STRONG CONVERGENCE OF THREE-STEP ITERATION PROCESSES FOR MULTIVALUED MAPPINGS IN SOME $CAT(k)$ SPACES

Saeed Shabani[3]

In this paper, we prove the strong convergence of the three-step iteration processes for some generalized nonexpansive multivalued mappings in the framework of $CAT(1)$ spaces. The obtained results extend some recent known results.

Keywords: Hybrid iterative process, generalized nonexpansive mapping, Common fixed point, strong convergence theorem.
MSC2010: 47H04; 47H09; 47H10.

1. Preliminaries

Let (X, d) be a metric space, and $x \in X$, $E \subset X$. The distance from x to E is defined by $dist(x, E) = inf\{d(x, y) : y \in E\}$. The diameter of E is defined by $diam(E) = sup\{d(x, y) : x, y \in E\}$. The set E is called proximal if for each $x \in E$, there exists an element $y \in E$ such that $d(x, y) = dist(x, E)$. we denote by $CB(E)$ the collection of all nonempty closed bounded subsets of E. The Hausdorff metric H on $CB(E)$ is defined by

$$H(A, B) = \max\{\sup_{x\in A} dist(x, B), \sup_{y\in B} dist(y, A)\},$$

for all $A, B \in CB(E)$. Let $T : X \to CB(E)$ be a multivalued mapping. An element $x \in X$ is said to be a fixed point of T if $x \in Tx$. The set of fixed points of T will be denoted by $F(T)$. The multivalued mapping $T : X \to CB(E)$ is said to

(i) be nonexpansive if

$$H(Tx, Ty) \leq d(x, y), \qquad x, y \in E;$$

(ii) be quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$H(Tx, Tp) \leq d(x, p), \qquad x \in E, \quad p \in F(T);$$

(iii) be hemicompact if for any sequence $\{x_n\}$ in E such that

$$\lim_{n\to\infty} dist(x_n, Tx_n) = 0,$$

there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = q \in E$.

In 2014, Thakur et al. [**?**] introduced the iterative process as follows:

Department of Mathematics, Izeh Branch, Islamic Azad University, Izeh, Iran, e-mail: shabani60@gmail.com, s.shabani@srbiau.ac.ir

Let E be a nonempty convex subset of a Banach space X and $T : E \to E$ be a nonexpansive mapping and $\alpha_n, \beta_n, \gamma_n \in (0,1)$. The sequence $\{x_n\}$ define by $x_1 \in E$ and

$$\begin{aligned} z_n = &\quad (1-\alpha_n)x_n + \alpha_n T x_n, \\ y_n = &\quad (1-\beta_n)z_n + \beta_n T z_n, \\ x_{n+1} = &\quad (1-\gamma_n)T x_n + \gamma_n T y_n, \end{aligned} \tag{1.1}$$

In their work, it was proved that this process compared to other processes such as the Mann, the Ishikawa, the Noor, the Agarwal et al. and the Abbas et al. converges faster.

In this paper, we extend (1.1) to multivalued quasi-nonexpansive mappings in some $CAT(k)$ spaces (see (2.2)).

Roughly speaking, $CAT(k)$ spaces are geodesic spaces of bounded curvature. The precise definition is given below. The study of fixed point theory in $CAT(k)$ was initiated by kirk [**?**, **?**]. His works were followed by many authors (see, e.g., [4–16]).

Let (X,d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, briefly, a geodesic from x to y) is a map c from a closed interval $[0,l] \subset R$ to X such that $c(0) = x, c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0,l]$. In particular, c is an isometry and $d(x,y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique, this geodesic is denoted by $[x,y]$. This means that $z \in [x,y]$ if and only if there exists $\alpha \in [0,1]$ such that

$$d(x,z) = td(x,y), \qquad d(y,z) = (1-t)d(x,y)$$

In this case, we write $(1-t)x \oplus ty$.

