vee, *, \rightarrow, 0, 1)$  is a residuated lattice.  $U(a) = \{a, b, 1\}$ ,  $U(b) = \{b, 1\}$ ,  $U(1) = \{1\}$ . Then we get  $U(a) \wedge U(1) \subseteq U(b)$  but  $U(a) \not\subseteq U(1) \rightarrow U(b)$ .

**Remark 3.6.** It is to be noted, generally,  $(U_A, \wedge, \vee, \rightarrow, U(1), U(0))$  is not a Heyting algebra.

**Lemma 3.7.** Let  $A$  be a residuated lattice. The following conditions are equivalent:

- (1)  $A$  is  $G(RL)$ -algebra,
- (2)  $z \rightarrow (y \rightarrow x) = (z \rightarrow y) \rightarrow (z \rightarrow x)$ , for all  $x, y, z \in A$ .

**Proof.** ( $1 \Rightarrow 2$ ): Let  $A$  be  $G(RL)$ -algebra then  $x = x^2$ , for all  $x \in A$ . Hence  $x \rightarrow (x \rightarrow y) = x^2 \rightarrow y = x \rightarrow y$ , for all  $x, y \in A$  and so by Lemma 2.1, we have

$$\begin{aligned} (z \rightarrow (y \rightarrow x)) &\rightarrow ((z \rightarrow y) \rightarrow (z \rightarrow x)) = \\ (z \rightarrow (y \rightarrow x)) &\rightarrow ((z \rightarrow y) \rightarrow (z \rightarrow (z \rightarrow x))) \geq \\ (z \rightarrow (y \rightarrow x)) &\rightarrow (y \rightarrow (z \rightarrow x)) = \\ (z \rightarrow (y \rightarrow x)) &\rightarrow (z \rightarrow (y \rightarrow x)) = 1 \end{aligned}$$

Hence  $z \rightarrow (y \rightarrow x) \leq (z \rightarrow y) \rightarrow (z \rightarrow x)$ . On the other hand by Lemma 2.1, we have  $y \leq z \rightarrow y$  and so  $zy \leq z(z \rightarrow y)$ . Then  $(z \rightarrow y) \rightarrow (z \rightarrow x) = z(z \rightarrow y) \rightarrow x \leq zy \rightarrow x = z \rightarrow (y \rightarrow x)$ . Therefore  $z \rightarrow (y \rightarrow x) = (z \rightarrow y) \rightarrow (z \rightarrow x)$

( $2 \Rightarrow 1$ ): Let  $z \rightarrow (y \rightarrow x) = (z \rightarrow y) \rightarrow (z \rightarrow x)$ , for all  $x, y, z \in A$ . Now let  $x \in A$  then  $1 = x \rightarrow (x \rightarrow x^2) = (x \rightarrow x) \rightarrow (x \rightarrow x^2) = x \rightarrow x^2$ . Then we get  $x = x^2$ . Therefore  $A$  is  $G(RL)$ -algebra.  $\square$

**Theorem 3.8.** Let  $\Delta = (A, \rightarrow, *, \wedge, \vee, 0, 1)$  be a residuated lattice and  $\Delta^{GT} = (U_A, \rightarrow, U(1))$ , where for  $U(x), U(y) \in U_A$ ,  $U(x) \rightarrow U(y) := U(x \rightarrow y)$ . Then the following conditions are equivalent

- (1)  $\Delta$  is  $G(RL)$ -algebra ,
- (2)  $\Delta^{GT}$  is  $GT$ -algebra.

**Proof.** At first we note that if  $U(x_1) = U(x_2)$  and  $U(y_1) = U(y_2)$  then by Proposition 3.1, we get  $x_1 = x_2, y_1 = y_2$ . Hence  $U(x_1) \rightarrow U(y_1) = U(x_1 \rightarrow y_1) = U(x_2 \rightarrow y_2) = U(x_2) \rightarrow U(y_2)$ . Therefore  $\rightarrow$  is well defined.

