THE ENTROPY OF COUNTABLE DYNAMICAL SYSTEMS

A. Ebrahimzadeh[1], M. Ebrahimi[2]

In [2, 3]*, entropy of dynamical systems on a probability space and on an algebraic structure with countable partitions is introduced. In this paper we prove some results about it and compare to the finite case. The aim of this note is to express conditions under which the entropy of a dynamical system is zero.*

Keywords: Countable partition; Entropy; Measure-preserving transformation; m- preserving transformation

1. Introduction

One of the applied branches of mathematics is the entropy of a dynamical system. It has been widely applied since in a variety of fields, including physics, chemistry, engineering, biology, economics, anthropology, general systems theory, information theory, psychotherapy, sociology, urban planning, and others. The entropy concept originated in thermodynamics in the mid-nineteenth century. Shannon in the 1940s was concerned with the problems of the transmission of information in the presence of noise. We assume the reader is familiar with the definition of measure [6], dynamical system [7]. The entropy of a finite partition, p, of a probability space $(X, \beta.m)$ is defined as

$$H(p) = -\sum_{i=1}^{n} m(A_i) log m(A_i),$$

where $p = \{A_1, ...A_n\} \subset \beta$. The entropy of dynamical system on an algebraic structure with finite partitions is defined by Riecan [5]. The entropy of dynamical system on a probability space and on the algebraic structure with countable partitions is introduced in [2, 3]. In this paper we define $A \doteq B$ for two countable partitions A, B of a probability space (X, β, m) and for two countable partitions A, B in the algebraic structure. We prove some properties about the entropy and conditional entropy and compare to the finite case. At the end we express some conditions under which the entropy of a dynamical system is zero.

2. The Entropy on Probability Space with Countable Partitions

Definition 2.1. *Let (X, β, m) be a probability space, a partition of (X, β, m) is a disjoint collection of elements of β whose union is X.*

Department of Mathematics, Science and Research Branch, Islamic Azad University, Kerman, Iran, Email: abolfazl35@yahoo.com

Department of Mathematics, Faculty of Mathematics and Computer Science, Shahid Bahonar University of Kerman, Kerman, Iran, Email: mohamad_ebrahimi@mail.uk.ac.ir

Definition 2.2. *Let* $A = \{A_i : i \in \mathbf{N}\}$, $B = \{B_j : j \in \mathbf{N}\}$ *be countable partitions of* (X, β, m). *Their join is the partition*

$$A \vee B = \{A_i \cap B_j : i, j \in \mathbf{N}\}.$$

Definition 2.3. *Let* $A = \{A_i : i \in \mathbf{N}\}$ *be a countable partition of probability space* (X, β, m). *The entropy of* A *is defined in* [2] *by*

$$H(A) = -log \sup_{i \in \mathbf{N}} m(A_i).$$

Definition 2.4. *Let* $A = \{A_i : i \in \mathbf{N}\}$, $B = \{B_i : i \in \mathbf{N}\}$ *be two countable partitions of probability space* (X, β, m). *The entropy of* A *given* B *is defined in* [2] *by*

$$H(A|B) = -\log \frac{\sup_{i,j \in \mathbf{N}} m(A_i \bigcap B_j)}{\sup_{j \in \mathbf{N}} m(B_j)}.$$

Definition 2.5. *Let* (X_1, β_1, m_1), (X_2, β_2, m_2) *be two probability spaces. A transformation* $T : X_1 \longrightarrow X_2$ *is measure preserving if*

i) $T^{-1}\beta_2 \subset \beta_1$;

ii) $m_1(T^{-1}B_2) = m_2(B_2)$, $\forall B_2 \in \beta_2$.