The space (X,d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x to y, for each $x, y \in X$. A subset $Y \subset X$ is said to be convex if Y includes every geodesic segment joining any two of its points.
In a geodesic space (X,d) for $x, y, z \in X$ and $t \in [0,1]$, one has

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z).$$

Let $D \in (0, \infty]$, then (X,d) is called a $D-$geodesic space if any two points of X with their distance smaller than D are joined by a geodesic segment. Notice that (X,d) is a geodesic space if and only if it is a $D-$geodesic space.
Let $n \in N$, we denote by $\langle .,. \rangle$ the Euclidean scalar product in R^n, that is,

$$\langle x, y \rangle = \Sigma_{i=1}^n x_i y_i, \quad x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n).$$

Let S^n denote the $n-$dimensional sphere defined by

$$S^n = \{x = (x_1, \ldots, x_{n+1}) \in R^{n+1} \ : \ \langle x, y \rangle = 1\},$$

with metric $d(x,y) = arccos\langle x, y \rangle$, for all $(x,y) \in S^n \times S^n$ (see[**?**, Proposition 2.1]).
From now on, we assume that $k \geq 0$ and define

$$D_k := \begin{cases} \frac{\pi}{\sqrt{k}} & k > 0 \\ \\ +\infty & k = 0 \end{cases}$$

we denote by M_k^n the following metric spaces:

(i) if $k = 0$ then M_0^n is the Euclidean space R^n;
(ii) if $k > 0$ then M_k^n is obtained from S^n by multiplying the distance function by the constant $\frac{1}{\sqrt{k}}$.

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in M_k^2 such that

$$d_{M_k^2}(\bar{x_i}, \bar{x_j}) = d(x_i, x_j), \qquad i, j \in \{1, 2, 3\}.$$

By [**?**, Lemma 2.14] a comparison triangle for Δ always exists provided that the perimeter $d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_1) < 2D_k$. A point $\bar{p} \in [\bar{x}, \bar{y}]$ is called a comparison point for $p \in [x, y]$ if $d(x, p) = d_{M_k^2}(\bar{x}, \bar{p})$.
A geodesic triangle $\Delta(x, y, z)$ in X with perimeter less than $2D_k$ is said satisfy the $CAT(k)$ inequality if for any $p, q \in \Delta(x, y, z)$ and for their comparison points $\bar{p}, \bar{q} \in \bar{\Delta}(x, y, z)$, one has

$$d(p, q) \leq d_{M_k^2}(\bar{p}, \bar{q}).$$

A metric space (X, d) is called a $CAT(k)$ space if it is D_k-geodesic and any geodesic triangle $\Delta(x, y, z)$ in X with perimeter less than $2D_k$ satisfies the $CAT(k)$ inequality.
In this paper, we consider $CAT(k)$ space with $k \geq 0$. since most of the results for such spaces are easily deduced from those for $CAT(1)$ spaces, in what follows, we mainly focus on $CAT(1)$ spaces.

The following lemmas are needed.

Lemma 1.1. [**?**, Proposition 3.1]. *If (X, d) is a $CAT(1)$ space with $diam(X) < \pi/2$, then there is a constant $K > 0$ such that*

$$d((1-t)x \oplus ty, z)^2 \leq (1-t)d(x, z)^2 + td(y, z)^2 - \frac{K}{2}t(1-t)d(x, y)^2,$$

for any $t \in [0, 1]$ and any points $x, y, z \in X$.

Lemma 1.2. [**?**]. *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences such that*

(i) $0 \leq \alpha_n, \beta_n < 1$;
(ii) $\beta_n \to 0$;
(iii) $\sum \alpha_n \beta_n = \infty$.

Let $\{\lambda_n\}$ be a nonnegative real sequence such that $\sum_{n=1}^{\infty} \alpha_n \beta_n (1-\beta_n) \lambda_n$ is bounded. Then $\{\lambda_n\}$ has a subsequence which converges to zero.

