BIANCHI IDENTITIES IN THE JET POLYMOMENTUM HAMILTON GEOMETRY

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Dedicated to Professor Constantin Udrişte
for his entire prolific scientific activity

In this paper we describe the local Ricci and Bianchi identities for an h-normal N-linear connection $D\Gamma(N)$ on the dual 1-jet space $J^{1*}(\mathcal{T}, M)$. To reach this aim, we firstly give the expressions of the local distinguished (d-) adapted components of torsion and curvature tensors produced by $D\Gamma(N)$, and then we analyze their attached local Ricci identities. The derived deflection d-tensor identities are also presented. Finally, we expose the local expressions of the Bianchi identities (in the particular case of an h-normal N-linear connection of Cartan type), which geometrically connect the local torsion and curvature d-tensors of the linear connection $D\Gamma(N)$.

Keywords: dual 1-jet spaces, nonlinear connections, h-normal N-linear connections of Cartan type, Ricci and Bianchi identities.

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1. Introduction

According to Olver's opinion [23], we consider that the 1-jet spaces and their duals are natural houses for the study of classical and quantum field theories. For such a reason, the differential geometry of 1-jet spaces was intensively studied, in a contravariant approach, by a lot of authors: Saunders [30], Asanov [1], Neagu and Udrişte (see [20], [21], [22]), and many others.

In the last decades, numerous physicists and geometers were preoccupied by the development of that so-called the *covariant Hamiltonian geometry of physical fields*, which is a multi-parameter, or multi-time, extension of the classical Hamiltonian formulation from Mechanics. In such a perspective, we point out that the covariant Hamiltonian geometry of physical fields appears in the literature of specialty in three distinct variants: **(1)** ► the *multisymplectic geometry* − developed by Gotay, Isenberg, Marsden, Montgomery and their co-workers (see [10], [9]) on a finite-dimensional multisymplectic phase space; **(2)** ► the *polysymplectic geometry* − elaborated by Giachetta, Mangiarotti and Sardanashvily (see [7], [8]), which emphasizes the relations between the equations of first order Lagrangian field theory on

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fiber bundles and the covariant Hamilton equations on a finite-dimensional polysymplectic phase space; **(3)** ► the *De Donder-Weyl Hamiltonian geometry* − studied by Kanatchikov (see [11], [12], [13]) as opposed to the conventional field-theoretical Hamiltonian formalism, which requires the space + time decomposition and leads to the picture of a field as a mechanical system with infinitely degrees of freedom.

From a geometrical point of view, following the ideas initially stated by Asanov [1], a multi-time Lagrange contravariant geometry on 1-jet spaces (in the sense of d-linear connections, d-torsions and d-curvatures) was recently developed by Neagu and Udrişte in [20], [21] and [22]. This 1-jet geometrical theory is a natural multi-time extension of the classical Lagrangian geometry on tangent bundles, initiated and developed by Miron and Anastasiei [15].

On the other hand, suggested by the field theoretical extension of the basic structures of classical Analytical Mechanics within the framework of the De Donder-Weyl covariant Hamiltonian formulation, the geometrical studies of Miron [14], Atanasiu [3], [2] and others led to the development of the Hamilton geometry on cotangent bundles, which is synthesized in the book [16]. Note that the Miron-Atanasiu Hamiltonian geometrical ideas from cotangent bundles represent the point start for the development of the jet covariant Riemann-Hamilton geometry depending on polymomenta, which is presented in the Atanasiu-Neagu papers [4] and [5]. In this paper we are going on the jet multi-time Hamiltonian geometrical studies from [4] and [5].

2. Components of N-linear connections on dual 1-jet bundle $J^{1*}(\mathcal{T}, M)$

Let $\mathcal{T}$ and M be a *temporal* (resp. *spatial*) real, smooth manifold of dimension m (resp. n), whose coordinates are $(t^a)_{a=\overline{1,m}}$, respectively $(x^i)_{i=\overline{1,n}}$. Note that, throughout this paper, the indices a, b, c, ... run from 1 to m, while the indices i, j, k, ... run from 1 to n. The Einstein convention of summation is also adopted all over this work.

Let $J^{1*}(\mathcal{T}, M)$ be the dual 1-jet fibre bundle, whose coordinates (t^a, x^i, p_i^a) are induced from $\mathcal{T}$ and M. The coordinate transformations from the product manifold $\mathcal{T}\times M$ produce on $J^{1*}(\mathcal{T}, M)$ the following coordinate transformations:

$$\widetilde{t}^a = \widetilde{t}^a\left(t^b\right), \quad \widetilde{x}^i = \widetilde{x}^i\left(x^j\right), \quad \widetilde{p}_i^a = \frac{\partial x^j}{\partial \widetilde{x}^i}\frac{\partial \widetilde{t}^a}{\partial t^b}p_j^b,$$

where $\det\left(\partial\widetilde{t}^a/\partial t^b\right) \neq 0$ and $\det\left(\partial\widetilde{x}^i/\partial x^j\right) \neq 0$.

Definition 2.1. *A pair of local functions on $E^* = J^{1*}(\mathcal{T}, M)$, denoted by*

$$N = \left(\underset{1}{N}{}_{(i)b}^{(a)},\ \underset{2}{N}{}_{(i)j}^{(a)}\right),$$

whose local components obey the transformation rules

$$\underset{1}{\widetilde{N}}{}_{(j)c}^{(b)}\frac{\delta\widetilde{t}^c}{\delta t^a} = \underset{1}{N}{}_{(k)a}^{(c)}\frac{\delta\widetilde{t}^b}{\delta t^c}\frac{\partial x^k}{\partial\widetilde{x}^j} - \frac{\partial\widetilde{p}_j^b}{\partial t^a},$$

$$\underset{2}{\widetilde{N}}{}_{(j)k}^{(b)}\frac{\partial\widetilde{x}^k}{\partial x^i} = \underset{2}{N}{}_{(k)i}^{(c)}\frac{\delta\widetilde{t}^b}{\delta t^c}\frac{\partial x^k}{\partial\widetilde{x}^j} - \frac{\partial\widetilde{p}_j^b}{\partial x^i},$$

is called a ***nonlinear connection*** *on* E^**. The components* $\underset{1}{N}{}^{(a)}_{(i)b}$ *(resp.* $\underset{2}{N}{}^{(a)}_{(i)j}$*) are called the* ***temporal*** *(resp.* ***spatial****)* ***components*** *of* N*.*

Example 2.1. *Let* $h_{ab}\left(t^f\right)$ *(resp.* $\varphi_{ij}\left(x^k\right)$*) be a semi-Riemannian metric on the temporal manifold* $\mathcal{T}$ *(resp. spatial manifold* M*). Taking into account the local transformation rules of the Christoffel symbols* $\chi^a_{bc}(t)$ *(resp.* $\Gamma^k_{ij}(x)$*) of the metrics* $h_{ab}(t)$ *(resp.* $\varphi_{ij}(x)$*), then the pair of local functions*

$$N_0 = \left(\underset{1}{\overset{0}{N}}{}^{(a)}_{(i)b},\ \underset{2}{\overset{0}{N}}{}^{(a)}_{(i)j}\right),$$

where

$$\underset{1}{\overset{0}{N}}{}^{(a)}_{(i)b} = \chi^a_{bc}p^c_i, \qquad \underset{2}{\overset{0}{N}}{}^{(a)}_{(i)j} = -\Gamma^k_{ij}p^a_k,$$

represents a nonlinear connection on E^**. This is called the* ***canonical nonlinear connection attached to the metrics*** $h_{ab}(t)$ ***and*** $\varphi_{ij}(x)$*.*

