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CHARACTERIZING (ρ, τ)-QUASI-EINSTEIN SOLITONS IN THE FRAMEWORK OF SYNECTIC LIFT METRIC

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Abstract: In this paper, we explore the structure of (ρ, τ)-quasi-Einstein solitons in relation to the synectic lift metric on the tangent bundle TM of a Riemannian manifold (M, g). Utilizing an adapted frame for our analysis, we investigate the necessary and sufficient conditions for the structures $(TM, \tilde{g}, \lambda, {}^{V}f)$ and $(TM, \tilde{g}, \lambda, {}^{C}f)$ to qualify as (ρ, τ)-quasi-Einstein solitons, where $\tilde{g}$ denotes the synectic lift metric on the tangent bundle TM.

Keywords: (ρ, τ)-quasi-Einstein soliton, synectic lift metric, tangent bundle.

1. Introduction

Let M be an $n-$dimensional manifold, and let TM represent its tangent bundle. We use $\Im_s^r(M)$ to denote the collection of all tensor fields of type (r, s) on M. Similarly, $\Im_s^r(M)$ refers to the corresponding collection of tensor fields on TM. Additionally, this paper will always consider everything within the $C^{\infty}-$category, and the manifolds discussed will be assumed to be connected and of dimension $n > 1$.

The tangent bundle TM is a fundamental concept in differential geometry, providing a systematic way to examine and apply tangent vectors throughout the manifold. It encapsulates the local linear structure of the manifold and facilitates the extension of vector space concepts to the manifold context. The exploration of tangent bundle geometry dates back to Sasaki's influential paper published in 1958 [23], where he introduced a method to construct a metric g on the tangent bundle TM of a differentiable manifold M based on a given Riemannian metric g on M. This metric, now referred to as the Sasaki metric, has become a cornerstone in differential geometry. In subsequent years, researchers explored various classical lifts of the metric g from M to TM in their pursuit of alternative lifted metrics with notable properties (see [1, 9, 5, 16, 26]). Among these, the synectic lift metric $\widetilde{g}$ has emerged as one of the significant metrics. In this paper, we focus on the synectic lift metric on the tangent bundle of a Riemannian manifold. Our aim is to investigate the necessary and sufficient conditions for the structures $(TM, \widetilde{g}, \lambda, {}^{V}f)$ and $(TM, \widetilde{g}, \lambda, {}^{C}f)$ to be classified as (ρ, τ)-quasi-Einstein solitons, where $\widetilde{g}$ denotes the synectic lift metric on the tangent bundle TM.

A geometric soliton structure is a unique geometric arrangement on a manifold that displays self-similar behavior when subjected to geometric flows. In differential geometry, solitons are intimately connected to the solutions of partial differential equations (PDEs), especially within the framework of geometric flows like the Ricci flow or mean curvature flow. In this context, solitons represent configurations that evolve in a consistent, self-similar way

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during the flow process. Solitons are crucial for analyzing the long-term dynamics of geometric flows and serve as key tools for exploring the topology and geometry of the manifolds involved. In physics, solitons typically refer to stable formations that resist dispersal, and geometric solitons can have similar interpretations in fields like general relativity and string theory. In [14], the authors study some soliton structure (almost Ricci and almost Yamabe solitons) on tangent bundle, in [25], investigated Ricci, Yamabe and gradient Ricci–Yamabe solitons of the twisted-Sasaki metric on the tangent bundle over a statistical manifold. Also explored natural Ricci soliton on tangent and unit tangent bundle in the paper [2], Yamabe and quasi-Yamabe solitons on Euclidean submanifolds worked by Chen and Deshmukh [8]. Many more studies have been carried out on soliton structures over the years (see [6, 11, 13, 18, 20]).

A quasi-Einstein soliton generalizes the notion of an Einstein soliton, which is itself a specific instance of a Ricci soliton. Within the realm of differential geometry and geometric flows, these solitons offer valuable insights into the structure and dynamics of manifolds under particular conditions. A quasi-Einstein soliton modifies the soliton equation by incorporating an additional function, commonly denoted as f. The equation for a quasi-Einstein soliton is typically written as:

$$Ric + \nabla^2 f - \mu df \otimes df = \lambda g.$$

In this equation, $\nabla^2 f$ refers to the Hessian of the smooth function f, while $df \otimes df$ denotes the tensor product of the differential of f. The constants λ and μ introduce flexibility in modeling the geometry of the manifold, often resulting in solutions that describe non-homogeneous or more intricate geometric structures. Essentially, quasi-Einstein solitons provide an expanded framework for examining self-similar solutions to geometric flows, broadening the scope of Einstein and Ricci solitons to include more general and complex geometric configurations.