( $1 \Rightarrow 2$ ): Let  $\Delta$  be  $G(RL)$ -algebra. By Lemma 3.7,  $z \rightarrow (y \rightarrow x) = (z \rightarrow y) \rightarrow (z \rightarrow x)$ , for all  $x, y, z \in A$ . Then (T1), (T2) and (T3) hold because:

$$\begin{aligned} \text{(T1): } U(1) \rightarrow U(x) &= U(1 \rightarrow x) = U(x), \\ \text{(T2): } U(x) \rightarrow U(x) &= U(x \rightarrow x) = U(1), \\ \text{(T3): } U(z) \rightarrow (U(y) \rightarrow U(x)) &= U(z \rightarrow (y \rightarrow x)) \\ &= U((z \rightarrow y) \rightarrow (z \rightarrow x)) \\ &= (U(z) \rightarrow U(y)) \rightarrow (U(z) \rightarrow U(x)) \end{aligned}$$

Hence  $\Delta^{GT}$  is  $GT$ -algebra.

(2  $\Rightarrow$  1): Let  $\Delta^{GT}$  be  $GT$ -algebra. By (T3) we have,

$$U(z) \rightarrow (U(y) \rightarrow U(x)) = (U(z) \rightarrow (y)) \rightarrow (U(z) \rightarrow U(x)), \text{ for all } x, y, z \in A.$$

Hence we get  $U(z \rightarrow (y \rightarrow x)) = U((z \rightarrow y) \rightarrow (z \rightarrow x))$ . By Proposition 3.1, we get  $z \rightarrow (y \rightarrow x) = (z \rightarrow y) \rightarrow (z \rightarrow x)$ . Using Lemma 3.7, we get  $\Delta$  is  $G(RL)$ -algebra.  $\square$

**Remark 3.9.** Using Theorem 3.3, the set of upper sets of  $A$  is closed with respect to finite intersection, but is not closed with respect to finite union:

**Example 2.** Let  $A = \{0, a, b, c, 1\}$ . Define  $*$  and  $\rightarrow$  as follow:

$*$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	a	b
c	0	a	a	c	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

Then  $(A, \wedge, \vee, *, \rightarrow, 0, 1)$  is a residuated lattice. We have  $U(a) = \{a, b, c, 1\}$ ,  $U(b) = \{b, 1\}$ ,  $U(c) = \{c, 1\}$ ,  $U(1) = \{1\}$ ,  $U(0) = A$ .

Then  $U(b) \cup U(c) = \{b, c, 1\}$  and  $U(b) \cup U(c) \neq U(x)$ , for all  $x \in A$ .

In the following, we see every filter is the union of upper sets of  $x$  where  $x \in F$ , but the converse is not true:

**Theorem 3.10.** Let  $F$  be a filter of residuated lattice  $A$ . Then

$$F = \bigcup \{U(x) \mid x \in F\}.$$

**Proof.** Let  $F$  be a filter of residuated lattice  $A$  and  $x \in F$ . We have  $x \leq x$  and so  $x \in U(x)$ . Hence

$$F \subseteq \bigcup \{U(x) \mid x \in F\}.$$

Now let  $x \in \bigcup\{U(x) \mid x \in F\}$ , then there exist  $y \in F$  such that  $x \in U(y)$ . Hence  $y \leq x$  and since  $y \in F$  we get  $x \in F$ . Therefore

$$\bigcup\{U(x) \mid x \in F\} \subseteq F. \square$$

**Remark 3.11.** Note that the converse of the above theorem dose not hold.

Consider  $A = ([0, 1], *, \Rightarrow, \min, \max, 0, 1)$  where  $x * y = \max(0, x + y - 1)$  and  $x \Rightarrow y = \min(1, 1 - x + y)$ , and let  $F = [1/2, 1]$ . An easy check shows that  $A$  is a residuated lattice and  $F = \bigcup\{U(x) \mid x \in F\} = U(1/2)$ . However  $F$  is not a filter: in fact it is not closed under  $*$  (for example  $1/2 * 1/2 = 0$ ).

By the above theorem we get:

**Corollary 3.12.** If  $F$  is a filter of residuated lattice  $A$ , then  $U(x) \subseteq F$ , for all  $x \in F$ .

It is to be noted that if we use upper sets of  $x * y$  instead of upper sets of  $x$  then the converse of the Theorem 3.10 holds:

**Theorem 3.13.** Let  $F$  be a subset of residuated lattice  $A$ . Then  $F$  is a filter of  $A$  if and only if  $F = \bigcup\{U(x * y) \mid x, y \in F\}$ .

