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QUASI-ANALYTIC SOLUTIONS OF FIRST-ORDER PARTIAL DIFFERENTIAL EQUATIONS USING THE ACCURATE ELEMENT METHOD

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O ecuaţie diferenţială cu derivate parţiale poate avea pe un domeniu bidimensional de integrare o ***soluţie analitică*** ***$\phi(x,t)$*** *(care înlocuită in ecuaţie va conduce la o identitate) sau o* ***soluţie numerică*** *(reprezentată de un şir de valori a căror exactitate este mai greu de cuantificat). Integrarea unei ecuaţii diferenţiale cu derivate parţiale bazată pe metoda AEM (Accurate Element Method) conduce la un număr redus de soluţii de formă polinomială, fiecare corespunzând unui singur subdomeniu de integrare (element). Un astfel de polinom – considerat a fi o* ***soluţie cvasi-analitică*** *– este adecuat unei verificări directe ce se efectuează (similar cu soluţia analitică)* ***înlocuindu-l în ecuaţia diferenţială.*** *Prin înlocuire va rezulta un reziduu a cărui valoare va permite acceptarea soluţiei găsite sau reluarea calculului cu parametri modificaţi, până se obţine un reziduu a cărui valoare se consideră admisibilă. Testul numeric introdus de AEM reprezintă o verificare directă a solutiei $\phi(x,t)$ care se poate face pentru oricare element* ***independent de etapele anterioare de calcul parcurse****.*

Scopul acestui articol introductiv este de a prezenta metoda şi câteva exemple simple. Analiza este deci limitată la ecuaţii diferenţiale cu derivate parţiale de ordinul unu cu coeficienţi constanţi, soluţiile cvasi-analitice fiind reprezentate de polinoame cu doua variabile de ***grad 5 având 21 de termeni, respectiv de grad 7 cu 36 de termeni****.*

It is usually considered that a PDE can have on a two-dimensional integration domain either an ***analytic solution*** ***$\phi(x,t)$*** *(which replaced in the PDE leads to an identity) or a* ***numeric solution*** *(represented by a string of numerical values whose accuracy is more difficult to quantify). The integration of a PDE by the Accurate Element Method leads to* ***Piecewise Polynomial Solutions*** *represented by a small number of polynomials, each one valid on a single sub-domain (element); they can be considered as* ***quasi-analytic solutions****. A quasi-analytic solution is suitable for* ***direct verification by replacing it in the PDE*** *that leads not to identities but to a quantifiable residual. Based on the value of the residual one can decide either to accept the solution or to resume the computation with modified parameters until an imposed allowable precision is reached. The numerical test introduced by the Accurate Element Method represent a direct and global verification of $\phi(x,t)$ on each element,* ***being independent on the various steps (the integration history) covered in order to obtain it****.*

The goal of this introductory paper is to present the method and some examples. Consequently it is restricted only to the first order PDEs with constant coefficients and to quasi-analytic solutions represented by ***a 5th degree polynomial with 21 terms and by a 7th degree polynomial with 36 terms***

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1. Integration of ODEs using the Accurate Element Method (*AEM*)

The Accurate Element Method (*AEM*) has been initially developed as a method leading to quasi-analytic solutions of *Ordinary Differential Equations* (*ODE*s). Before applying AEM to Partial Differential Equations (*PDE*s) we have to summarize the main characteristics of the AEM as they apply to *ODE*s [2,3].

Let us consider the first-order ODE

$$E_1(x)\frac{d\phi}{dx} + E_0(x)\phi + E_F(x) = 0 \tag{1.1}$$

where $E_1(x)$, $E_0(x)$, $E_F(x)$ can be any type of functions of *x*. This ODE will be integrated between a **starting** (*initial*) point x_S and a **target** (*final*) point x_T leading to

$$\int_{xS}^{xT} E_1(x)\frac{d\phi}{dx}dx + \int_{xS}^{xT} E_0(x)\ \phi(x)\ dx + \int_{xS}^{xT} E_F(x)\,dx = 0 \tag{1.2}$$

If the coefficients E_1 and E_0 are constants, this integral can be written as

$$E_1\,\phi_T = E_1\,\phi_S - E_0\int_{xS}^{xT} \phi(x)\ dx - \text{IntegEF} \tag{1.3}$$

While $\text{IntegEF} = \int_{xS}^{xT} E_F(x)\,dx$ is trivial, the only difficulty, typical for this type of equation, is related to the integral

$$\text{Int}\phi = \int_{xS}^{xT} \phi(x)\ dx \tag{1.4}$$

that includes the *unknown* function $\phi(x)$ under the integral sign. This integral can be solved by replacing $\phi(x)$ with an approximation function.

If only the two end values are considered – ϕ_S (*starting value*, usually known) and ϕ_T (*target value*, usually unknown) – the approximation can be a linear interpolation. Instead of this linear function, *AEM* introduces a *higher-order polynomial* referred to as a *Concordant Function* (CF), depending on **the same two end unknowns** (ϕ_S and ϕ_T). For instance, if a five degree polynomial is used, four additional equations are required to determine the coefficients. The *Accurate Element Method* establishes **accurately** these equations **by using the governing equation itself**. When the ODE is applied at both ends (*Starting* and *Target*) of the integration domain we have as a result two equations [2,3]. The remaining two required equations are obtained *from the first derivative of the ODE (1.1)* applied also at the both ends. Based on this approach it is possible to obtain *Concordant Functions* of high or very high order[2], by using higher order derivatives of the ODE (1.1).

It results a *Concordant Function* – perfectly adapted to the ODE (1.1) – *that depends on* ϕ_S*,* ϕ_T*, and a free term*. For any chosen *CF* it results by performing the integral (1.4)

[2] Concordant Functions represented by 15 degree polynomials have been succesfully used

$$\mathrm{Int}\phi = \int_{xS}^{xT} \phi(x)\, dx = K_S \phi_S + K_T \phi_T + K_F \qquad (1.5)$$

where K_S, K_T, K_F are three known coefficients. Finally, from (1.3) it results

$$E_1\, \phi_T = E_1\, \phi_S - E_0 \left(K_S \phi_S + K_T \phi_T + K_F \right) - \mathrm{IntegEF} \qquad (1.6)$$

The target value ϕ_T can be usually expressed as

$$\phi_T = f(\phi_S) \qquad (1.7) \qquad or \qquad \phi_T = f(\phi_S, \phi_T) \qquad (1.8)$$

In the first case (1.7) the method is considered *explicit*, while in the second as *implicit*, because the unknown parameter ϕ_T *is involved in both left and right terms*. The equation (1.6) is of this last type; consequently it results that **AEM is an implicit method, therefore stable**. For instance an integral of an ODE *using a single element* starting from x_S=0 and having as target **x_T=10000** proved to be perfectly stable **(**see[3], page 118). It is important to underline that for a linear ODE the *Accurate Element Method* **obtains ϕ_T directly from (1.6), without any iterative approach or any procedure for solving a system of equations** usually involved in the implicit methods.