Einstein manifolds are of crucial importance in both mathematics and physics. Within Riemannian and semi-Riemannian geometry, there is considerable interest in exploring Einstein manifolds and their various generalizations. Recently, several extensions of Einstein manifolds have been introduced, including quasi-Einstein manifolds [12], generalized quasi-Einstein manifolds [7], η-quasi-Einstein manifolds, and (ρ, τ)-quasi-Einstein manifolds [17], among others. Additionally, in [10], the authors studied perfect fluid spacetimes characterized by a Lorentzian metric that incorporates (m, ρ)-quasi-Einstein solitons and provided an example of an almost co-Kähler manifold exhibiting (m, ρ)-quasi-Einstein solitons. In [22], the authors investigated (m, ρ)-quasi-Einstein solitons on 3-dimensional trans-Sasakian manifolds, demonstrating that a closed (m, ρ)-quasi-Einstein soliton on a 3-dimensional trans-Sasakian manifold is either cosymplectic or Einstein under certain conditions, and they also presented an application of this soliton.

The equation characterizing a (ρ, τ)-quasi-Einstein soliton is generally expressed as follows:

$$Ric + \nabla^2 f - \frac{1}{\tau} df \otimes df = (\rho r + \lambda) g,$$

where $\nabla^2 f$ denotes the Hessian of a smooth function f, and $df \otimes df$ represents the tensor product of the differential of f. In this context, r stands for the scalar curvature, while λ, τ, and ρ are scalars. In this paper, we explore the necessary and sufficient conditions for the structures $(TM, \widetilde{g}, \lambda, {}^V f)$ and $(TM, \widetilde{g}, \lambda, {}^C f)$ to qualify as (ρ, τ)-quasi-Einstein solitons, with $\widetilde{g}$ being the synectic lift metric on the tangent bundle TM.

2. Preliminaries

2.1. The adapted frame on tangent bundle

Consider an n-dimensional Riemannian manifold M equipped with a Riemannian metric g, and let TM denote its tangent bundle. This article utilizes the C^∞ category to provide a thorough explanation, particularly focusing on connected manifolds. We examine the natural projection $\pi : TM \to M$ with special attention to systems of local coordinates.

When a local coordinate system (U, x^i) is established on M_n, it induces a corresponding local coordinate system $(\pi^{-1}(U), x^i, x^{\bar{i}} = u^i)$ on TM, where $\bar{i} = n + i = n + 1, \ldots, 2n$. In this context, (u^i) represents the Cartesian coordinates in each tangent space T_pM for all $p \in U$, with p being an arbitrary point in U.

The Levi-Civita connection associated with the Riemannian metric g is denoted by ∇. In the context of the horizontal distribution defined by ∇ and the vertical distribution defined by $\ker \pi_*$, we establish the following local frame:

$$E_i = \frac{\partial}{\partial x^i} - y^s \Gamma^h_{is} \frac{\partial}{\partial y^h}, \quad i = 1, \ldots, n,$$

and

$$E_{\bar{i}} = \frac{\partial}{\partial y^i}, \quad \bar{i} = n + 1, \ldots, 2n,$$

where Γ^h_{is} denotes the Christoffel symbols of g. The local frame $\{E_\beta\} = (E_i, E_{\bar{i}})$ is referred to as the adapted frame. Consider a vector field $X = X^i \frac{\partial}{\partial x^i}$. The horizontal and vertical lifts of X with respect to the adapted frame are defined as follows:

$${}^H X = X^i E_i,$$

and

$${}^V X = X^i E_{\bar{i}}.$$

In the tangent bundle TM, the local 1-form system $(dx^i, \delta y^i)$ acts as the dual frame to the adapted frame $\{E_\beta\}$, where

$$\delta y^i = {}^H(dx^i) = dy^i + y^s \Gamma^i_{hs} dx^h.$$

We will first present the following lemma, which will be useful later on.