In what follows, we fix a nonlinear connection on E^*, and we consider the *adapted bases* of the nonlinear connection N, defined by

$$\left\{\frac{\delta}{\delta t^a}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial p^a_i}\right\} \subset \mathcal{X}(E^*), \quad \left\{dt^a, dx^i, \delta p^a_i\right\} \subset \mathcal{X}^*(E^*), \tag{1}$$

where

$$\frac{\delta}{\delta t^a} = \frac{\partial}{\partial t^a} - \underset{1}{N}{}^{(b)}_{(j)a}\frac{\partial}{\partial p^b_j},$$
$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \underset{2}{N}{}^{(b)}_{(j)i}\frac{\partial}{\partial p^b_j},$$
$$\delta p^a_i = dp^a_i + \underset{1}{N}{}^{(a)}_{(i)b}dt^b + \underset{2}{N}{}^{(a)}_{(i)j}dx^j.$$

It is important to note that the transformation rules of the elements of the adapted bases (1) are tensorial ones:

$$\begin{gathered}\frac{\delta}{\delta t^a} = \frac{\partial \tilde{t}^b}{\partial t^a}\frac{\delta}{\delta \tilde{t}^b}, \quad \frac{\delta}{\delta x^i} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\delta}{\delta \tilde{x}^j}, \quad \frac{\partial}{\partial p^a_i} = \frac{\partial \tilde{t}^b}{\partial t^a}\frac{\partial x^i}{\partial \tilde{x}^j}\frac{\partial}{\partial \tilde{p}^b_j},\\ dt^a = \frac{\partial t^a}{\partial \tilde{t}^b}d\tilde{t}^b, \quad dx^i = \frac{\partial x^i}{\partial \tilde{x}^j}d\tilde{x}^j, \quad \delta p^a_i = \frac{\partial t^a}{\partial \tilde{t}^b}\frac{\partial \tilde{x}^j}{\partial x^i}\delta\tilde{p}^b_j.\end{gathered} \tag{2}$$

Remark 2.1. *The simple tensorial transformation rules* (2) *of the adapted bases* (1) *determined us to describe in what follows all geometrical objects on the dual* 1*-jet space* $J^{1*}(\mathcal{T}, M)$ *in adapted local components.*

In order to develop the geometrical theory of N-linear connections on the dual 1-jet space E^*, we need the following result:

Proposition 2.1. (i) *The Lie algebra* $\mathcal{X}(E^*)$ *of vector fields decomposes as*

$$\mathcal{X}(E^*) = \mathcal{X}(\mathcal{H}_{\mathcal{T}}) \oplus \mathcal{X}(\mathcal{H}_M) \oplus \mathcal{X}(\mathcal{V}),$$

where

$$\mathcal{X}(\mathcal{H}_{\mathcal{T}}) = \text{Span}\left\{\frac{\delta}{\delta t^a}\right\}, \quad \mathcal{X}(\mathcal{H}_M) = \text{Span}\left\{\frac{\delta}{\delta x^i}\right\}, \quad \mathcal{X}(\mathcal{V}) = \text{Span}\left\{\frac{\partial}{\partial p^a_i}\right\}.$$

(ii) *The Lie algebra* $\mathcal{X}^*(E^*)$ *of covector fields decomposes as*

$$\mathcal{X}^*(E^*) = \mathcal{X}^*(\mathcal{H}_{\mathcal{T}}) \oplus \mathcal{X}^*(\mathcal{H}_M) \oplus \mathcal{X}^*(\mathcal{V}),$$

where

$$\mathcal{X}^*(\mathcal{H}_{\mathcal{T}}) = \text{Span}\{dt^a\}, \quad \mathcal{X}^*(\mathcal{H}_M) = \text{Span}\{dx^i\}, \quad \mathcal{X}^*(\mathcal{V}) = \text{Span}\{\delta p_i^a\}.$$

Let us consider that $h_{\mathcal{T}}$, h_M (horizontal) and v (vertical) are the canonical projections of the above decompositions. In this context, we introduce the following geometrical concept:

Definition 2.2. *A linear connection* $D : \mathcal{X}(E^*) \times \mathcal{X}(E^*) \to \mathcal{X}(E^*)$ *is called an* N***-linear connection*** *on* E^* *if and only if* $Dh_{\mathcal{T}} = 0, Dh_M = 0$ *and* $Dv = 0$.

It is obvious that the local description of the N-linear connection D on E^* is accomplished by *nine* unique adapted components

$$D\Gamma(N) = \begin{pmatrix} A^a_{bc},\ A^i_{jc},\ -A^{(a)(j)}_{(i)(b)c},\ H^a_{bk},\ H^i_{jk},\ -H^{(a)(j)}_{(i)(b)k}, \\ C^{a(k)}_{b(c)},\ C^{i(k)}_{j(c)},\ -C^{(a)(j)(k)}_{(i)(b)(c)} \end{pmatrix}, \tag{3}$$

which are locally defined by the relations:

$$\begin{aligned}
D_{\frac{\delta}{\delta t^c}}\frac{\delta}{\delta t^b} &= A^a_{bc}\frac{\delta}{\delta t^a},\ D_{\frac{\delta}{\delta t^c}}\frac{\delta}{\delta x^j} = A^i_{jc}\frac{\delta}{\delta x^i},\ D_{\frac{\delta}{\delta t^c}}\frac{\partial}{\partial p^b_j} = -A^{(a)(j)}_{(i)(b)c}\frac{\partial}{\partial p^a_i},\\
D_{\frac{\delta}{\delta x^k}}\frac{\delta}{\delta t^b} &= H^a_{bk}\frac{\delta}{\delta t^a},\ D_{\frac{\delta}{\delta x^k}}\frac{\delta}{\delta x^j} = H^i_{jk}\frac{\delta}{\delta x^i},\ D_{\frac{\delta}{\delta x^k}}\frac{\partial}{\partial p^b_j} = -H^{(a)(j)}_{(i)(b)k}\frac{\partial}{\partial p^a_i},\\
D_{\frac{\partial}{\partial p^c_k}}\frac{\delta}{\delta t^b} &= C^{a(k)}_{b(c)}\frac{\delta}{\delta t^a},\ D_{\frac{\partial}{\partial p^c_k}}\frac{\delta}{\delta x^j} = C^{i(k)}_{j(c)}\frac{\delta}{\delta x^i},\ D_{\frac{\partial}{\partial p^c_k}}\frac{\partial}{\partial p^b_j} = -C^{(a)(j)(k)}_{(i)(b)(c)}\frac{\partial}{\partial p^a_i}.
\end{aligned}$$

Example 2.2. *Let* $N_0 = \left(\overset{0}{\underset{1}{N}}{}^{(a)}_{(i)b},\ \overset{0}{\underset{2}{N}}{}^{(a)}_{(i)j}\right)$ *be the canonical nonlinear connection produced by the semi-Riemannian metrics* (h_{ab}, φ_{ij})*. Taking into account the transformation rules of the Christoffel symbols* χ^a_{bc} *and* Γ^i_{jk}*, by local computations, we can show that the local components*

$$B\Gamma(N_0) = \left(\chi^a_{bc},\ 0,\ -A^{(a)(j)}_{(i)(b)c},\ 0,\ \Gamma^i_{jk},\ -H^{(a)(j)}_{(i)(b)k},\ 0,\ 0,\ 0\right),$$

where

$$A^{(a)(j)}_{(i)(b)c} = -\delta^j_i\chi^a_{bc}, \quad H^{(a)(j)}_{(i)(b)k} = \delta^a_b\Gamma^j_{ik},$$

verify the transformation rules of the components of an N*-linear connection (for more details, see* [5]*). Consequently,* $B\Gamma(N_0)$ *is an* N_0*-linear connection on* E^**, which is called the* ***Berwald connection*** *of the metric pair* (h_{ab}, φ_{ij}).