2. Integration of PDEs using the Accurate Element Method (*AEM*)

2.1 PDE in global coordinates

A PDE can be expressed in global coordinates X–T (Fig.1) as

$$M \frac{\partial \phi}{\partial X} + N \frac{\partial \phi}{\partial T} + P\,\phi + Q_G(X,T) = 0 \qquad (2.1G)$$

where here M, N, P are three constants (N×M>0) and the free term $Q_G(X,T)$ is a known function of *X* and *T*.

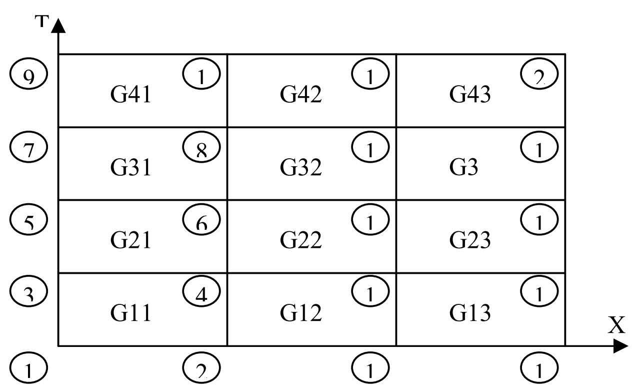


Fig.1

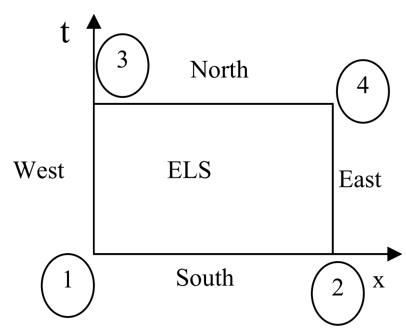


Fig.2

The case solved here is an *initial-boundary value problem* with known initial and boundary conditions represented by:

1. *Initial conditions* (T=0)

$$\Psi_G(X) = A_0 + A_1 X + A_2 X^2 + A_3 X^3 + A_4 X^4 + A_5 X^5 + A_6 X^6 ... \quad (2.2)$$

2.*Boundary* conditions (X=0):

$$\Omega_G(T) = B_0 + B_1 T + B_2 T^2 + B_3 T^3 + B_4 T^4 + B_5 T^5 + B_6 T^6 ... \quad (2.3)$$

2.2 PDE in local coordinates

The numerical integration on the rectangle represented in Fig.1 will be performed by dividing the domain in a small number of elements. The approach is simplified if each element is analyzed by using a *local coordinate system x-t* (Fig.2). The coordinates of the four nodes of a rectangular element having the dimensions *B* and *H* are:

Node 1(x_1=0,t_1=0);*Node 2*(x_2=B,t_2=0);*Node 3*(x_3=0,t_3=H);*Node 4*(x_4=B, t_4=H)(2.4)

In the local system the PDE will be written as

$$\boxed{M \frac{\partial \phi}{\partial x} + N \frac{\partial \phi}{\partial t} + P\,\phi + Q(x,t) = 0} \quad (2.1L)$$

where the free term Q(x,t) is a known two variable polynomial

$$Q(x,t) = q_1 + q_2 x + q_3 t + q_4 x^2 + q_5 xt + q_6 t^2 + ... \quad (2.5)$$

The initial conditions and boundary conditions are given by

Initial conditions (t=0): $\Psi(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4 + \alpha_5 x^5 + \alpha_6 x^6 ...$ (2.6)

Boundary conditions (x=0): $\Omega(t) = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3 + \beta_4 t^4 + \beta_5 t^5 + \beta_6 t^6 ...$ (2.7)

Remark. The free term (2.5) in local coordinates can be obtained from (2.1) by replacing

$X=x+X_1$ (2.8) ; $T=t+T_1$ (2.9)

where X_1 and T_1 are the coordinates of the node 1 of ELS in the *global axes system*. Similarly the condition (2.6) results from (2.2) using (2.8) and the condition (2.7) from (2.3) using (2.9).

The equation (2.1L) will be integrated on the rectangular element *ELS* (Fig.2) having the area $A = B \times H$

$$\int_A \left(M \frac{\partial \phi}{\partial x} + N \frac{\partial \phi}{\partial t} + P\,\phi \right) dA + \int_A Q(x,t)\, dA = 0 \quad (2.10)$$

According to §1, AEM replaces – in order to perform the left side integrals – the unknown two variables function $\phi(x,t)$ by a *Concordant Function*.

3. Concordant Function CF5-21

3.1 The Concordant Function: a complete two variables polynomial

A Concordant Function is a *complete two variables polynomial*, namely it includes all the possible terms that correspond to a chosen degree: *1* constant term + *2* linear terms (x,t)+ *3* second degree terms (x^2,xt,t^2) and so an. The total number of terms NT for a complete function results from

$$NT = (G+1)(G+2)/2 \qquad (3.1)$$

where **G** represents the maximum degree of the polynomial function. For instance a five-degree Concordant Function (G=5) having **NT=21 terms,** noted as **CF5-21,** is given in the *local system* by

$$\phi(x,t)=C_1+C_2\mathbf{x}+C_3\mathbf{t}+C_4\mathbf{x^2}+C_5\mathbf{xt}+C_6\mathbf{t^2}+C_7\mathbf{x^3}+C_8\mathbf{x^2t}+C_9\mathbf{xt^2}+C_{10}\mathbf{t^3}+C_{11}\mathbf{x^4}+C_{12}\mathbf{x^3t}+ +C_{13}\mathbf{x^2t^2}+C_{14}\mathbf{xt^3}+C_{15}\mathbf{t^4}+C_{16}\mathbf{x^5}+C_{17}\mathbf{x^4t}+C_{18}\mathbf{x^3t^2}+C_{19}\mathbf{x^2t^3}+C_{20}\mathbf{xt^4}+C_{21}\mathbf{t^5} \qquad (3.2)$$

In order to obtain the 21 coefficients of (3.2) there are necessary 21 equations. The first equation is represented by the integral (2.10), consequently only 20 equations remain to be established.

3.2 Equations based on the initial conditions: Decoupled Coefficients

The first kind of equations is those that impose rigorously the initial-boundary conditions.