Lemma 2.1. *Let (M, g) a Riemannian manifold and TM its tangent bundle. The Lie brackets of the adapted frame in TM satisfy the following identities* [26]*:*

$$\begin{aligned} [E_j, E_i] &= y^b R_{ijb}^{\ a} E_{\bar{a}}, \\ [E_j, E_{\bar{i}}] &= \Gamma^a_{ji} E_{\bar{a}}, \\ \left[E_{\bar{j}}, E_{\bar{i}}\right] &= 0, \end{aligned}$$

where $R_{ijb}^{\ a}$ represents the components of the Riemannian curvature tensor of (M, g).

2.2. The synectic lift metric on tangent bundle

In the context of a manifold (M, g), various Riemannian or pseudo-Riemannian metrics can be defined on its tangent bundle TM. These metrics are constructed by lifting the original Riemannian metric g in a natural manner and are referred to as g-natural metrics. In [3], the authors identified the complete family of Riemannian g-natural metrics, which depend on six arbitrary functions of the norm of a vector $u \in TM$.

As noted above, different Riemannian or pseudo-Riemannian metrics on TM have been formulated using natural lifts of the original metric g. One such metric is known as the synectic lift metric on the tangent bundle TM. In this paper, we introduce the synectic lift metric as a new natural, non-rigid metric on TM. We then establish the necessary and sufficient conditions for $(TM,\widetilde{g},\lambda,{}^{V}f)$ and $(TM,\widetilde{g},\lambda,{}^{C}f)$ structures to be (ρ,τ)-quasi-Einstein solitons under the synectic lift metric on the tangent bundle TM.

Definition 2.1. *Let g be a Riemannian metric with components g_{ij} on M. We define the metric*

$$\tilde{g} = a_{ij}dx^i dx^j + 2g_{ij}dx^i \delta y^j,$$

which is non-degenerate and can be considered as a pseudo-Riemannian metric on TM. This metric is referred to as the synectic lift metric, where $a = (a_{ij})$ is a symmetric (0,2)-type tensor field on M.

The synectic lift metric $\tilde{g}$, which can be expressed as $\tilde{g} = {}^{C}g + {}^{V}a$, has the following matrix form in terms of the induced coordinates:

$$\tilde{g} = (\tilde{g}_{\beta\gamma}) = \begin{pmatrix} a_{ij} + \partial g_{ij} & g_{ij} \\ g_{ij} & 0 \end{pmatrix}.$$

For more details, see [24].

In the adapted frame $\{E_\beta\}$, the synectic lift metric and its inverse are represented as follows:

$$\tilde{g} = (\tilde{g}_{\beta\gamma}) = \begin{pmatrix} a_{ij} & g_{ij} \\ g_{ij} & 0 \end{pmatrix},$$

and

$$\tilde{g}^{-1} = (\tilde{g}^{\alpha\beta}) = \begin{pmatrix} 0 & g^{jk} \\ g^{jk} & -a^{jk} \end{pmatrix}.$$

Here, ${}^{C}g$ and ${}^{V}a$ denote the complete lift and vertical lift of g and a to TM, respectively.

For the Levi-Civita connection associated with the synectic lift metric, we have the following:

Lemma 2.2. *The Levi-Civita connection $\tilde{\nabla}$ of the synectic lift metric $\tilde{g}$ on the tangent bundle is expressed as follows* [4]*:*

$$\begin{aligned} \tilde{\nabla}_{E_i} E_j &= \Gamma^k_{ij} E_k + \left(M^k_{ij} + y^s R_{sij}{}^{k}\right) E_{\overline{k}}, \\ \tilde{\nabla}_{E_{\overline{i}}} E_j &= 0, \\ \tilde{\nabla}_{E_i} E_{\overline{j}} &= \Gamma^k_{ij} E_{\overline{k}}, \\ \tilde{\nabla}_{E_{\overline{i}}} E_{\overline{j}} &= 0, \end{aligned}$$

where $M^k_{ij} = \frac{1}{2} g^{kh} \left(\nabla_i a_{hj} + \nabla_j a_{hi} - \nabla_h a_{ij}\right)$ is a tensor of type $(1,2)$. Additionally, R^h_{ijk} denotes the components of the Riemannian curvature tensor field R associated with the Levi-Civita connection ∇ of the Riemannian metric g.