2. **Main result**

Lemma 2.1. *Let (X,d) be a $CAT(1)$ space with convex metric and E be a nonempty closed convex subset of X. Let $T : E \to CB(E)$ be a multivalued mapping with $F(T) \neq \emptyset$ and P_T is quasi-nonexpansive mapping where*

$$P_T(x) = \{y \in T(x) : d(x, y) = dist(x, T(x))\}.$$

For an initial point $x_0 \in E$, let $\{x_n\}$ be sequence generated by the following algorithm:

$$\begin{aligned} z_n &= (1-\alpha_n)x_n \oplus \alpha_n w_n, \\ y_n &= (1-\beta_n)z_n \oplus \beta_n w_n', \\ x_{n+1} &= (1-\gamma_n)w_n \oplus \gamma_n w_n'', \end{aligned} \tag{2.2}$$

where $w_n \in P_T x_n$, $w_n' \in P_T z_n$, $w_n'' \in P_T y_n$, and $\alpha_n, \beta_n, \gamma_n \in [a, b] \subset (0, 1)$. Then, $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F(T)$.

Proof. for all $n \geq 0$, we have

$$\begin{aligned} d(z_n, p) = & \ d((1-\alpha_n)x_n \oplus \alpha_n w_n, p) \\ \leq & \ (1-\alpha_n)d(x_n, p) + \alpha_n d(w_n, p) \\ \leq & \ (1-\alpha_n)d(x_n, p) + \alpha_n dist(w_n, P_{T_n}p) \\ \leq & \ (1-\alpha_n)d(x_n, p) + \alpha_n H(P_{T_n}x_n, P_{T_n}p) \\ \leq & \ (1-\alpha_n)d(x_n, p) + \alpha_n d(x_n, p) = d(x_n, p), \end{aligned}$$

and

$$\begin{aligned} d(y_n, p) = & \ d((1-\beta_n)z_n \oplus \beta_n w'_n, p) \\ \leq & \ (1-\beta_n)d(z_n, p) + \beta_n d(w'_n, p) \\ \leq & \ (1-\beta_n)d(x_n, p) + \beta_n dist(w'_n, P_{T_n}p) \\ \leq & \ (1-\beta_n)d(x_n, p) + \beta_n H(P_{T_n}z_n, P_{T_n}p) \\ \leq & \ (1-\beta_n)d(x_n, p) + \beta_n d(z_n, p) \\ \leq & \ (1-\beta_n)d(x_n, p) + \beta_n d(x_n, p) = d(x_n, p), \end{aligned}$$

and

$$\begin{aligned} d(x_{n+1}, p) = & \ d((1-\gamma_n)w_n \oplus \gamma_n w''_n, p) \\ \leq & \ (1-\gamma_n)d(w_n, p) + \gamma_n d(w''_n, p) \\ \leq & \ (1-\gamma_n)dist(w_n, P_{T_n}p) + \gamma_n dist(w''_n, P_{T_n}p) \\ \leq & \ (1-\gamma_n)H(P_{T_n}x_n, P_{T_n}p) + \gamma_n H(P_{T_n}y_n, P_{T_n}p) \\ \leq & \ (1-\gamma_n)d(x_n, p) + \gamma_n d(y_n, p) \\ \leq & \ (1-\gamma_n)d(x_n, p) + \gamma_n d(x_n, p) = d(x_n, p), \end{aligned}$$

This implies that $\{d(x_n, p)\}_{n=1}^{\infty}$ is bounded and decreasing. Hence $\lim_{n\to\infty} d(x_n, p)$ exists. □

We remark that there exist examples of mappings for which S_T is nonexpansive (see [**?**]), so that the assumption on T is not artificial.

Lemma 2.2. *Let (X,d) be a $CAT(1)$ space with convex metric and diam$(X) < \pi/2$ and E be a nonempty closed convex subset of X. Let $T : E \to CB(E)$ be a multivalued mapping with $F(T) \neq \emptyset$ and P_T is quasi-nonexpansive mapping. Let $\alpha_n, \beta_n, \gamma_n \in [a, b] \subset (0, 1)$ and $\{x_n\}$ be sequence generated by (2.2). Then,* $\lim_{n\to\infty} dist(x_n, P_T x_n) = 0$.