3.2.1 Initial conditions on the South edge 1-2 (t=0)

On the South edge *1-2* (Fig.2) is imposed the initial condition (2.6), supposed here to be a polynomial. Because in the local coordinates for this edge it corresponds ***t=0***, the CF5-21 given by (3.2) becomes the polynomial

$$\phi(x,t=0) = C_1+ C_2\mathbf{x} + C_4\mathbf{x^2} + C_7\mathbf{x^3} + C_{11}\mathbf{x^4}+ C_{16}\mathbf{x^5} \qquad (3.3)$$

If (3.3) and (2.6) are identified it results **directly** ND1=G+1=5+1=6 coefficients

$$C_1 = \alpha_0 \;\; ; \;\; C_2 = \alpha_1 \;\; ; \;\; C_4 = \alpha_2 \;\; ; \;\; C_7 = \alpha_3 \;\; ; \;\; C_{11} = \alpha_4 \;\; ; \;\; C_{16} = \alpha_5 \qquad (3.4)$$

3.2.2 Boundary conditions on the West edge 1-3 (x=0)

Along the West edge 1-3 of ELS (Fig.2) where x=0, the CF (3.2) becomes

$$\phi(x=0,t)=C_1+ C_3\mathbf{t}+ C_6\mathbf{t^2}+ C_{10}\mathbf{t^3}+ C_{15}\mathbf{t^4}+ C_{21}\mathbf{t^5} \qquad (3.5)$$

If the boundary conditions are continuous for x=0 and t=0, namely $\beta_0 = \alpha_0$, the constant C_1 is already known from (3.4). By identifying (3.5) and the boundary condition (2.7) it results **directly** only ND2=G=5 coefficients

$$C_3 = \beta_1 \quad ; \quad C_6 = \beta_2 \quad ; \quad C_{10} = \beta_4 \quad ; \quad C_{15} = \beta_5 \quad ; \quad C_{21} = \beta_6 \tag{3.6}$$

All the coefficients established until now result directly **without any connection to other information**; they will be referred as **decoupled coefficients.** The other remaining constants result usually by solving a system of equations, being therefore **coupled** (depending of one another). From the 20 necessary equations, 6+5=11 conditions have been already found. The last 9 equations are established by using the PDE (2.1L).

3.3 Conditions based on the PDE (2.1L) and its derivatives

As it was shown in §1, the AEM develops **accurate** conditions for obtaining the unknown coefficients **by using the governing equation itself**. Depending on the degree of the Concordant Function one has to use the *PDE* but also an adequate number of its derivatives in order to obtain accurately the remaining equations. These equations have to be applied predominantly at the nodes of the element[3]. If one choose to apply for CF5-21 these conditions in the nodes *2*, *3* and *4* (Fig.2), three equations become necessary for each node. The first is obviously PDE (2.1L), while the two other are its first order derivatives. The coefficients *M, N, P* being constants these derivatives are

$$\frac{\partial(\mathrm{PDE})}{\partial t} = M\frac{\partial^2\phi}{\partial x\partial t} + N\frac{\partial^2\phi}{\partial t^2} + P\frac{\partial\phi}{\partial t} + \frac{\partial Q}{\partial t} = 0 \tag{3.7}$$

$$\frac{\partial(\mathrm{PDE})}{\partial x} = M\frac{\partial^2\phi}{\partial x^2} + N\frac{\partial^2\phi}{\partial x\partial t} + P\frac{\partial\phi}{\partial x} + \frac{\partial Q}{\partial x} = 0 \tag{3.8}$$

The use of these equations is different at each node. At the node *2* the function and the derivative versus x are known from the initial condition (2.6), while in the node *3* the function and the derivative versus t are known from the boundary condition (2.7). On the contrary, in the Target node *4* nothing is known, neither the function, nor its derivatives.

Suppose the integration procedure is applied on the element G11 (Fig.1). The first step is to obtain the functions $Q(x,t), \Psi(x), \Omega(t)$ in local coordinates[4] by using the relations (2.8) and (2.9). The methodology for obtaining an equation is illustrated on the element G11 only for the node *3*, whose local coordinates are x_3=0 and t_3=H. If (2.1L) is transferred to this node it results

[3] The programer may transfer some of these equations in other points of the integration domain, therefore this group of conditions is not unique.

[4] Because G11 starts from the origin of the global coordinates system, X_1=x_1=0 and T_1=t_1=0.

$$M\left(\frac{\partial\phi}{\partial x}\right)_{x=0,t=H} + N\left(\frac{\partial\phi}{\partial t}\right)_{x=0,t=H} + P(\phi)_{x=0,t=H} + (Q)_{x=0,t=H} = 0 \tag{3.9}$$

While $(Q)_{x=0,t=H}$ is obviously known, the Concordant Function $\phi(x,t)$ has to coincide at the West side (*x=0*) with (2.7) so that $(\phi)_{x=0,t=H} = \Omega(t=H)$. Similarly, the derivative of ϕ versus *t* has to coincide with the derivative of $\Omega(t)$

$$\left(\frac{\partial\phi}{\partial t}\right)_{x=0,t=H} = \left(\frac{d\Omega}{dt}\right)_{t=H} = \beta_1 + 2\beta_2 H + 3\beta_3 H^2 + 4\beta_4 H^3 + 5\beta_5 H^4 + 6\beta_6 H^5 ... \tag{3.10}$$

Using these values it results from (3.9) *the value of the derivative of* ϕ *versus x*

$$\left(\frac{\partial\phi}{\partial x}\right)_{x=0,t=H} = (\phi'_x)_3 = -\frac{1}{M}\left[N\left(\frac{\partial\phi}{\partial t}\right)_{x=0,t=H} + P(\phi)_{x=0,t=H} + (Q)_{x=0,t=H}\right] \tag{3.11}$$

From the derivative of the *Concordant Function* (3.2) versus *x* it results for *x=0* and *t=H* the first equation with coupled terms

$$\left(\frac{\partial\phi}{\partial x}\right)_{x=0.t=H} = C_2 + HC_5 + H^2C_9 + H^3C_{14} + H^3C_{20} = (\phi'_x)_3 \tag{3.12}$$

By applying (3.12), *the function $\phi(x,t)$ starts to become concordant not only to the boundary conditions, but also to the PDE (2.1L)*. One continues with (3.7) from which it results $\left(\frac{\partial^2\phi}{\partial x\partial t}\right)_{x=0,t=H}$ and (3.8) from which one obtains $\left(\frac{\partial^2\phi}{\partial x^2}\right)_{x=0,t=H}$. The procedure is similar for the node *2*, for which are known from the initial condition (2.6), the function $(\phi)_{x=B,t=0} = \Psi(x=B)$ and its derivatives versus *x*.