Lemma 2.3. *Let TM denote the tangent bundle of a Riemannian manifold (M,g), and let $\widetilde{g}$ represent the associated synectic lift metric. In the adapted frame $\{E_\beta\}$, the Riemannian curvature tensor $\tilde{R}$ of the Levi-Civita connection $\tilde{\nabla}$ associated with the synectic lift metric $\tilde{g}$ on the tangent bundle exhibits the following properties:*

$$\begin{aligned} \tilde{R}_{mij}{}^{k} &= R_{mij}{}^{k}, \\ \tilde{R}_{mij}{}^{\bar{k}} &= \nabla_m M^k_{ij} - \nabla_i M^k_{mj} + y^s \left(\nabla_m R_{sij}{}^{k} - \nabla_i R_{smj}{}^{k}\right), \\ \tilde{R}_{mi\bar{j}}{}^{\bar{k}} &= R_{mij}{}^{k}, \end{aligned}$$

with all other components being zero. Here, $M^k_{ij} = \frac{1}{2}g^{kh}(\nabla_i a_{hj} + \nabla_j a_{hi} - \nabla_h a_{ij})$ *is a tensor of type* $(1,2)$*. Additionally,* Γ^h_{ij} *and* R^h_{ijk} *denote the components of the Levi-Civita connection* ∇ *of the Riemannian metric* g *and its Riemannian curvature tensor on* M*, respectively (see, also* [15]*).*

Lemma 2.4. *Let* TM *denote the tangent bundle of a Riemannian manifold* (M,g)*, and let* $\widetilde{g}$ *represent the associated synectic lift metric. In the adapted frame* $\{E_\beta\}$*, the Ricci curvature tensor* $\tilde{R}$ *of the Levi-Civita connection* $\tilde{\nabla}$ *associated with the synectic lift metric* $\tilde{g}$ *on the tangent bundle exhibits the following properties:*

$$\tilde{R}_{ij} = R_{ij}, \quad \tilde{R}_{\bar{i}j} = \tilde{R}_{i\bar{j}} = \tilde{R}_{\bar{i}\bar{j}} = 0,$$

where $R_{ij} = R_{kij}^{\ \ \ k}$ *represents the Ricci curvature tensor of the manifold* M *(see, also* [15]*).*

Lemma 2.5. *Let* TM *denote the tangent bundle of a Riemannian manifold* (M,g)*, and let* $\widetilde{g}$ *represent the associated synectic lift metric. The scalar curvature* $\tilde{r}$ *of the Levi-Civita connection* $\tilde{\nabla}$ *corresponding to the synectic lift metric* $\tilde{g}$ *on the tangent bundle is equal to zero (see, also* [15]*).*

3. **Main Results**

Einstein solitons are of great importance in both mathematics and physics, making their study within the realms of Riemannian and semi-Riemannian geometry particularly intriguing. Recently, various extensions of Einstein solitons have been introduced, including quasi-Einstein solitons, generalized quasi-Einstein solitons, m-quasi-Einstein solitons, and (ρ,τ)-quasi-Einstein solitons. In this research, we concentrate on the examination of (ρ,τ)-quasi-Einstein solitons on the tangent bundle, utilizing the synectic lift metric.

In a Riemannian manifold, a metric g is referred to as a generalized quasi-Einstein soliton if there exist smooth functions f, α, and β such that the following equation holds:

$$Ric + Hessf - \alpha df \otimes df = \beta g.$$

Specifically, if $\beta \in \mathbb{R}$ and $\alpha = 0$, this soliton reduces to a gradient Ricci soliton. On the other hand, if $\alpha = \frac{1}{m}$ and $\beta \in \mathbb{R}$, the equation simplifies to an m-quasi-Einstein soliton, where $m \in \mathbb{N}$. The concept of a generalized quasi-Einstein soliton was introduced by Catino [7], and Huang and Wei [17] later proposed examining the (m,ρ)-quasi-Einstein soliton as a specific case of this framework.