**Proof.** Let  $F$  be a filter of  $A$ . We have,

$$F = \bigcup\{U(x) \mid x \in F\} \subseteq \bigcup\{U(x * y) \mid x, y \in F\}$$

Now let  $a \in \bigcup\{U(x * y) \mid x, y \in F\}$ . Then there exist  $b, c \in F$  such that  $a \in U(b * c)$ , hence  $b * c \leq a$ . Since  $b, c \in F$  we get  $a \in F$ .

Conversely, let  $F$  be a subset of  $A$  such that  $F = \bigcup\{U(x * y) \mid x, y \in F\}$ . Obviously,  $1 \in \bigcup\{U(x * y) \mid x, y \in F\} = F$ . Let  $a, b \in A$  be such that  $a, a \rightarrow b \in F$ . Since  $a * (a \rightarrow b) = a \wedge b \leq b$ , hence  $b \in U(a * (a \rightarrow b))$ . Then  $b \in \bigcup\{U(x * y) \mid x, y \in F\} = F$ , that is,  $F$  is a filter of  $A$ .  $\square$

Using proof of the above theorem we get:

**Corollary 3.14.** Let  $F$  be a subset of residuated lattice  $A$ . Then  $F$  is a filter of  $A$  if and only if  $U(x * y) \subseteq F$  for all  $x, y \in F$ .

**Theorem 3.15.** Let  $A$  be a residuated lattice. Then

- (1)  $A$  is a  $G(RL)$ -algebra if and only if  $U(x)$  is a filter, for all  $x \in A$ .
- (2)  $A$  is a Boolean algebra if and only if  $U(x)$  is a Boolean filter, for all  $x \in A$ .

**Proof.**(1) Let  $A$  be  $G(RL)$ -algebra and  $x \in A$ . Obviously,  $1 \in U(x)$ . Consider  $a, a \rightarrow b \in U(x)$ . Hence  $x \rightarrow a = 1$  and  $x \rightarrow (a \rightarrow b) = 1$ . Using Theorem 2.3,  $\{1\}$

is a  $G$ -filter of  $A$  and so  $\{1\}$  is a positive implicative filter. Therefore  $x \rightarrow b = 1$ . This means  $b \in U(x)$ . Thus  $U(x)$  is a filter of  $A$ .

Conversely, let  $U(x)$  be a filter, for all  $x \in A$ . We have  $x \rightarrow (x \rightarrow x^2) = 1$  and  $x \rightarrow x = 1$ , hence  $x \rightarrow x^2 \in U(x)$  and  $x \in U(x)$ , for all  $x \in A$ . Since  $U(x)$  is a filter,  $x^2 \in U(x)$ . Therefore  $x^2 = x$  and so  $A$  is a  $G(RL)$ -algebra.

(2) Let  $A$  be a Boolean algebra and  $x \in A$ . Hence  $A$  is a  $G(RL)$ -algebra. Using part (1) and Theorem 2.3,  $U(x)$  is a Boolean filter. Conversely, let  $U(x)$  be a Boolean filter, for all  $x \in A$ . Since  $(x \rightarrow y) \rightarrow x \in U((x \rightarrow y) \rightarrow x)$  and  $U((x \rightarrow y) \rightarrow x)$  is a Boolean filter, by Theorem 2.3 we get  $x \in A((x \rightarrow y) \rightarrow x)$ , for all  $x, y \in A$ . Then  $(x \rightarrow y) \rightarrow x \leq x$ . Using Lemma 2.1,  $x \leq (x \rightarrow y) \rightarrow x$ . Hence  $x = (x \rightarrow y) \rightarrow x$ , for all  $x, y \in A$  and so  $A$  is a Boolean algebra.

**Example 3.** Let  $A = \{0, a, b, 1\}$ . Define  $*$  and  $\rightarrow$  as follow:

$*$	0	$a$	$b$	1
0	0	0	0	0
$a$	0	0	0	$a$
$b$	0	0	$a$	$b$
1	0	$a$	$b$	1

$\rightarrow$	0	$a$	$b$	1
0	1	1	1	1
$a$	$b$	1	1	1
$b$	$a$	$b$	1	1
1	0	$a$	$b$	1

Then  $(A, \wedge, \vee, *, \rightarrow, 0, 1)$  is a residuated lattice but is not a  $G(RL)$ -algebra and  $U(a) = \{a, b, 1\}$  is not a filter.