As it was specified above nothing is known in node *4*, except the values of the free term Q(x,t) and of its derivatives. Because the unknown values of the node *4* are involved in the concordance procedure, the function thus obtained includes the target values in both left and right sides. Consequently **the AEM procedure for solving PDEs is implicit, therefore stable no matter the dimensions of the elements**.

4. Target Edges Polynomial Solutions and Target Value

The strategy developed by AEM concerning the **Ordinary Differential Equations** is based on two particularities of the method: the possibility to obtain a *quasi-analytic solution* valid on each element and the high precision of the results that can be obtained at the target point. Because the strategy applied for the integration of *PDE*s is on many aspects a two-dimensional extension of the *ODE* approach developed in [2,4,5], one can extend the analysis to the same parameters.

4.1 Residual function

Suppose that for a given *PDE* a function $\tilde{\phi}(x,t)$ that fulfils the boundary conditions exists and is known. The way to verify if this function represents a solution is to replace it in (2.1L)

$$R(x,t) = M\frac{\partial\tilde{\phi}}{\partial x} + N\frac{\partial\tilde{\phi}}{\partial t} + P\,\tilde{\phi} + Q(x,t) \qquad (4.1)$$

If the result R(x,t) – referred as *residual function* – is **zero** the function $\tilde{\phi}(x,t)$ represents an *analytic or exact solution* of the PDE. If the residual function is different from zero, the analysis of its value represents the best way to appreciate the precision of the numerical result. A very small residual will indicate that $\tilde{\phi}$ (x,t) is a good solution.

4.2 Residuals on the Target Edges

In §1 there was considered as working parameters for a (one-dimensional) ODE the following values of the function: $\boldsymbol{\phi_S}$ (start) and $\boldsymbol{\phi_T}$ (target). For the two-dimensional *PDE* (2.1L) *initial value problem*, two edges of the element *1-2* (South) and *1-3* (West) (fig.2) where the *initial-boundary conditions are known*, will be referred as *Starting Edges*. Also the values of the function at the *nodes 1,2,3* [known from (2.6) and (2.7)] will be considered as *Starting Values*

$$\phi_1 = \Psi(x=0,t=0) = \Omega(x=0,t=0) \;\; ; \phi_2 = \Psi(x=B,t=0) \; ; \; \phi_3 = \Omega(x=0,t=H) \quad (4.2)$$

On the contrary, for the other two edges of the element [*3-4* (North) and *2-4* (East)] the function $\tilde{\phi}(x,t)$ that verifies the PDE is unknown and its value $\boldsymbol{\phi_4(x_4,t_4)}$ at the *node 4* is also unknown. These two last edges will be referred as *Target Edges* and $\boldsymbol{\phi_4}$ as *Target Value*.

It is necessary to precise that $\tilde{\phi}(x,t)$ obtained from the *AEM* procedure described in §3 *cannot lead to R(x,t)≡0 on the whole integration domain*. In fact on the starting edges the residuals $R_{12}(x,t=0)$ and $R_{13}(x=0,t)$ are usually different from zero, because the CF is rigorously adapted – due to (3.4) and (3.6)– to the imposed boundary conditions **that do not depend on the PDE to be integrated**, which is not involved in these relations**.** Consequently, the verification on the whole integration domain of the residual according to (4.1) will fail implicitly. Therefore the verification will be performed **only on the Target Edges** where (4.1) becomes

Target Edge 34: $$RX = R(x, t = t_4) = \left(M\frac{\partial\tilde{\phi}}{\partial x} + N\frac{\partial\tilde{\phi}}{\partial t} + P\tilde{\phi} \right)_{t=t4} + Q(x, t = t_4) \qquad (4.3)$$

Target Edge 24: $$RT = R(x = x_4, t) = \left(M\frac{\partial\tilde{\phi}}{\partial x} + N\frac{\partial\tilde{\phi}}{\partial t} + P\tilde{\phi} \right)_{x=x4} + Q(x = x_4, t) \qquad (4.4)$$

Both residuals (4.3) and (4.4) are **one-dimensional functions,** which simplify the analysis necessary to obtain a numerical criterion that certifies the quality of the solution.

Good *Target Edges Polynomial Solutions* are very important for the usual case when the integration is performed on more than one element. Suppose the integration does not stop at the edge 3-4 (Fig.1), but has to be continued along the *T* axis, namely on the following element *G12*. In this case the *North Target Edge Polynomial Solution* on the edge *3-4* [noted $(\Psi_{NTE})_{G11}$] will be used as *initial-condition for G12* [$(\Psi)_{G12} = (\Psi_{NTE})_{G11}$].

The accuracy of the *Target Edge Polynomial Solution* (4.3) becomes decisive for a proper continuation of the integration procedure, because thus *the problem is reduced to a known one*, namely *similar* to that used for *G11.* Consequently, in local coordinates the integration algorithm remains the same.

4.3 Numerical tests concerning the Target Edges residuals

As it was shown in [5], the residual RX (4.3) can be divided in a number of points NP leading to x_i(i=1,2,…NP) abscissas, based on which one can calculate a mean square root value given by

$$RX_{MS} = \frac{1}{NP}\sqrt{\sum_{i=1}^{i=NP}[R(x_i, t = t_4)]^2} \qquad (4.5)$$

This value has to be compared with an *allowable residual*: $RX_{MS} < R_{allow}$ (4.6)

For the East *Target Edge residual* (Fig.2, edge 2-4), the analysis is performed along the ordinate *t*, for the constant abscissa $x=x_4$. The condition concerning the mean square root value becomes

$$RT_{MS} = \frac{1}{NP}\sqrt{\sum_{i=1}^{i=NP}[R(x = x_4, t_i)]^2} < R_{allow} \tag{4.7}$$

The parameter R_{allow} is a conventional value that remains to be established. Some numerical tests solved by the author have shown that a value of RX_{MS} smaller than 10^{-10}-10^{-11} indicate a very good result, but greater values like $R_{allow} \approx 10^{-7}$-$10^{-8}$ can also be accepted. *These values of R_{allow} are obviously disputable*. A value like $R_{MS} = 10^{-3}$ shows that *the result $\tilde{\phi}(x,t)$ leading to such value has to be rejected* and a new computation using a greater number of elements or/and a higher degree Concordant Function has to be performed. This approach gives to the user a powerful and global tool to verify the validity of the whole computation, no matter how many elements or *CF*s were involved. It is important to underline that the numerical tests (4.6) and/or (4.7) are performed for any element *independently of the results obtained on the previous elements*. Consequently they represent **a verification of the whole procedure, concerned only by the final result, being therefore independent on the various steps covered in order to obtain $\tilde{\phi}$ (x,t)**.