Now, let $D\Gamma(N)$ be an N-linear connection on E^*, locally defined by (3). The linear connection $D\Gamma(N)$ induces a linear connection on the set of d-tensors on the dual 1-jet fibre bundle $E^* = J^{1*}(\mathcal{T}, M)$, in a natural way. Thus, starting with a d-vector field X and a d-tensor field T, locally expressed by

$$\begin{aligned} X &= X^a \frac{\delta}{\delta t^a} + X^i \frac{\delta}{\delta x^i} + X_{(i)}^{(a)} \frac{\partial}{\partial p_i^a}, \\ T &= T_{cj(b)(l)...}^{ai(k)(d)...} \frac{\delta}{\delta t^a} \otimes \frac{\delta}{\delta x^i} \otimes \frac{\partial}{\partial p_l^d} \otimes dt^c \otimes dx^j \otimes \delta p_k^b \otimes ..., \end{aligned}$$

we can define the covariant derivative

$$\begin{aligned} & D_X T = X^g D_{\frac{\delta}{\delta t^g}} T + X^s D_{\frac{\delta}{\delta x^s}} T + X_{(s)}^{(g)} D_{\frac{\partial}{\partial p_s^g}} T = \\ & = \left\{ X^g T_{cj(b)(l).../g}^{ai(k)(d)...} + X^s T_{cj(b)(l)...|s}^{ai(k)(d)...} + \right. \\ & \left. + X_{(s)}^{(g)} T_{cj(b)(l)...}^{ai(k)(d)...} |_{(g)}^{(s)} \right\} \frac{\delta}{\delta t^a} \otimes \frac{\delta}{\delta x^i} \otimes \frac{\partial}{\partial p_l^d} \otimes dt^c \otimes dx^j \otimes \delta p_k^b \otimes ..., \end{aligned}$$

where

- the $\mathcal{T}$-*horizontal covariant derivative* of $D\Gamma(N)$:

$$(h_{\mathcal{T}}) \begin{cases} T_{cj(b)(l).../g}^{ai(k)(d)...} = \dfrac{\delta T_{cj(b)(l)...}^{ai(k)(d)...}}{\delta t^g} + T_{cj(b)(l)...}^{fi(k)(d)...} A_{fg}^a + \\ + T_{cj(b)(l)...}^{ar(k)(d)...} A_{rg}^i + T_{cj(f)(l)...}^{ai(r)(d)...} A_{(r)(b)g}^{(f)(k)} + ... - \\ - T_{fj(b)(l)...}^{ai(k)(d)...} A_{cg}^f - T_{cr(b)(l)...}^{ai(k)(d)...} A_{jg}^r - T_{cj(b)(r)...}^{ai(k)(f)...} A_{(l)(f)g}^{(d)(r)} - ..., \end{cases}$$

- the M-*horizontal covariant derivative* of $D\Gamma(N)$:

$$(h_M) \begin{cases} T_{cj(b)(l)...|s}^{ai(k)(d)...} = \dfrac{\delta T_{cj(b)(l)...}^{ai(k)(d)...}}{\delta x^s} + T_{cj(b)(l)...}^{fi(k)(d)...} H_{fs}^a + \\ + T_{cj(b)(l)...}^{ar(k)(d)...} H_{rs}^i + T_{cj(f)(l)...}^{ai(r)(d)...} H_{(r)(b)s}^{(f)(k)} + ... - \\ - T_{fj(b)(l)...}^{ai(k)(d)...} H_{cs}^f - T_{cr(b)(l)...}^{ai(k)(d)...} H_{js}^r - T_{cj(b)(r)...}^{ai(k)(f)...} H_{(l)(f)s}^{(d)(r)} - ..., \end{cases}$$

- the *vertical covariant derivative* of $D\Gamma(N)$:

$$(v) \begin{cases} T_{cj(b)(l)...}^{ai(k)(d)...}|_{(g)}^{(s)} = \dfrac{\partial T_{cj(b)(l)...}^{ai(k)(d)...}}{\partial p_s^g} + T_{cj(b)(l)...}^{fi(k)(d)...} C_{f(g)}^{a(s)} + \\ + T_{cj(b)(l)...}^{ar(k)(d)...} C_{r(g)}^{i(s)} + T_{cj(f)(l)...}^{ai(r)(d)...} C_{(r)(b)(g)}^{(f)(k)(s)} + ... - \\ - T_{fj(b)(l)...}^{ai(k)(d)...} C_{c(g)}^{f(s)} - T_{cr(b)(l)...}^{ai(k)(d)...} C_{j(g)}^{r(s)} - T_{cj(b)(r)...}^{ai(k)(f)...} C_{(l)(f)(g)}^{(d)(r)(s)} - .... \end{cases}$$

Remark 2.2. *If $T = Y$ is a d-vector field on E^*, locally expressed by*

$$Y = Y^a \frac{\delta}{\delta t^a} + Y^i \frac{\delta}{\delta x^i} + Y_{(i)}^{(a)} \frac{\partial}{\partial p_i^a},$$

then the following expressions of the local covariant derivatives hold good:

$$(h_{\mathcal{T}})\begin{cases} Y^a_{/c} = \dfrac{\delta Y^a}{\delta t^c} + Y^b A^a_{bc}, \\ Y^i_{/c} = \dfrac{\delta Y^i}{\delta t^c} + Y^j A^i_{jc}, \\ Y^{(a)}_{(i)/c} = \dfrac{\delta Y^{(a)}_{(i)}}{\delta t^c} - Y^{(b)}_{(j)} A^{(a)(j)}_{(i)(b)c}, \end{cases} \qquad (h_M)\begin{cases} Y^a_{|k} = \dfrac{\delta Y^a}{\delta x^k} + Y^b H^a_{bk}, \\ Y^i_{|k} = \dfrac{\delta Y^i}{\delta x^k} + Y^j H^i_{jk}, \\ Y^{(a)}_{(i)|k} = \dfrac{\delta Y^{(a)}_{(i)}}{\delta x^k} - Y^{(b)}_{(j)} H^{(a)(j)}_{(i)(b)k}, \end{cases}$$

$$(v)\begin{cases} Y^a|^{(k)}_{(c)} = \dfrac{\partial Y^a}{\partial p^c_k} + Y^b C^{a(k)}_{b(c)}, \\ Y^i|^{(k)}_{(c)} = \dfrac{\partial Y^i}{\partial p^c_k} + Y^j C^{i(k)}_{j(c)}, \\ Y^{(a)}_{(i)}|^{(k)}_{(c)} = \dfrac{\partial Y^{(a)}_{(i)}}{\partial p^c_k} - Y^{(b)}_{(j)} C^{(a)(j)(k)}_{(i)(b)(c)}. \end{cases}$$

3. Components of h-normal N-linear connections on dual 1-jet spaces

Because the number of components which characterize an N-linear connection on E^* is big one (nine local components), we are constrained to study only a particular class of N-linear connections on E^*, which must be characterized by a reduced number of components. In this direction, let us fix on the temporal manifold $\mathcal{T}$ a semi-Riemannian metric h_{ab}, together with its Christoffel symbols χ^a_{bc}. Let $\mathbb{J}$ be the *h-normalization d-tensor field* on E^*, locally expressed by [5]

$$\mathbb{J} = J^{(i)}_{(a)bj} \delta p^a_i \otimes dt^b \otimes dx^j,$$

where $J^{(i)}_{(a)bj} = h_{ab}\delta^i_j$. In this context, we introduce the following geometrical concept:

Definition 3.1. *An N-linear connection $D\Gamma(N)$ on E^*, whose local components* (3) *verify the relations*

$$A^a_{bc} = \chi^a_{bc}, \qquad H^a_{bi} = 0, \qquad C^{a(i)}_{b(c)} = 0, \qquad D\mathbb{J} = 0,$$

*is called an h-**normal N-linear connection** on the dual* 1*-jet fibre bundle E^*.*