Proof.

$$\begin{aligned} d(z_n, p)^2 = & \ d((1-\alpha_n)x_n \oplus \alpha_n w_n, p)^2 \\ \leq & \ (1-\alpha_n)d(x_n, p)^2 + \alpha_n d(w_n, p)^2 - \tfrac{K}{2}\alpha_n(1-\alpha_n)d(x_n, w_n)^2 \\ \leq & \ (1-\alpha_n)d(x_n, p)^2 + \alpha_n dist(w_n, P_{T_n}p)^2 - \tfrac{K}{2}\alpha_n(1-\alpha_n)d(x_n, w_n)^2 \\ \leq & \ (1-\alpha_n)d(x_n, p)^2 + \alpha_n H(P_{T_n}x_n, P_{T_n}p)^2 - \tfrac{K}{2}\alpha_n(1-\alpha_n)d(x_n, w_n)^2 \\ \leq & \ (1-\alpha_n)d(x_n, p)^2 + \alpha_n d(x_n, p)^2 - \tfrac{K}{2}\alpha_n(1-\alpha_n)d(x_n, w_n)^2 \\ \leq & \ d(x_n, p)^2 - \tfrac{K}{2}\alpha_n(1-\alpha_n)d(x_n, w_n)^2, \end{aligned}$$

and

$$\begin{aligned}
d(y_n,p)^2 =& \; d((1-\beta_n)z_n \oplus \beta_n w'_n, p)^2\\
\leq& \; (1-\beta_n)d(z_n,p)^2 + \beta_n d(w'_n,p)^2 - \tfrac{K}{2}\beta_n(1-\beta_n)d(z_n,w'_n)^2\\
\leq& \; (1-\beta_n)d(z_n,p)^2 + \beta_n dist(w'_n,P_{T_n}p)^2 - \tfrac{K}{2}\beta_n(1-\beta_n)d(z_n,w'_n)^2\\
\leq& \; (1-\beta_n)d(z_n,p)^2 + \beta_n H(P_{T_n}z_n,P_{T_n}p)^2 - \tfrac{K}{2}\beta_n(1-\beta_n)d(z_n,w'_n)^2\\
\leq& \; (1-\beta_n)d(z_n,p)^2 + \beta_n d(z_n,p)^2 - \tfrac{K}{2}\beta_n(1-\beta_n)d(z_n,w'_n)^2\\
=& \; d(z_n,p)^2 - \tfrac{K}{2}\beta_n(1-\beta_n)d(z_n,w'_n)^2\\
\leq& \; d(z_n,p)^2\\
\leq& \; d(x_n,p)^2 - \tfrac{K}{2}\alpha_n(1-\alpha_n)d(x_n,w_n)^2.
\end{aligned}$$