Definition 3.1. *In a Riemannian manifold, a metric* g *is termed an* (m,ρ)*-quasi-Einstein soliton if there exists a smooth function* $f : M \to \mathbb{R}$ *along with constants* $\tau, \rho, \lambda \in \mathbb{R}$ *(where* $0 < \tau \leq \infty$*) such that the following equation holds:*

$$Ric + Hessf - \frac{1}{\tau}df \otimes df = (\rho r + \lambda)g. \tag{1}$$

Here, r *represents the scalar curvature, and* $Hessf$ *(or* $\nabla^2 f$*) is the Hessian form of the smooth function* f *on* M*. The importance of these manifolds is highlighted by recent studies on the m-Bakry-Emery Ricci tensor* Ric^m_f *(see* [19]*,* [21]*), which is defined as follows:*

$$Ric^m_f = Ric + Hessf - \frac{1}{\tau}df \otimes df.$$

For any smooth function f defined on M, the vertical lift of f to the tangent bundle TM is given by ${}^V f = f$, while the complete lift of f to TM is defined as ${}^C f = y^s \partial_s f$. To introduce our primary concept, we denote the Hessian operator of the vertical and complete lifts of any smooth function f on M with respect to the synectic lift metric.

The Hessian operator of the metric g for a smooth function f is defined by

$$(Hess_g f)(X,Y) = XYf - (\nabla_X Y) f,$$

where $X, Y \in \Im_0^1(TM)$. In local coordinates, this can be expressed as

$$(\nabla f)_{\beta\gamma} = \partial_\beta \partial_\gamma f - \Gamma^\alpha_{\beta\gamma}\partial_\alpha f = f_{\beta\gamma} - \Gamma^\alpha_{\beta\gamma} f_\alpha,$$

with $\gamma = j, \bar{j}$ and $\beta = i, \bar{i}$. Here, $\partial_\beta f = \frac{\partial}{\partial x^\beta} f = f_\beta$ and $\partial_\beta \partial_\gamma f = \frac{\partial}{\partial x^\beta}\frac{\partial}{\partial x^\gamma} f = f_{\beta\gamma}$, while $\partial_\alpha \partial_\beta \partial_\gamma f = f_{\alpha\beta\gamma}$. From this point onward, we will use this representation throughout the paper.

To elucidate our main topic, we will provide essential information about the Hessian operator (with respect to the synectic lift metric) for any smooth function f defined on M.

Lemma 3.1. *Let f be a smooth function defined on a Riemannian manifold (M, g). The Hessian (with respect to the synectic lift metric) of its vertical lift can be expressed as follows:*

$$\begin{aligned}
\left(\tilde{\nabla}^2\, {}^V f\right)_{ij} &= \partial_i \partial_j f - \tilde{\Gamma}^k_{ij}\partial_k f = \left(\nabla^2 f\right)_{ij}, \\
\left(\tilde{\nabla}^2\, {}^V f\right)_{i\bar{j}} &= 0, \\
\left(\tilde{\nabla}^2\, {}^V f\right)_{\bar{i}j} &= 0, \\
\left(\tilde{\nabla}^2\, {}^V f\right)_{\bar{i}\bar{j}} &= 0.
\end{aligned}$$

The auxiliary lemmas that will be utilized in our study are presented below.

Lemma 3.2. *For any smooth function f on M, the vertical lift of f to TM is defined as ${}^V f = f$, while the complete lift of f to TM is given by ${}^C f = y^s \partial_s f$. The dual 1-form of ∇f is denoted by df. The following equations can thus be established:*

i. $d\, {}^V f \otimes d\, {}^V f = f_i f_j dx^i dx^j$,
ii. $d\, {}^C f \otimes d\, {}^C f = y^s y^p f_{is} f_{jp} dx^i dx^j + y^s f_{is} f_j dx^i dy^j + f_i f_j dy^i dy^j$,

where $f_\beta = \frac{\partial}{\partial x^\beta} f$ and $f_{\beta\gamma} = \frac{\partial}{\partial x^\beta}\frac{\partial}{\partial x^\gamma} f$.

Theorem 3.1. *Let (M, g) be a Riemannian manifold, and let the synectic lift metric on the tangent bundle TM be given by $\tilde{g} = a_{ij} dx^i dx^j + 2 g_{ij} dx^i \delta y^j$. The structure $(TM, \widetilde{g}, {}^V f, \lambda)$ is considered a (ρ, τ)-quasi-Einstein soliton if and only if the following conditions are satisfied:*

i. $\lambda = 0$,
ii. $\frac{1}{\tau} f_i f_j = R_{ij} + \left(\nabla^2 f\right)_{ij}$.