Theorem 3.1. *The adapted components of an h-normal N-linear connection $D\Gamma(N)$ verify the following identities:*

$$\begin{gathered} A^a_{bc} = \chi^a_{bc}, \quad H^a_{bi} = 0, \quad C^{a(i)}_{b(c)} = 0, \\ A^{(a)(j)}_{(i)(b)c} = \delta^a_b A^j_{ic} - \delta^j_i \chi^a_{bc}, \quad H^{(a)(j)}_{(i)(b)k} = \delta^a_b H^j_{ik}, \\ C^{(a)(j)(k)}_{(i)(b)(c)} = \delta^a_b C^{j(k)}_{i(c)}. \end{gathered} \tag{4}$$

Proof. It is obvious that the first three relations come immediately from the definition of an h-normal N-linear connection. To prove the other three relations, we emphasize that, taking into account the definition of the local $\mathcal{T}$-horizontal ("$_{/g}$"),

M-horizontal ("$_{|s}$") and vertical ("$|_{(g)}^{(s)}$") covariant derivatives produced by $D\Gamma(N)$, the condition $D\mathbb{J}=0$ is equivalent to

$$J^{(i)}_{(a)bj/g}=0,\qquad J^{(i)}_{(a)bj|s}=0,\qquad J^{(i)}_{(a)bj}|^{(s)}_{(g)}=0.$$

Consequently, the condition $D\mathbb{J}=0$ provides the local identities

$$h_{bf}A^{(f)(i)}_{(j)(a)c}=h_{ab}A^i_{jc}-\delta^i_j\left(\frac{\partial h_{ab}}{\partial t^c}-h_{ag}\chi^g_{bc}\right),$$

$$h_{bf}H^{(f)(i)}_{(j)(a)k}=h_{ba}H^i_{jk},\quad h_{bf}C^{(f)(i)(k)}_{(j)(a)(c)}=h_{ba}C^{i(k)}_{j(c)}.$$

Contracting now the above relations by h^{be}, we obtain the last required identities from (4). □

Remark 3.1. *The above theorem says us that an h-normal N-linear connection on E^* is an N-linear connection determined by **four** effective components (instead of nine in the general case):*

$$D\Gamma(N)=\left(\chi^a_{bc},\ A^i_{jc},\ H^i_{jk},\ C^{i(k)}_{j(c)}\right).$$

The other five components either vanish or are provided by the relations (4)*. Consequently, we can assert that the Berwald N_0-linear connection associated to the pair of metrics (h_{ab},φ_{ij}) is an h-normal N_0-linear connection on E^*, whose four effective components are*

$$B\Gamma(N_0)=\left(\chi^a_{bc},\ 0,\ \Gamma^i_{jk},\ 0\right).$$

4. Adapted components of torsion and curvature tensors

The study of the adapted components of the torsion and curvature tensors of an arbitrary N-linear connection $D\Gamma(N)$ on E^* was done in [5]. In that context, one proves that the torsion tensor $\mathbb{T}$ is determined by *twelve* effective local adapted d-tensors, while the curvature tensor $\mathbb{R}$ is determined by *eighteen* local adapted d-tensors. In what follows, we study the adapted components of the torsion and curvature tensors for an h-normal N-linear connection $D\Gamma(N)$.

Theorem 4.1. *The torsion tensor $\mathbb{T}$ of an h-normal N-linear connection $D\Gamma(N)$ is determined by **nine** effective local adapted d-tensors (instead of twelve in the general case):*

	h_T	h_M	v
h_Th_T	0	0	$R^{(f)}_{(r)ab}$
h_Mh_T	0	T^r_{aj}	$R^{(f)}_{(r)aj}$
vh_T	0	0	$P^{(f)\ (j)}_{(r)a(b)}$
h_Mh_M	0	T^r_{ij}	$R^{(f)}_{(r)ij}$
vh_M	0	$P^{r(j)}_{i(b)}$	$P^{(f)\ (j)}_{(r)i(b)}$
vv	0	0	$S^{(f)(i)(j)}_{(r)(a)(b)}$

(5)

where

$$T^r_{aj}=-A^r_{ja},\quad T^r_{ij}=H^r_{ij}-H^r_{ji},\quad P^{r(j)}_{i(b)}=C^{r(j)}_{i(b)},$$

$$P^{(f)\ (j)}_{(r)a(b)} = \frac{\partial \underset{1}{N}{}^{(f)}_{(r)a}}{\partial p^b_j} + \delta^f_b A^j_{ra} - \delta^j_r \chi^f_{ba}, \quad P^{(f)\ (j)}_{(r)i(b)} = \frac{\partial \underset{2}{N}{}^{(f)}_{(r)i}}{\partial p^b_j} + \delta^f_b H^j_{ri},$$

$$R^{(f)}_{(r)ab} = \frac{\delta \underset{1}{N}{}^{(f)}_{(r)a}}{\delta t^b} - \frac{\delta \underset{1}{N}{}^{(f)}_{(r)b}}{\delta t^a}, \quad R^{(f)}_{(r)aj} = \frac{\delta \underset{1}{N}{}^{(f)}_{(r)a}}{\delta x^j} - \frac{\delta \underset{2}{N}{}^{(f)}_{(r)i}}{\delta t^a},$$

$$R^{(f)}_{(r)ij} = \frac{\delta \underset{2}{N}{}^{(f)}_{(r)i}}{\delta x^j} - \frac{\delta \underset{2}{N}{}^{(f)}_{(r)j}}{\delta x^i}, \quad S^{(f)(i)(j)}_{(r)(a)(b)} = -\left(\delta^f_a C^{i(j)}_{r(b)} - \delta^f_b C^{j(i)}_{r(a)}\right).$$

Proof. Particularizing the general local expressions from [5], which generally give those twelve d-components of the torsion tensor of an N-linear connection, for an h-normal N-linear connection $D\Gamma(N)$, we deduce that the adapted components T^a_{bc}, T^a_{bj} and $P^{a(k)}_{b(c)}$ vanish, while the other nine are given by the formulas from theorem. □

Remark 4.1. *All torsion d-tensors of the Berwald h-normal N_0-linear connection $B\Gamma(N_0)$ (associated to the metrics h_{ab} and φ_{ij}) are zero, except*

$$R^{(f)}_{(r)ab} = \chi^f_{gab} p^g_r, \quad R^{(f)}_{(r)ij} = -\mathfrak{R}^s_{rij} p^f_s,$$

where $\chi^f_{gab}(t)$ (resp. $\mathfrak{R}^s_{rij}(x)$) are the local curvature tensors of the semi-Riemannian metric h_{ab} (resp. φ_{ij}).

Theorem 4.2. *The curvature $\mathbb{R}$ of an h-normal N-linear connection $D\Gamma(N)$ is characterized by **7** effective local d-tensors (instead of **18** in the general case):*

	$h_{\mathcal{T}}$	h_M	v
$h_{\mathcal{T}}h_{\mathcal{T}}$	χ^d_{abc}	R^l_{ibc}	$-R^{(d)(i)}_{(l)(a)bc} = \delta^i_l \chi^d_{abc} - \delta^d_a R^i_{lbc}$
$h_M h_{\mathcal{T}}$	0	R^l_{ibk}	$-R^{(d)(i)}_{(l)(a)bk} = -\delta^d_a R^i_{lbk}$
$w h_{\mathcal{T}}$	0	$P^{l\ (k)}_{ib(c)}$	$-P^{(d)(i)\ (k)}_{(l)(a)b(c)} = -\delta^d_a P^{i\ (k)}_{lb(c)}$
$h_M h_M$	0	R^l_{ijk}	$-R^{(d)(i)}_{(l)(a)jk} = -\delta^d_a R^i_{ljk}$
$w h_M$	0	$P^{l\ (k)}_{ij(c)}$	$-P^{(d)(i)\ (k)}_{(l)(a)j(c)} = -\delta^d_a P^{i\ (k)}_{lj(c)}$
ww	0	$S^{l(j)(k)}_{i(b)(c)}$	$-S^{(d)(i)(j)(k)}_{(l)(a)(b)(c)} = -\delta^d_a S^{i(j)(k)}_{l(b)(c)}$