4.4 The accuracy estimation of the Target Value $\phi_4(x_4,t_4)$

The first question that has to receive an answer is: *the value of $\phi_4(x_4,t_4)$ is reliable or not ?* At least two different ways to give an answer can be considered:

1. Compute the *Target Value* $\phi_4(x_4,t_4)$ by using an increasing number of elements in order to follow the convergence of the results. This can be done by comparing the values obtained for $\phi_4(x_4,t_4)$ using *NE* and *NE+Δ(NE)* elements, which allows to obtain an estimated error given by

$$\textit{Estimated Target Error} = \frac{\phi(x_4,t_4)_{(NE+\Delta NE)} - \phi(x_4,t_4)_{(NE)}}{\phi(x_4,t_4)_{(NE+\Delta NE)} + \phi(x_4,t_4)_{(NE)}} \tag{4.8}$$

Obviously, this is only an *estimated error* because it is related to a previous value $\phi_4(x_4,t_4)_{(NE)}$, *not to the actual value which is unknown*.

2. Compare the values of $\phi_4(x_4,t_4)$ obtained by using different Concordant Functions.

5. First-order PDEs with P=0 integrated using CF5-21

If the coefficient P=0, the PDE (2.1G) becomes

$$M\frac{\partial\phi}{\partial X}+N\frac{\partial\phi}{\partial T}+Q_G(X,T)=0 \tag{5.1}$$

For the examples analyzed in this paragraph the integration domain is quite large, being extended along *X* from $X_{Left}=\boldsymbol{0}$ to $X_{Right}=\boldsymbol{1}$ and along *T* from $T_{Start}=\boldsymbol{0}$ up to $T_{Target}=\boldsymbol{10}$. The computations are performed with CF5-21 using a small number of elements, with *large or very large dimensions*.

The *initial-boundary* conditions *for all the examples analyzed below* are imposed by the following polynomials taken at random

$$\Psi_G(X)=10-3X+X^2+2X^3+3X^4 \tag{5.2}$$

$$\Omega_G(T)=10-2T+3T^2-0.5T^3+0.2T^4 \tag{5.3}$$

5.1 Example 1

The well-known one-way wave *PDE*

$$\frac{\partial\phi}{\partial X}+2\frac{\partial\phi}{\partial T}=0 \tag{5.4}$$

will be integrated on the rectangular domain **X=1, T=10**. By comparing (5.1) and (5.4) it results the **M=1, N=2** and the free term $\boldsymbol{Q_G(X,T)=0}$. The *initial-boundary* conditions are (5.2) and(5.3).

As it was shown in §3.3 the Accurate Element Method is an *implicit* method, consequently one can use elements having quite large dimensions that can be considered as improper by other methods. In order to follow the influence of the elements dimensions on the results, the integration of *PDE* (5.4) has been performed on a single column along *T*, using different number of elements between NE=1 and NE=10. The results, together with the dimensions (B×H) of each element, are given in Table 1.

Table 1

Exact Target Value ϕ_{TE} = 749.2					
NE	B×H	Target Value $\phi_{TE}(X=1,T=10)$	Residual North	Residual East	Actual error
(1)	(2)	(3)	(4)	(5)	(6)
1	1×10	748.0424382716078	7.6×10^{-4}	3.9×10^{-1}	-1.5×10^{-3}
2	1×5	747.7472991695568	7.7×10^{-4}	1.1×10^{-1}	-1.9×10^{-3}
3	1×3.33	749.2102287345520	7.3×10^{-5}	1.2×10^{-3}	1.4×10^{-5}
4	1×2.5	749.2011443978998	1.3×10^{-4}	4×10^{-4}	1.5×10^{-6}
5	**1×2**	**749.2000000000011**	$\mathbf{3\times10^{-13}}$	$\mathbf{3.9\times10^{-13}}$	$\mathbf{1.4\times10^{-15}}$

6	1×1.66	749.2000001699947	1×10^{-7}	4×10^{-8}	$2.2\times10^{--10}$
7	1×1.43	749.2000000113434	4×10^{-9}	7×10^{-10}	1.5×10^{-11}
8	1×1.25	749.1999999939001	1.4×10^{-9}	1.1×10^{-10}	-8.1×10^{-12}
9	1×1.11	749.1999999992441	1.5×10^{-9}	5.5×10^{-11}	-1×10^{-12}
10	1×1	749.2000000104398	9.1×10^{-10}	1.2×10^{-11}	1.4×10^{-11}

5.1.1 Because the analytic solution of (5.1) can be obtained [8], the Exact Target Value for *X=1* and *T=10* has been inserted on the top of Table 1. Nevertheless there are two other more important parameters in the *columns (4) and (5)* that have to be inspected by the user: the **North and East residuals**. These parameters indicate how good the resulted Target Value is.

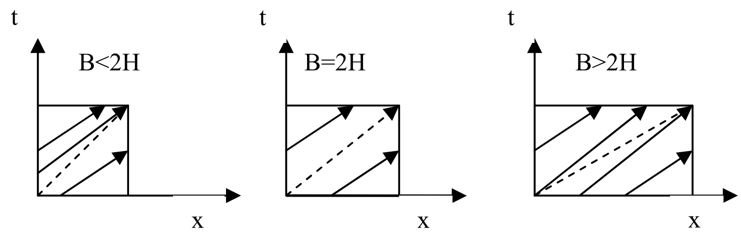
Both North and East residuals have small values when the Number of Elements [NE, column (1)] increases. The residual values are $\approx10^{-4}$ up to *NE=4*, then decrease abruptly to a very small value ($\mathbf{10^{-13}}$) for *NE=5.* If the computation continues by increasing *NE*, the results are less accurate, though their values remain still very good. The explanation of this behavior is connected to the *characteristic curves* of the PDE (5.4) that can be obtained by integrating the ordinary differential equation [1,7,8]

$$\frac{dt}{dx} = \frac{N}{M} = \frac{2}{1} = 2 \tag{5.5}$$

The integration of (5.5) leads to a family of curves, which – in this particular case – is represented by a family of straight parallel lines given by

$$t = K + 2x \tag{5.6}$$

In the origin (for x=0 and t=0) it results K=0, therefore **t=2x**. If **x=B=1** it results **t=H=2**, which **corresponds to NE=5** (Table1). This means that for NE=5 the line **t=2x** represents *the diagonal of the rectangular element* (Fig.3b). As a consequence the information provided by the initial conditions (5.2) on the South edge and by the boundary conditions (5.3) on the West edge is separated from each other. For any other ratio H/B different from 2 (Fig.3a and Fig.3c) the information is superposed leading to greater residuals than for NE=5. Nevertheless the results remain very good also for NE>5, as it results from Table 1. Because the Exact Target Value is known in this case, the last column of Table1 shows that the *Actual errors* are between $10^{-10}\ldots10^{-12}$. Such negligible errors **allow concluding that all the Target Values corresponding to NE>4 can be considered as accurate**. It is important to observe that for all these computations the main parameters that are available for the user, namely the residuals, are between $10^{-7}\ldots10^{-11}$.