and

$$\begin{aligned}
d(x_{n+1},p)^2 =& \; d((1-\gamma_n)w_n + \gamma_n w''_n, p)^2\\
\leq& \; (1-\gamma_n)d(w_n,p)^2 + \gamma_n d(w''_n,p)^2 - \tfrac{K}{2}\gamma_n(1-\gamma_n)d(w_n,w''_n)^2\\
\leq& \; (1-\gamma_n)d(w_n,p)^2 + \gamma_n dist(w''_n,P_{T_n}p)^2 - \tfrac{K}{2}\gamma_n(1-\gamma_n)d(w_n,w''_n)^2\\
\leq& \; (1-\gamma_n)d(w_n,p)^2 + \gamma_n H(P_{T_n}y_n,P_{T_n}p)^2 - \tfrac{K}{2}\gamma_n(1-\gamma_n)d(w_n,w''_n)^2\\
\leq& \; (1-\gamma_n)d(w_n,p)^2 + \gamma_n d(y_n,p)^2 - \tfrac{K}{2}\gamma_n(1-\gamma_n)d(w_n,w''_n)^2\\
\leq& \; (1-\gamma_n)dist(w_n,P_{T_n}p)^2 + \gamma_n d(y_n,p)^2 - \tfrac{K}{2}\gamma_n(1-\gamma_n)d(w_n,w''_n)^2\\
\leq& \; (1-\gamma_n)H(P_{T_n}x_n,P_{T_n}p)^2 + \gamma_n d(y_n,p)^2 - \tfrac{K}{2}\gamma_n(1-\gamma_n)d(w_n,w''_n)^2\\
\leq& \; (1-\gamma_n)d(x_n,p)^2 + \gamma_n d(y_n,p)^2 - \tfrac{K}{2}\gamma_n(1-\gamma_n)d(w_n,w''_n)^2\\
\leq& \; (1-\gamma_n)d(x_n,p)^2 + \gamma_n d(y_n,p)^2\\
\leq& \; (1-\gamma_n)d(x_n,p)^2 + \gamma_n d(x_n,p)^2 - \tfrac{K}{2}\gamma_n\alpha_n(1-\alpha_n)d(x_n,w_n)^2\\
=& \; d(x_n,p)^2 - \tfrac{K}{2}\gamma_n\alpha_n(1-\alpha_n)d(x_n,w_n)^2
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\tfrac{K}{2}a^2(1-b)d(x_n,w_n)^2 \leq& \; \tfrac{K}{2}\gamma_n\alpha_n(1-\alpha_n)d(x_n,w_n)^2\\
\leq& \; d(x_n,p)^2 - d(x_{n+1},p)^2.
\end{aligned} \tag{2.3}$$

so

$$\sum_{n=0}^{\infty} \frac{K}{2}a^2(1-b)d(x_n,w_n)^2 < \infty,$$

Thus we obtain that $\lim_{n\to\infty} d(x_n,w_n)^2 = 0$. Hence $\lim_{n\to\infty} dist(x_n,P_T x_n) = 0$. □

Recall that a mapping $T: K \to X$ is said to satisfy Condition (I) (see [**?**]), if there exists a nondecreasing function $f: [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for each $r > 0$ such that

$$dist(x,Tx) \geq f(dist(x,F(T))),$$

for $x \in K$.

Theorem 2.1. *Let (X,d) be a $CAT(1)$ space with convex metric and $diam(X) < \pi/2$ and E be a nonempty closed convex subset of X. Let $T: E \to CB(E)$ be a multivalued mapping with $F(T) \neq \emptyset$ and P_T is quasi-nonexpansive mapping. Let $\alpha_n, \beta_n, \gamma_n \in [a,b] \subset (0,1)$ and $\{x_n\}$ be sequence generated by (2.2). If T satisfies condition (I), Then, $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 2.2, $\lim_{n\to\infty} dist(x_n,Tx_n) \leq \lim_{n\to\infty} dist(x_n,P_T x_n) = 0$. Since T satisfies condition (I), we have $\lim_{n\to\infty} dist(x_n,F(T)) = 0$. The rest of the proof follows the proof of Theorem 3.2 in [**?**]. □

In 2011, Falset et al. [?] introduced Condition (E) which is weaker than nonexpansive and stronger than quasi-nonexpansive: A mapping $T : E \to CB(E)$ satisfy Condition (E_μ), if

$$dist(x, Ty) \leq \mu dist(Tx, x) + d(x, y)$$

holds, for all $x, y \in K$. It is said T satisfies Condition (E) whenever T satisfies (E_μ) for some $\mu \geq 1$. The following example denotes a generalized nonexpansive multivalued mapping satisfying the condition(E) which is not a nonexpansive multivalued mapping.

Example 2.1. *Define a mapping T on the closed interval $[0, 5]$ by*

$$T(x) = \begin{cases} \left[0, \frac{x}{5}\right] & x \neq 5 \\ \\ \{1\} & x = 5 \end{cases}$$

Then the mapping T has the required properties (see[?]).