(6)

where

$$R^d_{abc} := \chi^d_{abc} = \frac{\delta \chi^d_{ab}}{\delta t^c} - \frac{\delta \chi^d_{ac}}{\delta t^b} + \chi^f_{ab}\chi^d_{fc} - \chi^f_{ac}\chi^d_{fb},$$

$$R^l_{ibc} = \frac{\delta A^l_{ib}}{\delta t^c} - \frac{\delta A^l_{ic}}{\delta t^b} + A^r_{ib}A^l_{rc} - A^r_{ic}A^l_{rb} + C^{l(r)}_{i(f)} R^{(f)}_{(r)bc},$$

$$R^l_{ibk} = \frac{\delta A^l_{ib}}{\delta x^k} - \frac{\delta H^l_{ik}}{\delta t^b} + A^r_{ib}H^l_{rk} - H^r_{ik}A^l_{rb} + C^{l(r)}_{i(f)} R^{(f)}_{(r)bk},$$

$$P^{l\ (k)}_{ib(c)} = \frac{\partial A^l_{ib}}{\partial p^c_k} - C^{l(k)}_{i(c)/b} + C^{l(r)}_{i(f)} P^{(f)\ (k)}_{(r)b(c)},$$

$$R^l_{ijk} = \frac{\delta H^l_{ij}}{\delta x^k} - \frac{\delta H^l_{ik}}{\delta x^j} + H^r_{ij}H^l_{rk} - H^r_{ik}H^l_{rj} + C^{l(r)}_{i(f)} R^{(f)}_{(r)jk},$$

$$P^{l\ (k)}_{ij(c)} = \frac{\partial H^l_{ij}}{\partial p^c_k} - C^{l(k)}_{i(c)|j} + C^{l(f)}_{i(r)} P^{(r)\ (k)}_{(f)j(c)},$$

$$S^{l(j)(k)}_{i(b)(c)} = \frac{\partial C^{l(j)}_{i(b)}}{\partial p^c_k} - \frac{\partial C^{l(k)}_{i(c)}}{\partial p^b_j} + C^{r(j)}_{i(b)} C^{l(k)}_{r(c)} - C^{r(k)}_{i(c)} C^{l(j)}_{r(b)}.$$

Proof. The general formulas that express the local curvature d-tensors of an arbitrary N-linear connection (for more details, see [5]), applied to the particular case of an h-normal N-linear connection $D\Gamma(N)$, imply the above formulas and the relations from the Table (6). □

Remark 4.2. *In the case of the Berwald h-normal N_0-linear connection $B\Gamma(N_0)$ (associated to the pair of metrics (h_{ab}, φ_{ij})), all curvature d-tensors are zero, except*

$$R^d_{abc} = \chi^d_{abc}, \quad R^{(d)(i)}_{(l)(a)bc} = -\delta^i_l \chi^d_{abc}, \quad R^l_{ijk} = \mathfrak{R}^l_{ijk}, \quad R^{(d)(l)}_{(i)(a)jk} = \delta^d_a \mathfrak{R}^l_{ijk},$$

where $\chi^f_{gab}(t)$ (resp. $\mathfrak{R}^s_{rij}(x)$) are the local curvature tensors of the semi-Riemannian metric h_{ab} (resp. φ_{ij}).

5. Local Ricci identities. Non-metrical deflection d-tensor identities

Let us consider now the following more particular geometrical concept:

Definition 5.1. *An h-normal N-linear connection, whose local components*

$$C D\Gamma(N) = \left(\chi^a_{bc},\ A^i_{jc},\ H^i_{jk},\ C^{i(k)}_{j(c)}\right),$$

verify the relations

$$H^i_{jk} = H^i_{kj}, \qquad C^{i(k)}_{j(c)} = C^{k(i)}_{j(c)},$$

*is called an h-**normal N-linear connection of Cartan type** or a $CD\Gamma(N)$-**linear connection** on $E^* = J^{1*}(\mathcal{T}, M)$.*

Remark 5.1. *The torsion tensor $\mathbb{T}$ of an h-normal N-linear connection of Cartan type $CD\Gamma(N)$ is characterized only by **eight** adapted local d-tensors because the torsion components $T^i_{jk} = H^i_{jk} - H^i_{kj}$ from the Table (5) are vanishing.*

Example 5.1. *Taking into account that the Christoffel symbols $\Gamma^i_{jk}(x)$ of the spatial metric $\varphi_{ij}(x)$ are symmetric, it follows that the Berwald h-normal N_0-linear connection $B\Gamma(N_0)$ is of Cartan type.*

Theorem 5.1. *The following local **Ricci identities** for a $CD\Gamma(N)$-linear connection are true:*

- *the $h_{\mathcal{T}}$-Ricci identities:*

$$X^a_{/b/c} - X^a_{/c/b} = X^f \chi^a_{fbc} - X^a|^{(r)}_{(f)} R^{(f)}_{(r)bc},$$

$$X^a_{/b|k} - X^a_{|k/b} = -X^a_{|r} T^r_{bk} - X^a|^{(r)}_{(f)} R^{(f)}_{(r)bk},$$

$$X^a_{|j|k} - X^a_{|k|j} = -X^a|^{(r)}_{(f)} R^{(f)}_{(r)jk},$$

$$X^a_{/b}|^{(k)}_{(c)} - X^a|^{(k)}_{(c)/b} = -X^a|^{(r)}_{(f)} P^{(f)\ (k)}_{(r)b(c)},$$

$$X^a_{|j}|^{(k)}_{(c)} - X^a|^{(k)}_{(c)|j} = -X^a_{|r}C^{r(k)}_{j(c)} - X^a|^{(r)}_{(f)}P^{(f)\ (k)}_{(r)j(c)},$$
$$X^a|^{(j)}_{(b)}|^{(k)}_{(c)} - X^a|^{(k)}_{(c)}|^{(j)}_{(b)} = -X^a|^{(r)}_{(f)}S^{(f)(j)(k)}_{(r)(b)(c)};$$

- *the* h_M*-Ricci identities:*

$$X^i_{/b/c} - X^i_{/c/b} = X^r R^i_{rbc} - X^i|^{(r)}_{(f)}R^{(f)}_{(r)bc},$$
$$X^i_{/b|k} - X^i_{|k/b} = X^r R^i_{rbk} - X^i_{|r}T^r_{bk} - X^i|^{(r)}_{(f)}R^{(f)}_{(r)bk},$$
$$X^i_{|j|k} - X^i_{|k|j} = X^r R^i_{rjk} - X^i|^{(r)}_{(f)}R^{(f)}_{(r)jk},$$
$$X^i_{/b}|^{(k)}_{(c)} - X^i|^{(k)}_{(c)/b} = X^r P^{i\ (k)}_{rb(c)} - X^i|^{(r)}_{(f)}P^{(f)\ (k)}_{(r)b(c)},$$
$$X^i_{|j}|^{(k)}_{(c)} - X^i|^{(k)}_{(c)|j} = X^r P^{i\ (k)}_{rj(c)} - X^i_{|r}C^{r(k)}_{j(c)} - X^i|^{(r)}_{(f)}P^{(f)\ (k)}_{(r)j(c)},$$
$$X^i|^{(j)}_{(b)}|^{(k)}_{(c)} - X^i|^{(k)}_{(c)}|^{(j)}_{(b)} = X^r S^{i(j)(k)}_{r(b)(c)} - X^i|^{(r)}_{(f)}S^{(f)(j)(k)}_{(r)(b)(c)};$$