a b c

Fig.3

5.1.2 For the case NE=5 the analytic (exact) solution in local coordinates of the last element (B=1,H=2) is given by

$$\phi_{exact}(x,t) = 749.2 - 719.2\,\mathbf{x} + 359.6\,\mathbf{t} + 271.2\,\mathbf{x^2} - 271.2\,\mathbf{xt} + 67.8\,\mathbf{t^2} - 47.2\,\mathbf{x^3}$$

$$+70.8\,\mathbf{x^2t} - 35.4\,\mathbf{xt^2} + 5.9\,\mathbf{t^3} + 3.2\,\mathbf{x^4} - 6.4\,\mathbf{x^3t} + 4.8\,\mathbf{x^2t^2} - 1.6\,\mathbf{xt^3} + 0.2\,\mathbf{t^4} \qquad (5.7)$$

For x=B=1 and t=H=2 it results from (5.7) the *Exact Target Value* given in Table1. The solution obtained by AEM using CF5-21 (based on the coefficients C_1, C_2… C_{21} resulted from the computation) can be written as

$$\tilde{\phi}(x,t) = \phi_{exact}(x,t) + 10^{-13} \times (0.44\,\mathbf{x^4t} + 8.23\mathbf{x^3t^2} + 1.13\mathbf{x^2t^3} - 0.15\mathbf{xt^4}) \qquad (5.8)$$

This polynomial function can be considered in two ways:

1. Because the last parenthesis of (5.8) is multiplied by 10^{-13} leading to a negligible value, one can eliminate this parenthesis in which case it results the exact solution. Such a case is quite seldom and requires the user's decision.

2. Consider the solution as it is furnished by the program, namely (5.8). Obviously, in this case (5.8) is no more an exact solution, but the error is negligible. In such case a very good solution like (5.8) will be referred as *accurate solution*. **The Accurate Element Method can lead to such accurate solutions.**

The chosen solution (exact or accurate) *is valid only on the domain represented by a single element*. Because an analytic solution is valid on the whole integration domain, the solution furnished by *AEM* is referred as a **quasi-analytic solution.**

5.1.3 An important function resulted during the computation is the *Target Edge Solution*. For instance the *North Edge* solution, which corresponds in the local coordinates to *t=1*, results from the solution (5.7)

$$\phi_{NTE}(x, t = 1) = 1790 - 1416\,x + 432\,x^2 - 60\,x^3 + 3.2\,x^4 \qquad (5.9)$$

Remarks. 1. The exact Target Value that corresponds to X=1, T=10 results in local coordinates from (5.9) for x=1, $\phi_{NTE}(x=1, t=1) = 749.2$ (see Table 1).

2. The function (5.9) is valid for the last element. For any intermediary element ϕ_{NTE} becomes the initial condition for the next element (See §4.2).

5.2 Example 2

The PDE
$$\frac{\partial \phi}{\partial X} + 2\frac{\partial \phi}{\partial T} + Q_G(X,T) = 0 \qquad (5.10)$$

where $Q_G(X,T) = -2-20X-22T-12X^2-28XT-18T^2-6X^3-2X^2T-2XT^2-3T^3=0$ (5.11)

will be integrated on the rectangular domain **X=1, T=10** with the *initial-boundary* conditions (5.2) and (5.3).

The results, based on an integration strategy similar to that used in Example 1, are given in Table 2.

5.2.1 The characteristic curves (5.6) are still valid for this example. Consequently the computation with *NE=5* elements leads to the best results as in *Example 1*. The Target Value that corresponds to this case [column (3)] can also be considered as accurate.

Table 2

Exact Target Value ϕ_{TE} = 4910.7					
NE	B×H	Target Value ϕ_{TE}(X=1,T=10)	Residual North	Residual East	Actual error
(1)	(2)	(3)	(4)	(5)	(6)
1	1×10	4917.587191358002	1.8×10^{-6}	8.8×10^{-3}	1.4×10^{-3}
2	1×5	4909.596783653137	1.1×10^{-7}	4.7×10^{-5}	-2.2×10^{-4}
3	1×3.33	4910.610781325277	1.2×10^{-7}	3.7×10^{-6}	-1.8×10^{-5}
4	1×2.5	4910.700312824404	6.8×10^{-9}	3.2×10^{-8}	6.3×10^{-8}
5	**1×2**	**4910.700000000015**	8.9×10^{-16}	7.6×10^{-16}	3.1×10^{-15}
6	1×1.66	4910.700000112351	1.3×10^{-11}	6.4×10^{-12}	2.3×10^{-11}
7	1×1.43	4910.700000011681	7.9×10^{-13}	1.7×10^{-13}	2.4×10^{-12}
8	1×1.25	4910.699999993970	2.5×10^{-13}	2.4×10^{-14}	-1.2×10^{-12}
9	1×1.11	4910.699999998376	3.1×10^{-13}	1.2×10^{-14}	-3.3×10^{-13}
10	1×1	4910.700000008856	2.1×10^{-13}	2.9×10^{-15}	1.8×10^{-12}

5.2.2 For the case NE=5 the analytic (exact) solution in local coordinates of the last element (B=1,H=2) is given by

$$\phi_{exact}(x,t) = 749.2 + 2146.8\,\mathbf{x} + 359.6\,\mathbf{t} - 396.8\,\mathbf{x^2} + 614.8\,\mathbf{xt} + 67.8\,\mathbf{t^2} + 56.8\,\mathbf{x^3} - 71.2\,\mathbf{x^2t}$$
$$+54.6\,\mathbf{xt^2} + 5.9\,\mathbf{t^3} - (1.9/3)\,\mathbf{x^4} - (12.8/3)\,\mathbf{x^3t} - 2.7\,\mathbf{x^2t^2} + 1.4\,\mathbf{xt^3} + 0.2\,\mathbf{t^4} \qquad (5.12)$$

The *Exact Target Value* given in Table 2 results from (5.12) for x=B=1 and t=H=2. The solution obtained from by AEM from the computation using CF5-21 (3.2) is

$$\tilde{\phi}(x,t) = \phi_{exact}(x,t) - 10^{-12} \times (0.91\,\mathbf{x^4 t} + 1.49\mathbf{x^3 t^2} - 1.36\mathbf{x^2 t^3} + 0.098\mathbf{xt^4}) \qquad (5.13)$$

The remarks made in §5.1.2 concerning the exact and accurate solutions are still valid here.