**Theorem 3.16.** Let  $A$  be a  $G(RL)$ -algebra. Then  $y \in U(x)$  if and only if  $U(x) = U(x * y)$ , for all  $x, y \in A$ .

**Proof.** Let  $A$  be a  $G(RL)$ -algebra and  $y \in U(x)$ . It is clear that  $U(x) \subseteq U(x * y)$ . Now let  $z \in U(x * y)$ : since  $A$  is  $G(RL)$ -algebra, using Lemma 3.7 we get

$$x \rightarrow z = 1 \rightarrow (x \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z) = x \rightarrow (y \rightarrow z) = (x * y) \rightarrow z = 1.$$

This means  $z \geq x$  and so  $z \in U(x)$ . Therefore  $U(x * y) \subseteq U(x)$ .

Conversely, let  $U(x) = U(x * y)$ . Since  $x * y \leq y$ , then  $y \in U(x * y) = U(x)$ .  $\square$

By the above Theorem we get:



**Corollary 3.17.** Let  $A$  be a  $G(RL)$ -algebra. For all  $x, y \in A$  we have

$$y \notin U(x) \text{ if and only if } U(x) \subset U(x * y).$$

**Example 4.** In Example 1, we have  $A$  is a  $G(RL)$ -algebra and

$$c \notin U(b) = \{b, 1\} \text{ and } U(c) = \{c, 1\} \subset U(b * c) = \{a, b, c, 1\}.$$

**Example 5.** Let  $A = \{0, a, b, c, d, 1\}$ . Define  $*$  and  $\rightarrow$  as follows:

$*$	1	$a$	$b$	$c$	$d$	0
1	1	$a$	$b$	$c$	$d$	0
$a$	$a$	$b$	$b$	$d$	0	0
$b$	$b$	$b$	$b$	0	0	0
$c$	$c$	$d$	0	$c$	$d$	0
$d$	$d$	0	0	$d$	0	0
0	0	0	0	0	0	0

$\rightarrow$	1	$a$	$b$	$c$	$d$	0
1	1	$a$	$b$	$c$	$d$	0
$a$	1	1	$a$	$c$	$c$	$d$
$b$	1	1	1	$c$	$c$	$c$
$c$	1	$a$	$b$	1	$a$	$b$
$d$	1	1	$a$	1	1	$a$
0	1	1	1	1	1	1

Then  $(A, \wedge, \vee, *, \rightarrow, 0, 1)$  is a residuated lattice, but is not a  $G(RL)$ -algebra. We get  $U(d) \subset U(a * d)$  but  $a \in U(d)$ .

#### 4. The Krull dimension of a residuated lattice

**Definition 4.1.** Let  $A$  be a residuated lattice, and

$$\Lambda : F_0 = \{1\} \subset F_1 \subset F_2 \subset \dots \subset F_n = A$$

be a chain of distinct filters in  $A$ . If  $n$  is finite we say that  $\Lambda$  is a finite chain in  $A$  and  $n$  is the length of  $\Lambda$ . Otherwise,  $\Lambda$  is called to be an infinite chain in  $A$ .  $\square$

**Definition 4.2.** We define

$$\text{Krull dimension of } A = \text{Max} \{n \mid n \text{ is length of an chain of distinct filters in } A\}$$

and denoted by  $Kdim(A)$ . The chain of distinct filters with the maximal length is called a maximal chain of distinct filters in  $A$ .

We recall a residuated lattice  $A$  is called simple if its only filters are  $\{1\}$  and  $A$ . It is easy to see that:

**Proposition 4.3.** Let  $A$  be a residuated lattice and let  $F$  be a filter of  $A$ . Then

- (1)  $Kdim(A) = 1$  if and only if  $A$  is a simple residuated lattice.
- (2)  $F$  is a maximal filter if and only if  $Kdim A/F = 1$ .