- *the* v*-Ricci identities:*

$$X^{(a)}_{(i)/b/c} - X^{(a)}_{(i)/c/b} = X^{(a)}_{(r)}R^r_{ibc} - X^{(f)}_{(i)}\chi^a_{fbc} - X^{(a)}_{(i)}|^{(r)}_{(f)}R^{(f)}_{(r)bc},$$
$$X^{(a)}_{(i)/b|k} - X^{(a)}_{(i)|k/b} = X^{(a)}_{(r)}R^r_{ibk} - X^{(a)}_{(i)|r}T^r_{bk} - X^{(a)}_{(i)}|^{(r)}_{(f)}R^{(f)}_{(r)bk},$$
$$X^{(a)}_{(i)|j|k} - X^{(a)}_{(i)|k|j} = X^{(a)}_{(r)}R^r_{ijk} - X^{(a)}_{(i)}|^{(r)}_{(f)}R^{(f)}_{(r)jk},$$
$$X^{(a)}_{(i)/b}|^{(k)}_{(c)} - X^{(a)}_{(i)}|^{(k)}_{(c)/b} = X^{(a)}_{(r)}P^{r\ (k)}_{ib(c)} - X^{(a)}_{(i)}|^{(r)}_{(f)}P^{(f)\ (k)}_{(r)b(c)},$$
$$X^{(a)}_{(i)|j}|^{(k)}_{(c)} - X^{(a)}_{(i)}|^{(k)}_{(c)|j} = X^{(a)}_{(r)}P^{r\ (k)}_{ij(c)} - X^{(a)}_{(i)|r}C^{r(k)}_{j(c)} - X^{(a)}_{(i)}|^{(r)}_{(f)}P^{(f)\ (k)}_{(r)j(c),}$$
$$X^{(a)}_{(i)}|^{(j)}_{(b)}|^{(k)}_{(c)} - X^{(a)}_{(i)}|^{(k)}_{(c)}|^{(j)}_{(b)} = X^{(a)}_{(r)}S^{r(j)(k)}_{i(b)(c)} - X^{(a)}_{(i)}|^{(r)}_{(f)}S^{(f)(j)(k)}_{(r)(b)(c)},$$

where

$$X = X^a\frac{\delta}{\delta t^a} + X^i\frac{\delta}{\delta x^i} + X^{(a)}_{(i)}\frac{\partial}{\partial p^a_i}$$

is an arbitrary d-vector field on the dual 1*-jet space* $E^* = J^{1*}(\mathcal{T}, M)$*.*

Proof. Let (Y_A) and (ω^A), where $A \in \left\{a, i, {}^{(a)}_{(i)}\right\}$, be on $E^* = J^{1*}(\mathcal{T}, M)$ the dual bases adapted to the nonlinear connection N, and let $X = X^F Y_F$ be a d-vector field on E^*. In this context, using the following true equalities (applied for a $CD\Gamma(N)$-linear connection D):

(1) $D_{Y_C}Y_B = \Gamma^F_{BC}Y_F$,

(2) $[Y_B, Y_C] = R^F_{BC}Y_F$,

(3) $\mathbb{T}(Y_C, Y_B) = \mathbb{T}^F_{BC}Y_F = \{\Gamma^F_{BC} - \Gamma^F_{CB} - R^F_{CB}\}Y_F$,

(4) $\mathbb{R}(Y_C, Y_B)Y_A = \mathbb{R}^F_{ABC}Y_F$,

(5) $D_{Y_C}\omega^B = -\Gamma^B_{FC}\omega^F$,

(6) $[\mathbb{R}(Y_C, Y_B)X] \otimes \omega^B \otimes \omega^C = \{D_{Y_C} D_{Y_B} X -$

$-D_{Y_B} D_{Y_C} X - D_{[Y_C,Y_B]} X\} \otimes \omega^B \otimes \omega^C,$

by a direct calculation, we find that

$$X^A_{:B:C} - X^A_{:C:B} = X^F \mathbb{R}^A_{FBC} - X^A_{:F} \mathbb{T}^F_{BC}, \tag{7}$$

where "$_{:G}$" represents one from the local covariant derivatives "$_{/b}$", "$_{|j}$" or "$|^{(j)}_{(b)}$" produced by the h-normal N-linear connection of Cartan type $CD\Gamma(N)$.

Taking into account in (7) that the indices $A, B, C, \ldots$ belong to the set

$$\left\{a, i, {}^{(a)}_{(i)}\right\},$$

and using the particular features of an h-normal N-linear connection of Cartan type $CD\Gamma(N)$ (i.e., the torsion d-components T^i_{jk} are zero; we have the curvature relations from the Table (6)), by complicated computations, we find what we were looking for (see also the Table (5)). □

In order to find an interesting application of the preceding Ricci identities, let us consider the *canonical Liouville-Hamilton d-tensor field of polymomenta* on $E^* = J^{1*}(T, M)$, which is given by

$$\mathbb{C}^* = p^a_i \frac{\partial}{\partial p^a_i}.$$

In this context, for an h-normal N-linear connection of Cartan type $CD\Gamma(N)$, we can construct the *non-metrical deflection d-tensors*, setting

$$\Delta^{(a)}_{(i)b} = p^a_{i/b}, \quad \Delta^{(a)}_{(i)j} = p^a_{i|j}, \quad \vartheta^{(a)(j)}_{(i)(b)} = p^a_i|^{(j)}_{(b)},$$

where "$_{/b}$", "$_{|j}$" and "$|^{(j)}_{(b)}$" are the local covariant derivatives produced by $CD\Gamma(N)$.

By direct local computations, we deduce that the non-metrical deflection d-tensors of $CD\Gamma(N)$ have the expressions:

$$\Delta^{(a)}_{(i)b} = -\underset{1}{N}{}^{(a)}_{(i)b} - A^r_{ib} p^a_r + \chi^a_{fb} p^f_i, \quad \Delta^{(a)}_{(i)j} = -\underset{2}{N}{}^{(a)}_{(i)j} - H^r_{ij} p^a_r,$$
$$\vartheta^{(a)(j)}_{(i)(b)} = \delta^a_b \delta^j_i - C^{r(j)}_{i(b)} p^a_r.$$

Applying now the preceding (v)-set of Ricci identities (attached to an h-normal N-linear connection of Cartan type) to the components of the canonical Liouville-Hamilton d-vector field of polymomenta, we get

Corollary 5.1. *The following the* ***deflection d-tensor identities****, associated to an h-normal N-linear connection of Cartan type, are true:*

$$\begin{cases} \Delta^{(a)}_{(i)b/c} - \Delta^{(a)}_{(i)c/b} = p^a_r R^r_{ibc} - p^f_i \chi^a_{fbc} - \vartheta^{(a)(r)}_{(i)(f)} R^{(f)}_{(r)bc} \\ \Delta^{(a)}_{(i)b|k} - \Delta^{(a)}_{(i)k/b} = p^a_r R^r_{ibk} - \Delta^{(a)}_{(i)r} T^r_{bk} - \vartheta^{(a)(r)}_{(i)(f)} R^{(f)}_{(r)bk} \\ \Delta^{(a)}_{(i)j|k} - \Delta^{(a)}_{(i)k|j} = p^a_r R^r_{ijk} - \vartheta^{(a)(r)}_{(i)(f)} R^{(f)}_{(r)jk} \\ \Delta^{(a)}_{(i)b}|^{(k)}_{(c)} - \vartheta^{(a)(k)}_{(i)(c)/b} = p^a_r P^{r\ (k)}_{ib(c)} - \vartheta^{(a)(r)}_{(i)(f)} P^{(f)\ (k)}_{(r)b(c)} \\ \Delta^{(a)}_{(i)j}|^{(k)}_{(c)} - \vartheta^{(a)(k)}_{(i)(c)|j} = p^a_r P^{r\ (k)}_{ij(c)} - \Delta^{(a)}_{(i)r} C^{r(k)}_{j(c)} - \vartheta^{(a)(r)}_{(i)(f)} P^{(f)\ (k)}_{(r)j(c)} \\ \vartheta^{(a)(j)}_{(i)(b)}|^{(k)}_{(c)} - \vartheta^{(a)(k)}_{(i)(c)}|^{(j)}_{(b)} = p^a_r S^{r(j)(k)}_{i(b)(c)} - \vartheta^{(a)(r)}_{(i)(f)} S^{(f)(j)(k)}_{(r)(b)(c)}. \end{cases} \tag{8}$$