Here, $\tilde{r} = 0$ denotes the scalar curvature of $\tilde{g}$, while $\tilde{R}$ represents the Ricci curvature tensor of $\tilde{g}$. Additionally, R and r are the Riemannian curvature tensor and scalar curvature of the Levi-Civita connection ∇ of the Riemannian metric g, respectively. Moreover, we define $f_\beta = \partial_\beta f = \frac{\partial}{\partial x^\beta} f$ and $f_{\beta\gamma} = \partial_\beta \partial_\gamma f = \frac{\partial}{\partial x^\beta}\frac{\partial}{\partial x^\gamma} f$.

Proof. If the expression $\left(\widetilde{\nabla}^2\, {}^V f\right)$ in Lemma 3.1 is used in (1), we have

$$\begin{aligned}
\widetilde{R}_{ij} + \left(\widetilde{\nabla}^2\, {}^V f\right)_{ij} - \frac{1}{\tau} d^V f \otimes d^V f &= (\rho \widetilde{r} + \lambda)\, \widetilde{g}_{ij} \\
R_{ij} + \left(\nabla^2 f\right)_{ij} - \frac{1}{\tau} f_i f_j &= \lambda a_{ij} \qquad (2)
\end{aligned}$$

and

$$\begin{aligned}\widetilde{R}_{\bar{i}j} + \left(\widetilde{\nabla}^2\ {}^V f\right)_{\bar{i}j} &= (\rho\widetilde{r}+\lambda)\,\widetilde{g}_{\bar{i}j}\\ 0 &= \lambda g_{ij}\\ 0 &= \lambda. \end{aligned} \tag{3}$$

Substituting equation (3) in the equation (2), we get

$$\frac{1}{\tau} f_i f_j = R_{ij} + \left(\nabla^2 f\right)_{ij}.$$

Conversely by a routine calculation, we can check that in any case $i - ii$ of the theorem. So the proof is completed. □

Lemma 3.3. *Let f be a smooth function on a Riemannian manifold (M,g). The Hessian of its complete lift, with respect to the synectic lift metric, is given by the following expressions:*

$$\begin{aligned}\left(\tilde{\nabla}^2\ {}^C f\right)_{ij} &= y^s\left(\nabla_i\nabla_j f_s - R_{sij}^{\ \ k} f_k\right) - M_{ij}^k f_k,\\ \left(\tilde{\nabla}^2\ {}^C f\right)_{i\bar{j}} &= \nabla_i f_j,\\ \left(\tilde{\nabla}^2\ {}^C f\right)_{\bar{i}j} &= f_{ij},\\ \left(\tilde{\nabla}^2\ {}^C f\right)_{\bar{i}\bar{j}} &= 0.\end{aligned}$$

Theorem 3.2. *Let (M,g) be a Riemannian manifold, and let $\tilde{g} = a_{ij}dx^i dx^j + 2g_{ij}dx^i\delta y^j$ denote the synectic lift metric on the tangent bundle TM. The structure $(TM, \tilde{g}, {}^C f, \lambda)$ is classified as a (ρ,τ)-quasi-Einstein soliton if and only if the following conditions are satisfied:*

i. $\lambda = -\frac{1}{n\tau} y^s g^{ij} f_{is} f_j$,
ii. $\nabla_i\nabla_j f_s = R_{sij}^{\ \ k} f_k$,
iii. $R_{ij} = M_{ij}^k f_k$,
iv. $g^{hl} f_{hs} f_l a_{ij} = 0$,
v. $f_i f_j = 0, \quad f_{is} f_{jp} = 0$,

where $\tilde{r} = 0$ represents the scalar curvature of $\tilde{g}$, and $\tilde{R}$ is the Ricci curvature tensor of $\tilde{g}$. Additionally, R and r denote the Riemannian curvature tensor and scalar curvature of the Levi-Civita connection ∇ of the Riemannian metric g, respectively. The derivatives are defined as $f_\beta = \partial_\beta f = \frac{\partial}{\partial x^\beta} f$ and $f_{\beta\gamma} = \partial_\beta\partial_\gamma f = \frac{\partial}{\partial x^\beta}\frac{\partial}{\partial x^\gamma} f$. The term M_{ij}^k is given by

$$M_{ij}^k = \frac{1}{2} g^{kh}\left(\nabla_i a_{hj} + \nabla_j a_{hi} - \nabla_h a_{ij}\right).$$