Theorem 2.2. *Let (X,d) be a $CAT(1)$ space with convex metric and diam$(X) < \pi/2$ and E be a nonempty closed convex subset of X. Let $T : E \to CB(E)$ be a multivalued mapping with $F(T) \neq \emptyset$ and P_T satisfying the condition (E). Assume that (i) $0 \leq \alpha_n, \beta_n, \gamma_n < 1$; (ii) $\alpha_n \to 0$; (iii) $\sum \alpha_n \gamma_n = \infty$, and $\{x_n\}$ be sequence generated by (2.2). If T is hemicompact, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By (2.3), we have

$$\frac{K}{2} \sum_{n=0}^{\infty} \gamma_n \alpha_n (1 - \alpha_n) d(x_n, w_n)^2 < \infty.$$

By Lemma 1.2, there exist a subsequence $\{d(x_{n_k}, z_{n_k})\}$ of $\{d(x_n, z_n)\}$ such that $\lim_{k\to\infty} d(x_{n_k}, z_{n_k}) = 0$. Thus

$$\lim_{n\to\infty} dist(x_n, Tx_n) \leq \lim_{k\to\infty} dist(x_{n_k}, P_T x_{n_k}) = 0.$$

Since T is hemicompact, we may assume that $x_{n_k} \to q \in E$. By condition (E), for some $\mu \geq 1$, we have

$$\begin{aligned} dist(q, Tq) \leq & \ dist(q, S_T q) \\ \leq & \ d(q, x_{n_k}) + dist(x_{n_k}, S_T q) \\ \leq & \ 2d(q, x_{n_k}) + \mu dist(x_{n_k}, S_T x_{n_k}) \to 0 \quad as \quad k \to \infty. \end{aligned}$$

Therefore $q \in F(T)$. Since $\lim_{n\to\infty} d(x_n, p)$ exists by Lemma 2.1, it follows that $\{x_n\}$ converges strongly to q. □

Corollary 2.1. *Let (X,d) be a $CAT(1)$ space with convex metric and diam$(X) < \pi/2$ and E be a nonempty closed convex subset of X. Let $T : E \to CB(E)$ be a multivalued mapping with $F(T) \neq \emptyset$ and P_T is quasi-nonexpansive mapping. Assume that (i) $0 \leq \alpha_n, \beta_n, \gamma_n < 1$; (ii) $\alpha_n \to 0$; (iii) $\sum \alpha_n \gamma_n = \infty$, and $\{x_n\}$ be sequence generated by (2.2). If T is hemicompact and continuous, then $\{x_n\}$ converges strongly to a fixed point of T.*

3. Conclusion

In this work, we obtained some strong convergence results of the iterative algorithm (2.2) under weaker assumptions than those of Thakur et al. [?]. To do this work, we were inspired by [?, ?, ?, ?].

Our results, carry over results of [?] to generalized nonexpansive multivalued mappings in the framework of $CAT(k)$ spaces. Since most of the results for $CAT(k)$ spaces are deduced for $CAT(1)$ spaces, we used the framework of $CAT(1)$ spaces.