A proper filter  $F$  of residuated lattice  $A$  is called an obstinate filter if  $x, y \notin F$  imply  $x \rightarrow y \in F$  and  $y \rightarrow x \in F$  [1]. Following [1], every obstinate filter is maximal filter. Hence we get:

**Corollary 4.4.** If  $F$  is an obstinate filter of residuated lattice  $A$  then  $Kdim A/F = 1$ .

The next example shows that the converse of the above theorem is not true.

**Example 6.** In Example 1,  $F = \{b, 1\}$  is a filter. It is easy to see that  $Kdim A/F = 1$  but  $F$  is not an obstinate filter.

**Remark 4.5.** Let  $A$  and  $B$  be two residuated lattices and let us consider  $A \times B$  the residuated lattices product of  $A$  and  $B$ . If  $F_1, F_2$  are filters of  $A$  and  $B$  respectively then  $F_1 \times F_2$  is a filter of  $A \times B$  and, conversely, any filter of  $A \times B$  is of the form  $F_1 \times F_2$ ; where  $F_1, F_2$  are filters of  $A$  and  $B$  respectively [10].

**Theorem 4.6.** Let  $A$  and  $B$  be two residuated lattices and let  $Kdim A = m$ ,  $Kdim B = n$ . Then  $Kdim(A \times B) = Kdim A + Kdim B$ .

**Proof.** If  $A$  or  $B$  have an infinite chain then it is straightforward  $A \times B$  has an infinite chain and so  $Kdim(A \times B) = Kdim A + Kdim B$ . Now let  $A$  and  $B$  have no infinite chain and  $Kdim A = m \geq Kdim B = n$ . Then there exist two maximal chains  $\Lambda : F_0 = \{1\} \subset F_1 \subset F_2 \subset \dots \subset F_m = A$  and  $\Lambda' : G_0 = \{1\} \subset G_1 \subset G_2 \subset \dots \subset G_n = B$  of distinct filters in  $A$  and  $B$ , respectively. Now we consider the following chain of distinct filters in  $A \times B$ :

$$\{1\} \times \{1\} = F_0 \times G_0 \subset F_1 \times G_0 \subset F_1 \times G_1 \subset F_2 \times G_1 \subset \dots \subset F_n \times G_n = F_n \times B \subset F_{n+1} \times B \subset F_{n+2} \times B \subset \dots \subset F_m \times B = A \times B.$$

Hence  $Kdim(A \times B) \geq 2n + m - n = m + n$ . Let  $Kdim(A \times B) = z > m + n$ . Then there exists a chain of distinct filters in  $A \times B : \{1\} \times \{1\} = H'_0 \subset H'_1 \subset \dots \subset H'_z = A \times B$ .

Using the above remark there exist filters  $\{1\} = I_0, I_1, \dots, I_z = A$  and  $\{1\} =$

$J_0, J_1, \dots, J_z = B$  of  $A$  and  $B$ , respectively such that  $H'_i = I_i \times J_i$ , for all  $0 \leq i \leq z$ . Now consider  $\Gamma = \{i \mid I_i = I_{i+1}\}$ . We note that  $I_i \subseteq I_{i+1}$ , for all  $1 \leq i \leq z$ . Using  $Kdim A = m < z$  we get  $\Gamma \neq \emptyset$  and  $Card(\Gamma) \geq z - m$ . Let  $I_i = I_{i+1}$ , since  $H'_i \subset H'_{i+1}$  we get  $J_i \subset J_{i+1}$ . Hence there exists a chain such that  $\{1\} = K_0 \subset K_1 \subset \dots \subset K_t = B$ , where  $K_i \in \{J_0, J_1, \dots, J_z = B\}$  for all  $0 \leq i \leq t$  and  $t = Card(\Gamma)$ . But this is a contradiction since  $t \geq z - m > n = Kdim B$ .  $\square$

**Theorem 4.7.** If  $F$  and  $G$  are filters of residuated lattices of  $A$  and  $B$ , respectively then  $Kdim A \times B/F \times G = Kdim A/F + Kdim B/G$ .