Remark 5.2. *The deflection d-tensor identities* (8) *will be used in the near future for the construction of the* ***geometrical Maxwell equations*** *that will govern the abstract multi-time geometrical "electromagnetism" produced by a quadratic Hamiltonian depending on polymomenta (this is our work in progress).*

6. The local Bianchi identities of the $CD\Gamma(N)$-connections on the dual jet bundle $J^{1*}(\mathcal{T}, M)$

From the general theory of linear connections on a vector bundle, one knows that the torsions $\mathbb{T}$ and curvature $\mathbb{R}$ of a connection D on the dual 1-jet space $E^* = J^{1*}(T, M)$ are not independent. In other words, they are interrelated by the following general *Bianchi identities* (for any $X, Y, Z, U \in \mathcal{X}(E^*)$):

$$\sum_{\{X,Y,Z\}} \{(D_X \mathbb{T})(Y, Z) - \mathbb{R}(X, Y)Z + \mathbb{T}(\mathbb{T}(X, Y), Z)\} = 0,$$
$$\sum_{\{X,Y,Z\}} (D_X \mathbb{R})(Y, Z, U) + \mathbb{R}(\mathbb{T}(X, Y), Z)U = 0,$$

where $\Sigma_{\{X,Y,Z\}}$ means a cyclic sum. Obviously, working with a $CD\Gamma(N)$-linear connection and the local adapted basis of d-vector fields $(X_A) \subset \mathcal{X}(E^*)$ (associated to the given nonlinear connection N on E^*), the above Bianchi identities are locally described by the equalities:

$$\begin{aligned} &\sum_{\{A,B,C\}} \left\{\mathbb{R}^F_{ABC} - \mathbb{T}^F_{AB:C} - \mathbb{T}^G_{AB}\mathbb{T}^F_{CG}\right\} = 0, \\ &\sum_{\{A,B,C\}} \left\{\mathbb{R}^F_{DAB:C} + \mathbb{T}^G_{AB}\mathbb{R}^F_{DCG}\right\} = 0, \end{aligned} \tag{9}$$

where $\mathbb{R}(X_A, X_B)X_C = \mathbb{R}^D_{CBA}X_D$, $\mathbb{T}(X_A, X_B) = \mathbb{T}^D_{BA}X_D$, and "$_{:C}$" represents one from the local covariant derivatives "$_{/a}$", "$_{|i}$" or "$|^{(i)}_{(a)}$" of the $CD\Gamma(N)$-linear connection D (for similar details, see the works [15], [16] and [19]). Consequently, we find:

Theorem 6.1. *The following* ***thirty*** *effective local* ***Bianchi identities*** *for an h-normal N-linear connection of Cartan type $CD\Gamma(N)$ are true on the dual 1-jet space $E^* = J^{1*}(\mathcal{T}, M)$:*

- *the first set:*

  1. $\sum_{\{a,b,c\}} \chi^d_{abc} = 0,$
  2. $\mathcal{A}_{\{a,b\}} \left\{ T^l_{ar} T^r_{bk} - T^l_{ak/b} \right\} = R^l_{kab} - C^{l(r)}_{k(f)} R^{(f)}_{(r)ab},$
  3. $\mathcal{A}_{\{j,k\}} \left\{ C^{l(r)}_{k(f)} R^{(f)}_{(r)aj} + R^l_{jak} + T^l_{aj|k} \right\} = 0,$
  4. $\sum_{\{i,j,k\}} \left\{ C^{l(r)}_{k(f)} R^{(f)}_{(r)ij} - R^l_{ijk} \right\} = 0,$

- *the second set:*

  5. $\sum_{\{a,b,c\}} \left\{ R^{(d)}_{(l)ab/c} + P^{(d)\ \ (r)}_{(l)c(f)} R^{(f)}_{(r)ab} \right\} = 0,$
  6. $\mathcal{A}_{\{a,b\}} \left\{ R^{(d)}_{(l)ak/b} + P^{(d)\ \ (r)}_{(l)b(f)} R^{(f)}_{(r)ak} + R^{(d)}_{(l)br} T^r_{ak} \right\} =$
     $= R^{(d)}_{(l)ab|k} + P^{(d)\ \ (r)}_{(l)k(f)} R^{(f)}_{(r)ab},$
  7. $\mathcal{A}_{\{j,k\}} \left\{ R^{(d)}_{(l)aj|k} + P^{(d)\ \ (r)}_{(l)k(f)} R^{(f)}_{(r)aj} + R^{(d)}_{(l)kr} T^r_{aj} \right\} =$
     $= -R^{(d)}_{(l)jk/a} - P^{(d)\ \ (r)}_{(l)a(f)} R^{(f)}_{(r)jk},$
  8. $\sum_{\{i,j,k\}} \left\{ R^{(d)}_{(l)ij|k} + P^{(d)\ \ (r)}_{(l)k(f)} R^{(f)}_{(r)ij} \right\} = 0,$

- *the third set:*

  9. $T^l_{ak}|^{(p)}_{(e)} - C^{l(p)}_{r(e)} T^r_{ak} + P^{l\ \ (p)}_{ka(e)} + C^{l(p)}_{k(e)/a} - C^{l(r)}_{k(f)} P^{(f)\ \ (p)}_{(r)a(e)} + C^{r(p)}_{k(e)} T^l_{ar} = 0,$
  10. $\mathcal{A}_{\{j,k\}} \left\{ C^{l(p)}_{j(e)|k} + C^{l(r)}_{k(f)} P^{(f)\ \ (p)}_{(r)j(e)} + P^{l\ \ (p)}_{jk(e)} \right\} = 0,$

- *the fourth set:*

  11. $\mathcal{A}_{\{a,b\}} \left\{ P^{(d)\ \ (p)}_{(l)a(e)/b} + P^{(d)\ \ (r)}_{(l)b(f)} P^{(f)\ \ (p)}_{(r)a(e)} \right\} =$
      $= R^{(d)}_{(l)ab}|^{(p)}_{(e)} + R^{(d)(p)}_{(l)(e)ab} + S^{(d)(p)(r)}_{(l)(e)(f)} R^{(f)}_{(r)ab},$
  12. $\mathcal{A}_{\{a,k\}} \left\{ P^{(d)\ \ (p)}_{(l)a(e)|k} + P^{(d)\ \ (r)}_{(l)k(f)} P^{(f)\ \ (p)}_{(r)a(e)} \right\} =$
      $= R^{(d)}_{(l)ak}|^{(p)}_{(e)} + R^{(d)(p)}_{(l)(e)ak} + S^{(d)(p)(r)}_{(l)(e)(f)} R^{(f)}_{(r)ak} + R^{(d)}_{(l)ar} C^{r(p)}_{k(e)} - T^r_{ak} P^{(d)\ (p)}_{(l)r(e)},$
  13. $\mathcal{A}_{\{j,k\}} \left\{ P^{(d)\ (p)}_{(l)j(e)|k} + P^{(d)\ (r)}_{(l)k(f)} P^{(f)\ (p)}_{(r)j(e)} + R^{(d)}_{(l)kr} C^{r(p)}_{j(e)} \right\} =$
      $= R^{(d)}_{(l)jk}|^{(p)}_{(e)} + R^{(d)(p)}_{(l)(e)jk} + S^{(d)(p)(r)}_{(l)(e)(f)} R^{(f)}_{(r)jk},$