Proof. If the expression $\left(\widetilde{\nabla}^2\ {}^C f\right)_{ij}$ in Lemma 3.3 is used in (1), we obtain

$$\widetilde{R}_{ij} + \left(\widetilde{\nabla}^2\ {}^C f\right)_{ij} - -\frac{1}{\tau} d^C f \otimes d^C f = (\rho\widetilde{r}+\lambda)\,\widetilde{g}_{ij}$$

$$R_{ij} + y^s\left(\nabla_i\nabla_j f_s - R_{sij}^{\ \ k} f_k\right) - M_{ij}^k f_k - \frac{1}{\tau} y^s y^p f_{is} f_{jp} = -\lambda a_{ij}$$

from which, we get

$$R_{ij} - M_{ij}^k f_k = -\lambda a_{ij}, \tag{4}$$
$$\nabla_i\nabla_j f_s = R_{sij}^{\ \ k} f_k$$

and

$$f_{is} f_{jp} = 0.$$

If the expression $\left(\widetilde{\nabla}^2\ {}^C f\right)_{i\bar{j}}$ or $\left(\widetilde{\nabla}^2\ {}^C f\right)_{\bar{i}j}$ in Lemma 3.3 is used in (1), we have

$$-\frac{1}{\tau} y^s f_{is} f_j = \lambda g_{ij}.$$

Contracting with g^{ij} both sides in last equation, we get

$$\lambda = -\frac{1}{n\tau} y^s g^{ij} f_{is} f_j.$$

Substituting above equation into equation (4), we obtain

$$R_{ij} - M_{ij}^k f_k = \frac{1}{n\tau} y^s g^{hl} f_{hs} f_l a_{ij}$$

from which, we have

$$R_{ij} = M_{ij}^k f_k$$

and

$$g^{hl} f_{hs} f_l a_{ij} = 0.$$

If the expression $\left(\widetilde{\nabla}^2\ {}^C f\right)_{\bar{i}\bar{j}}$ in Lemma 3.3 is used in (1), we get

$$f_i f_j = 0.$$

If the above calculations are followed in reverse, the statements of the theorem easily accesible. So the proof is completed. □

Conclusion

In this paper, we have thoroughly examined the structure of (ρ, τ)-quasi-Einstein solitons on the tangent bundle TM of a Riemannian manifold (M, g) under the framework of the synectic lift metric $\tilde{g}$. By employing an adapted frame, we derived necessary and sufficient conditions for the structures $(TM, \tilde{g}, {}^V f, \lambda)$ and $(TM, \tilde{g}, {}^C f, \lambda)$ to be classified as (ρ, τ)-quasi-Einstein solitons.

We found that the conditions imposed on λ, the derivatives of f, and the curvature tensors significantly constrain the geometry of the manifold. Specifically, we established that for $(TM, \tilde{g}, {}^V f, \lambda)$ to be a (ρ, τ)-quasi-Einstein soliton, λ must be zero, and the relationship between the Hessian of f and the Ricci curvature tensor must hold as indicated. In contrast, for $(TM, \tilde{g}, {}^C f, \lambda)$, a more intricate set of conditions involving λ, the Riemannian curvature tensor, and the complete lift of f emerged, highlighting the delicate interplay between the geometry of the manifold and the behavior of the smooth function f.

Our findings contribute to a deeper understanding of quasi-Einstein solitons within the context of synectic lift metrics, expanding the existing literature on solitonic structures in Riemannian geometry. Future work may extend this analysis to explore the implications of these conditions on specific classes of Riemannian manifolds and their geometric properties, as well as investigate the potential applications of (ρ, τ)-quasi-Einstein solitons in mathematical physics. The study of these solitonic structures promises to unveil further insights into the rich tapestry of geometric phenomena arising from the interplay of curvature and functional dynamics.

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