**Proof.** We define  $\varphi : A \times B \longrightarrow A/F \times B/G$  such that  $\varphi(a, b) = ([a], [b])$ . It is easy to see that  $\varphi$  is well-defined, onto homomorphism and  $ker \varphi = F \times G$ . Using the Homomorphism Theorem we get  $A \times B/F \times G \cong A/F \times B/G$ . By the above theorem  $Kdim A \times B/F \times G = Kdim A/F + Kdim B/G$ .  $\square$

In the following, we concentrate on  $G(RL)$ -algebras.

**Theorem 4.8.** Let  $A$  be  $G(RL)$ -algebra. Then  $Kdim(A) = n$  if and only if there exists a maximal chain of distinct upper sets in  $A$  as follows

$$U(1) = \{1\} \subset U(x_1) \subset U(x_1 * x_2) \subset \dots \subset U(x_1 * x_2 * \dots * x_n) = A,$$

for some  $x_1, x_2, \dots, x_n \in A$ .

**Proof.** Let  $Kdim(A) = n$  then there exists a maximal chain

$$\Lambda : F_0 = \{1\} \subset F_1 \subset F_2 \subset \dots \subset F_n = A$$

of distinct filters in  $A$ . We have  $F_0 = \{1\} = U(1)$ . Since  $F_0 = \{1\} \subset F_1$ , then there exist  $x_1 \in F_1$  such that  $x_1 \neq 1$ . Since  $F_1$  is a filter by Corollary 3.12, we get  $F_0 = \{1\} \subset U(x_1) \subseteq F_1$ . By Theorem 3.15,  $U(x_1)$  is a filter. If  $F_1 \neq U(x_1)$ , this contradicts to the maximality and so  $F_1 = U(x_1)$ . Now since  $F_1 \subset F_2$ , there exist  $x_2 \in F_2 - F_1$ . Then  $F_1 = U(x_1) \subset U(x_1 * x_2) \subseteq F_2$  (we note that  $x_1, x_2 \in F$  and so  $x_1 * x_2 \in F$ ). Since  $U(x_1 * x_2)$  is a filter, if  $F_2 \neq U(x_1 * x_2)$ , this contradicts to the maximality and so  $F_2 = U(x_1 * x_2)$ . Continuing this process we get  $F_k = U(x_1 * x_2 * \dots * x_k)$ , for some  $x_1, x_2, \dots, x_k \in A$ .

Hence there exist  $x_1, x_2, \dots, x_k \in A$  such that  $F_k = U(x_1 * x_2 * \dots * x_k)$ , for all  $1 \leq k \leq n$ . Therefore there exists a maximal chain of distinct filters in  $A$  as follows

$$U(1) = \{1\} \subset U(x_1) \subset U(x_1 * x_2) \subset \dots \subset U(x_1 * x_2 * \dots * x_n) = A$$

Conversely, let  $\Lambda : U(1) = \{1\} \subset U(x_1) \subset U(x_1 * x_2) \subset \dots \subset U(x_1 * x_2 * \dots * x_n) = A$  be a maximal chain of distinct upper sets in  $A$ . Since  $A$  is  $G(RL)$ -algebra,  $U(x_1 * x_2 * \dots * x_n)$  is a filter of  $A$  and so  $Kdim A \geq n$ . Now let  $Kdim A = m$ . Then we get there exists a chain of distinct upper sets in  $A$  as follows

$$U(1) = \{1\} \subset U(y_1) \subset U(y_1 * y_2) \subset \dots \subset U(y_1 * y_2 * \dots * y_m) = A$$

Since  $\Lambda$  is a maximal chain of distinct upper sets in  $A$  we get  $n \geq m$ . Therefore  $n = m$ .  $\square$

### Conclusion and future research

In this paper, we defined the notion of upper sets in residuated lattices. We studied the connection between filters and upper sets in residuated lattices. We proved a residuated lattice is a (Boolean algebra)  $G(RL)$ -algebra if and if every upper set in residuated lattice is a (Boolean filter) filter. At last we defined the Krull dimension of residuated lattices and proved a maximal chain of distinct filters, in fact, is a maximal chain of distinct upper sets on  $G(RL)$ -algebras.

In our future work, we are going to develop the properties of the Krull dimension of residuated lattices and find useful results on other structures. We hope this work would serve as a foundation for further studies on the structure of residuated lattices and develop corresponding many-valued logical systems.

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