5.2.3 The *Target North Edge Solution* is given by

$$\phi_{NTE}(x, t=1) = 1790 + 3606\,x - 550\,x^2 + (196/3)\,x^3 - (1.9/3)\,x^4 \qquad (5.14)$$

6. First-order PDEs with P different from zero

6.1 Integration using CF5-21

6.1.1 Example 3

The PDE
$$\frac{\partial\phi}{\partial X} + 2\frac{\partial\phi}{\partial T} + \phi = 0 \qquad (6.1)$$

will be integrated on the rectangular domain **X=1, T=1** with the *initial-boundary* conditions (5.2) and (5.3).

In this case AEM, which uses polynomial Concordant Functions, cannot obtain a function leading to an exact solution. Nevertheless an accurate solution can be obtained with a modified strategy and – as it will result below – with a higher order Concordant Function.

Table 3

NE	Concordant Function CF5-21			Concordant Function CF7-36		
	Target Value ϕ_{TE}(X=1,T=1)	Residual North	Residual East	Target Value ϕ_{TE}(X=1,T=1)	Residual North	Residual East
(1)	(2)	(3)	(4)	(5)	(6)	(7)
2	5.52522400452	1×10^{-2}	5.5×10^{-3}	5.54028509483	7.1×10^{-3}	3.3×10^{-3}
8	5.56988306035	1×10^{-3}	7.5×10^{-5}	5.57120294336	5.4×10^{-4}	3.8×10^{-4}
18	5.57218922378	1.1×10^{-4}	2.6×10^{-5}	5.57245422435	3.4×10^{-5}	2.7×10^{-5}
32	5.57242210050	1.3×10^{-5}	2.2×10^{-5}	5.57249908115	1.6×10^{-6}	1.4×10^{-6}
50	5.57246993387	$3.2\times10^{--6}$	1.1×10^{-5}	5.57250041043	5.9×10^{-8}	5.9×10^{-8}
72	5.57248594283	1.4×10^{-6}	5.2×10^{-6}	5.57250043866	1.6×10^{-9}	3.6×10^{-9}
98	5.57249268026	7.5×10^{-7}	2.8×10^{-6}	5.57250043742	4.5×10^{-11}	8×10^{-10}
128	5.57249591879	4.3×10^{-7}	1.6×10^{-6}	5.57250043670	4×10^{-11}	1.7×10^{-10}
162	**5.57249** 763010	2.7×10^{-7}	1×10^{-6}	**5.572500436** 40	2×10^{-11}	1.7×10^{-10}
200	**5.57249** 860252	1.8×10^{-7}	6.7×10^{-7}	**5.572500436** 26	1.1×10^{-11}	8.8×10^{-11}

Table 4

NE	8	18	32	50	72	98	128	162	200
B	0.25	0.166	0.125	0.1	0.083	0.071	0.062	0.055	0.05
H	0.5	0.33	0.25	0.2	0.166	0.142	0.125	0.111	0.1
CF5-21	8×10^{-3}	4×10^{-4}	4×10^{-5}	8×10^{-6}	3×10^{-6}	1×10^{-6}	5×10^{-7}	3×10^{-7}	2×10^{-7}
CF7-36	5×10^{-3}	2×10^{-4}	8×10^{-6}	2×10^{-7}	5×10^{-9}	2×10^{-10}	1×10^{-10}	5×10^{-11}	2×10^{-11}

Because the term depending on P does not modify the characteristic curves [1,8], the elements used will have the best ratio resulted from (5.6), namely **H/B=2**. The target ordinate being T=1, even for a single row of elements along *X* one needs two elements, each one having **B=0.5** so that H=2B=1. The results obtained using CF5-21 on a single row (*NE=2*), are given in the column (2) of Table 3. Because both *North* and *East* residuals have great values (10^{-2}–10^{-3}), it is necessary to increase the number of elements in order to obtain reliable results. This will be done by maintaining the ratio H/B=2, so that if two rows and two columns are used, the number of elements becomes *NE=8*. The residuals obtained in this case are still unsatisfactory (10^{-3}–10^{-5}). The procedure of increasing the number of rows and columns has to continue. The dimensions B and H of the elements are given in Table 4.

The values obtained for an increasing number of elements are given in Table 3, columns (2), (3) and (4). They show that the results are convergent (see the fourth row of Table 4, where are given the *Estimated Target Errors*). The Target value obtained with NE=200 elements compared with that corresponding to NE=162 elements have *6 digits that coincide*. If a better result is needed it is necessary – in order to avoid the excessive increase of the number of elements – to use a higher order polynomial as Concordant Function.

6.1.2 Example 4

The PDE (2.1L) for which **M=1, N=2, P=1** and **Q_G(X,T)** is given by (5.11) will be integrated on the rectangular domain **X=1, T=1** with the *initial-boundary* conditions (5.2) and (5.3).

The strategy used is similar to that of *Example 3*, leading to better results. For instance the *North* and *East* residuals are 10^{-5} for NE=2 (a single row) and (10^{-8}–10^{-9}) for NE=72. A *6 digits coincidence* is reached between NE=50 and NE=72.

Table 5

NE	Concordant Function CF5-21			Concordant Function CF7-36		
	Target Value ϕ_{TE}(X=1,T=1)	Residual North	Residual East	Target Value ϕ_{TE}(X=1,T=1)	Residual North	Residual East
(1)	(2)	(3)	(4)	(5)	(6)	(7)
2	29.9417350066	3×10^{-5}	9.8×10^{-5}	29.9560889157	1.1×10^{-5}	7.4×10^{-6}

8	29.9592996771	2.9×10^{-6}	2.3×10^{-6}	29.9602462392	8.7×10^{-7}	7×10^{-7}
18	29.9602823212	3.4×10^{-7}	4.8×10^{-7}	29.9604532822	6×10^{-8}	5.3×10^{-8}
32	29.9604109493	5.7×10^{-8}	1.7×10^{-7}	29.9604618073	3.1×10^{-9}	3×10^{-9}
50	**29.9604** 417572	1.8×10^{-8}	7.2×10^{-8}	**29.96046207** 53	1.1×10^{-10}	1.5×10^{-10}
72	**29.9604** 524036	8×10^{-9}	3.4×10^{-8}	**29.96046207** 84	2.5×10^{-12}	1.6×10^{-11}

6.2 Integration using CF7-36

The two last examples will be integrated by using a higher degree Concordant Function **CF7-36** represented by a complete two variables polynomial of 7th degree with 36 terms. The 36 equations necessary to obtain the coefficients of CF7-36 are established on a basis similar to that given in §3. Besides the integral of the PDE and the initial-boundary conditions, a number of 6 equations[5] are written in the nodes *2,3,4* and also two equations in the node *1* (Fig.2).