- *the fifth set:*

  14. $\mathcal{A}_{\left\{ {(j) \atop (b)}, {(k) \atop (c)} \right\}} \left\{ C^{l(j)}_{i(b)}|^{(k)}_{(c)} + C^{r(k)}_{i(c)} C^{l(j)}_{r(b)} \right\} = S^{l(j)(k)}_{i(b)(c)} - C^{l(r)}_{i(f)} S^{(f)(j)(k)}_{(r)(b)(c)},$

- *the sixth set:*

15. $\mathcal{A}_{\left\{ {(j) \atop (b)}, {(k) \atop (c)} \right\}} \left\{ P^{(d)\ \ (j)}_{(l)a(b)}|^{(k)}_{(c)} + P^{(f)\ \ (j)}_{(r)a(b)} S^{(d)(k)(r)}_{(l)(c)(f)} - P^{(d)(j)\ \ (k)}_{(l)(b)a(c)} \right\} =$

$= -S^{(d)(j)(k)}_{(l)(b)(c)/a} - S^{(f)(j)(k)}_{(r)(b)(c)} P^{(d)\ \ (r)}_{(l)a(f)},$

16. $\mathcal{A}_{\left\{ {(j) \atop (b)}, {(k) \atop (c)} \right\}} \left\{ P^{(d)\ (j)}_{(l)i(b)}|^{(k)}_{(c)} + P^{(f)\ (j)}_{(r)i(b)} S^{(d)(k)(r)}_{(l)(c)(f)} - P^{(d)(j)\ (k)}_{(l)(b)i(c)} - \right.$

$\left. -C^{r(j)}_{i(b)} P^{(d)\ \ (k)}_{(l)r(c)} \right\} = -S^{(d)(j)(k)}_{(l)(b)(c)|i} - S^{(f)(j)(k)}_{(r)(b)(c)} P^{(d)\ (r)}_{(l)i(f)},$

- *the seventh set:*

17. $\sum_{\left\{ {(i) \atop (a)}, {(j) \atop (b)}, {(k) \atop (c)} \right\}} \left\{ S^{(d)(i)(j)}_{(l)(a)(b)}|^{(k)}_{(c)} + S^{(f)(i)(j)}_{(r)(a)(b)} S^{(d)(k)(r)}_{(l)(c)(f)} + S^{(d)(i)(j)(k)}_{(l)(a)(b)(c)} \right\} = 0,$

- *the eight set:*

18. $\sum_{\{a,b,c\}} \chi^d_{eab/c} = 0,$

19. $\chi^d_{eab|k} = 0,$

20. $\chi^d_{eab}|^{(k)}_{(c)} = 0,$

21. $\sum_{\{a,b,c\}} \left\{ R^l_{pab/c} + R^{(f)}_{(r)ab} P^{l\ \ (r)}_{pc(f)} \right\} = 0,$

22. $\mathcal{A}_{\{a,b\}} \left\{ R^l_{pak/b} + R^{(f)}_{(r)ak} P^{l\ \ (r)}_{pb(f)} + T^r_{ak} R^l_{pbr} \right\} = R^l_{pab|k} + R^{(f)}_{(r)ab} P^{l\ \ (r)}_{pk(f)},$

23. $\mathcal{A}_{\{j,k\}} \left\{ R^l_{paj|k} + R^{(f)}_{(r)aj} P^{l\ \ (r)}_{pk(f)} + T^r_{aj} R^l_{pkr} \right\} = -R^l_{pjk/a} - R^{(f)}_{(r)jk} P^{l\ \ (r)}_{pa(f)},$

24. $\sum_{\{i,j,k\}} \left\{ R^l_{pij|k} + R^{(f)}_{(r)ij} P^{l\ \ (r)}_{pk(f)} \right\} = 0,$

- *the nineth set:*

25. $\mathcal{A}_{\{a,b\}} \left\{ P^{l\ \ (p)}_{ia(e)/b} + P^{(f)\ \ (p)}_{(r)a(e)} P^{l\ (r)}_{ib(f)} \right\} = R^l_{iab}|^{(p)}_{(e)} + R^{(f)}_{(r)ab} S^{l(p)(r)}_{i(e)(f)},$

26. $\mathcal{A}_{\{a,k\}} \left\{ P^{l\ \ (p)}_{ia(e)|k} + P^{(f)\ \ (p)}_{(r)a(e)} P^{l\ \ (r)}_{ik(f)} \right\} =$

$= R^l_{iak}|^{(p)}_{(e)} + R^{(f)}_{(r)ak} S^{l(p)(r)}_{i(e)(f)} + C^{r(p)}_{k(e)} R^l_{iar} - T^r_{ak} P^{l\ \ (p)}_{ir(e)},$

27. $\mathcal{A}_{\{j,k\}} \left\{ P^{l\ \ (p)}_{ij(e)|k} + P^{(f)\ \ (p)}_{(r)j(e)} P^{l\ \ (r)}_{ik(f)} + C^{r(p)}_{j(e)} R^l_{ikr} \right\} =$

$= R^l_{ijk}|^{(p)}_{(e)} + R^{(f)}_{(r)jk} S^{l(p)(r)}_{i(e)(f)},$

- *the tenth set:*

28. $\mathcal{A}_{\left\{ {(j) \atop (b)}, {(k) \atop (c)} \right\}} \left\{ P^{l\ \ (j)}_{pa(b)}|^{(k)}_{(c)} + P^{(f)\ \ (j)}_{(r)a(b)} S^{l(k)(r)}_{p(c)(f)} \right\} =$

$= -S^{l(j)(k)}_{p(b)(c)/a} - S^{(f)(j)(k)}_{(r)(b)(c)} P^{l\ \ (r)}_{pa(f)},$

29. $\mathcal{A}_{\left\{ {(j) \atop (b)}, {(k) \atop (c)} \right\}} \left\{ P^{l\ \ (j)}_{pi(b)}|^{(k)}_{(c)} + P^{(f)\ \ (j)}_{(r)i(b)} S^{l(k)(r)}_{p(c)(f)} - C^{r(j)}_{i(b)} P^{l\ \ (k)}_{pr(c)} \right\} =$

$= -S^{l(j)(k)}_{p(b)(c)|i} - S^{(f)(j)(k)}_{(r)(b)(c)} P^{l\ \ (r)}_{pi(f)},$

- *the eleventh set:*

$$30.\ \sum_{\left\{ {(i) \atop (a)}, {(j) \atop (b)}, {(k) \atop (c)} \right\}} \left\{ S^{l(i)(j)}_{p(a)(b)}|^{(k)}_{(c)} + S^{(r)(i)(j)}_{(f)(a)(b)} S^{l(k)(f)}_{p(c)(r)} \right\} = 0,$$

where, if $\{A, B, C\}$ *are indices of type* $\left\{a, i, {(a) \atop (i)}\right\}$, *then* $\sum_{\{A,B,C\}}$ *represents a cyclic sum, and* $\mathcal{A}_{\{A,B\}}$ *represents an alternate sum.*

Proof. Taking into account that the indices $A, B, C, D...$ are of type

$$\left\{a, i, {(a) \atop (i)}\right\},$$

and the torsion $\mathbb{T}^C_{AB}$ and curvature $\mathbb{R}^D_{ABC}$ adapted components are given in the Tables (5) and (6), after laborious local computations, the formulas (9) imply the required Bianchi identities. □

Remark 6.1. *We point out that, in the particular single-time case*

$$(\mathcal{T}, h) = (\mathbb{R}, \delta = 1),$$

the last identity of our each set of local Bianchi identities reduces to one of the classical eleven Bianchi identities that characterize the N-linear connections in the classical Hamilton geometry on cotangent bundles (see [16]*).*

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