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REFERENCES

1. D. Thakur, B. S. Thakur and M. Postolache, *New iteration scheme for numerical reckoning fixed points of nonexpansive mappings,* J. Ineq. Appl., (2014), **2014:328**.
2. W.A. Kirk, *Geodesic geometry and fixed point theory. In:* Seminar of Mathematical Analysis (Malaga/Seville,2002/2003). Colecc. Abierta, vol. 64, pp. 195-225 Univ. Sevilla Secr. Publ., Seville (2003).
3. W.A. Kirk, *Geodesic geometry and fixed point theory II. In:* International Confere nce on Fixed Point Theory and Applications, pp. 113-142. Yokohama Publ., Yokohama, (2004).
4. S. Dhompongsa, A. Kaewkhao, and B. Panyanak, *Lims theorems for multivalued mappings in* $CAT(0)$ *spaces* J. Math. Anal. Appl., **312** (2005) 478-487.
5. P. Chaoha and A. Phon-on, *A note on fixed points sets in* $CAT(0)$ *spaces,* J. Math. Anal. Appl., **320** (2006) 983-987.
6. L. Leustean, *A quadratic rate of asymptotic regularity for CAT(0)-spaces,* J. Math. Anal. Appl., **325** (2007), 386-399.
7. N. Shahzad and J. Markin, *Invariant approximations for commuting mappings in CAT(0) and hyperconvex spaces,* J. Math. Anal. Appl., **337** (2008) 1457-1464.
8. R. Espinola and A. Fernandez-leon, $CAT(k)$ *spaces, weak convergence and fixed points,* J. Math. Anal. Appl., **353** (2009) 410-427.
9. B. Piatek, *Halpern iteration in* $CAT(k)$ *spaces,* Acta Mathematica Sinica, **27** (2011) 635-646.
10. J.S. Hea, D.H. Fang, G. Lopez and C. Li, *Manns algorithm for nonexpansive mappings in* $CAT(k)$ *spaces* Nonlinear Analysis, **75** (2012), 445-452.
11. G.S. Saluja, M. Postolache, *Strong and* Δ*-convergence theorems for two asymptotically nonexpansive mappings in the intermediate sense in* $CAT(0)$ *spaces,* Fixed Point Theory Appl., (2015), **2015:12**.
12. M. Abbas, S.H. Khan, M. Postolache, *Existence and approximation results for SKC mappings in* $CAT(0)$ *spaces,* J. Inequal. Appl., (2014), **2014:212**.
13. G.S. Saluja, M. Postolache, *Three-step iterations for total asymptotically nonexpansive mappings in* $CAT(0)$ *spaces,* Filomat (in printing)..
14. B.S. Thakur, D. Thakur, M. Postolache, *Modifed Picard-Mann hybrid iteration process for total asymptotically nonexpansive mappings,* Fixed Point Theory Appl., (2015), **2015:140**.
15. B. Panyanak, *On the Ishikawa iteration processes for multivalued mappings in some* $CAT(k)$ *spaces,* Fixed Point Theory Appl., (2014), **2014:1**.
16. G.S. Saluja, M. Postolache, A Kurdi, *Convergence of three-step iterations for nearly asymptotically nonexpansive mappings in* $CAT(k)$ *spaces,* J. Inequal. Appl., (2015), **2015:156**.
17. M. R. Bridson and A. Haefiger, *Metric Spaces of Non-Positive Curvature,* Springer-Verlag, Berlin Heidelberg, 1999..
18. S. Ohta, *Convexities of metric spaces,* Geom. Dedic., **125** (2007), 225-250.
19. B. Panyanak, *Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces,* Comput. Math. Appl., **54** (2007), 872-877.
20. N. Shahzad and H. Zegeye, *On Mann and Ishikawa iteration schemes for multi-valued mappings in Banach spaces,* Nonlinear Anal., **71** (2009), 838844.
21. H. F. Senter and W. G. Dotson, *Approximating fixed points of nonexpansive mappings,* Proc. Amer. Math. Soc., vol. **44** (1974). 375380.

22. T. Puttasontiphot, *Mann and Ishikawa iteration schemes for multivalued mappings in $CAT(0)$ spaces,* Appl. Math. Sci, **4** (2010), 3005-3018.
23. J. Garcia-Falset, E. Liorens-Fuster and T. Suzuki, *Fixed point theory for a class of generalized nonexpansive mapping,* J. Math. Anal. Appl., **375** (2011), 185-195.
24. A. Abkar and M. Eslamian, *Common fixed point results in $CAT(0)$ spaces,* Nonlinear Anal., **74** (2011), 1835-1840.
25. M.O. Olatinwo, M. Postolache, *Stability results for Jungck-type iterative processes in convex metric spaces,* Appl. Math. Comput., **12** (2012), 6727-6732.
26. Y. Yao, M. Postolache, Y.C. Liou, *Coupling Ishikawa algorithms with hybrid techniques for pseudo-contractive mappings,* Fixed Point Theory Appl., (2013), **2013:211**.