6.2.1 Example 3. The results obtained using CF7-36 are given in the columns (5), (6), (7) of *Table 3.* The comparison with the corresponding values obtained using CF5-21 given in the same Table (columns (2), (3) and (4), respectively) shows that all *the results obtained with CF7-36 are better*:

a. The residuals for the same number of elements (NE) are smaller for CF7-36. This results especially for the last case NE=200, for which the residuals corresponding to CF5-21 are 10^{-7}, while those obtained using CF7-36 are 10^{-11} (10 000 times smaller).

b. The residual values 10^{-11} resulted for NE=200 elements show that the corresponding Target value $\phi_{TE}=5.57250043626$ can be considered as *accurate*. In fact there is a 10 digits coincidence of this last value with that obtained using NE=162 elements. From (4.8) it results that

$$\textit{The Estimated Target Error} = 2\times10^{-11}$$

6.2.1 Example 4. The values obtained using CF7-36 are given in the columns (5), (6), (7) of *Table 5.*In this case the results are also better than those obtained by using CF5-21:

a. The residuals for NE=72 are 10^{-11}-10^{-12} for CF7-36 , while for CF5-21 are much greater (10^{-8}-10^{-9}).

b. There is **9 digits** coincidence between the Target values obtained with 50 and 72 elements for CF7-36 ($\phi_{TE}=29.9604620$), while for CF5-21 the coincidence is only with **6 digits** ($\phi_{TE}=29.9604$).

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[5] One equation for the PDE, two equations (3.7), (3.8) for its first derivatives, to which one adds three equations for the second derivatives.

Remark. The conclusion that the results improve always by increasing the degree of the Concordant Function can be delusive. The author's experience has shown that for the ordinary differential equations there are cases when the results may worsen when the degree of the Concordant Functions increases [4,5]. In the particular cases analyzed here the values obtained using CF7-36 – which can be considered as accurate – show that **it is useless to search for a hypothetical better solution** using a higher degree *CF*.

7. AEM versus other numerical methods

There are various methods used for the numerical integration of *PDE*s [1,7,9]. It is customary, when a new numerical method is developed, to compare it with the other known methods. This comparison has the purpose to both establish the comparative precision of the new method, as well as to compare the computation times. Such comparisons will not be made here because:

1. For the *Example 1* an *Example 2* the exact solutions are known, therefore any comparison with a numerical method is useless. For the Example *3* and *Example 4*, the powerful "internal" tests provided by *AEM* are enough to validate the results without any reference to other numerical methods:

a. By increasing the number of elements from *NE=50* to *NE=72* (Table 5), one can validate a reliable *6 digits* value for CF5-21 (ϕ_{TE} = **29.9604)** or a reliable *10 digits* value for CF7-36 (ϕ_{TE} = **29.96046207)**.

b. The Target value can also be verified by a *cross-comparison* between the results corresponding to similar values of the *Residuals* obtained using CF5-21 and CF7-36, respectively. For instance ϕ_{TE} = **29.9604** obtained with CF5-21 using NE=**72** elements (*Residuals*$\approx 10^{-8}$) is confirmed by CF7-36 with only **18** elements and similar residuals.

2. The time spent for the computation can be compared either directly using two different programs on the same computer or by comparing the number of operations. Both these approaches are useless because it is irrelevant to compare this program written by the author for the scientific purpose to develop and verify the method, with commercially available programs compiled and optimized by professional programmers.

8. Conclusions and further developments

The present paper has a limited goal, namely to present the extension to *PDE*s of the *Accurate Element Method* that has initially been developed for *ODE*s. The analysis was limited here to the **first-order *PDE*s with constant coefficients** followed by some illustrative examples. The goal was to show:

a. How the methodology developed by *AEM* allows a direct verification of the function-solution depending only on *PDE* to be solved and **not on the intermediary steps of the computation**.

b. The way to obtain numerical parameters that indicate the possible convergence of the Target Edge Solution and of the Target Value.

The very simple examples presented above have shown that *AEM* can give good (if not accurate) results by using a small number of elements whose dimensions can be considered as improper by other methods. The user can have permanently reliable information concerning the residuals having the possibility to accept or reject the result.

Some further developments – from which some are already operational – will be developed elsewhere:

a. The integration of *PDE*s with variable coefficients M(*X,T*), N(*X,T*), P(*X,T*).

b. The integration of *PDE*s who's initial and/or boundary conditions are discontinuous.

c. The extension of the variable coefficients solutions to the integration of non-linear problems.

d. The integration of second order hyperbolic *PDE*s.

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R E F E R E N C E S

[1]. *C. Berbente, S. Mitran, S. Zancu,* Metode Numerice (Numerical Methods) , Editura Tehnica, Bucharest, 1997.

[2]. *M. Blumenfeld*., The Accurate Element Method for solving Ordinary Differential Equations, Editura JIF, Bucharest 2005 (in English).

[3]. *M. Blumenfeld*., A New Method for Accurate Solving of Ordinary Differential Equations, Editura Tehnica, Bucharest 2001 (in English).

[4]. *M. Blumenfeld,* Accurate Element Method strategy for the integration of first order Ordinary Differential Equations, University Polytehnica Bucharest Sci.Bull., Series A, Vol. 69, No.2, 2007

[5]. *M. Blumenfeld*, Verification of the quasi-analytic solutions of Ordinary Differential Equations using the Accurate Element Method, University Polytehnica Bucharest Sci.Bull., Series A, Vol. 71, Iss. 2/2009.

[6]. *M. Blumenfeld, P.Cizmas*, The Accurate Element Method: A new paradigm for numerical solution of Ordinary Differential Equations, Proceedings of Romanian Academy, 4(3), 2003.

[7]. *S. Danaila, C.Berbente*, Metode Numerice in Dinamica Fluidelor (Numerical Methods in Fluid Dynamics), Editura Academiei Romane, Bucharest, 2003
[8]. *P.V.O'Neil*, Beginning Partial Differential Equations, John Wiley & Sonns, New-York, 1999.
[9]. *S. C. Chapra, R. P. Canale*, Numerical Methods for Engineers, McGraw-Hill, 2002.