Definition 2.6. [2] *A probability dynamical system is a complex* (X, β, m, T) *where* (X, β, m) *is a probability space and* $T : X \longrightarrow X$ *is a measure-preserving transformation.*

Definition 2.7. *Let* A *and* B *be countable partitions of* (X, β, m). *We write* $A \subset^o B$ *if for every* $A_i \in A$ *there exists* $C_j \in B$ *such that* $m(A_i \vartriangle C_j) = 0$. *We write* $A \doteq B$ *if* $A \subset^o B$ *and* $B \subset^o A$.

Proposition 2.1. *If* $A = \{A_i : i \in \mathbf{N}\}$, $B = \{C_j : j \in \mathbf{N}\}$ *and* $C = \{D_k : k \in \mathbf{N}\}$ *be countable partitions of* (X, β, m). *Then*

i) *If* $A \doteq B$ *then* $H(A) = H(B)$;

ii) *If* $A \doteq C$ *then* $H(A|B) = H(C|B)$;

iii) *If* $B \doteq C$ *then* $H(A|B) = H(A|C)$.

Proof.

i) Since $A \subset^o B$, for every $i \in \mathbf{N}$ there is $j \in \mathbf{N}$, such that

$$m(A_i \cap C_j) = m(C_j) = m(A_i).$$

So

$$\{m(A_i) : i \in \mathbf{N}\} \subseteq \{m(C_j) : j \in \mathbf{N}\}.$$

It follows that

$$\sup_{i \in \mathbf{N}} m(A_i) \leq \sup_{j \in \mathbf{N}} m(C_j).$$

On the other hand $B \subset^o A$ similarly implies $\sup_{j \in \mathbf{N}} m(C_j) \leq \sup_{i \in \mathbf{N}} m(A_i)$. Hence

$$\sup_{i \in \mathbf{N}} m(A_i) = \sup_{j \in \mathbf{N}} m(C_j).$$

ii) By definition it is sufficient to show that

$$\sup_{i,j \in \mathbf{N}} m(A_i \cap C_j) = \sup_{k,j \in \mathbf{N}} m(D_k \cap C_j).$$

Since $A \subset^o C$ then for every $i \in \mathbf{N}$ there is $k \in \mathbf{N}$, such that

$$m(D_k - A_i) = 0, m(A_i - D_k) = 0.$$

Hence we have

$$\begin{aligned} m(D_k \cap C_j) &= m(D_k \cap (A_i^C \cup A_i) \cap C_j) \\ &\leq m((D_k - A_i) \cap C_j) + m((D_k \cap A_i \cap C_j) \\ &\leq m(D_k - A_i) + m(A_i \cap C_j) \\ &= m(A_i \cap C_j). \end{aligned}$$

So for every $j \in \mathbf{N}$

$$m(D_k \cap C_j) \leq m(A_i \cap C_j). \tag{2.1}$$

On the other hand

$$\begin{aligned} m(A_i \cap C_j) &= m(A_i \cap (D_k \cup D_k) \cap C_j) \\ &= m((D_k \cap A_i \cap C_j) \cup (A_i \cap D_k \cap C_j)) \\ &\leq m(D_k \cap C_j) + m(A_i - D_k) \\ &= m(D_k \cap C_j). \end{aligned}$$

So for every $j \in \mathbf{N}$

$$m(A_i \cap C_j) \leq m(D_k \cap C_j). \tag{2.2}$$

Hence (2.1) and (2.2) imply that for every $i \in \mathbf{N}$ there exists $k \in \mathbf{N}$, such that for any $j \in \mathbf{N}$,

$$m(A_i \cap C_j) = m(D_k \cap C_j).$$

Thus

$$\{m(A_i \cap C_j) : i, j \in \mathbf{N}\} \subseteq \{m(D_k \cap C_j) : k, j \in \mathbf{N}\}.$$

It follows that

$$\sup_{i,j \in \mathbf{N}} m(A_i \cap C_j) \leq \sup_{k,j \in \mathbf{N}} m(D_k \cap C_j).$$

Now on the other hand $D \subset^o A$ similarly implies that

$$\sup_{i,j \in \mathbf{N}} m(D_k \cap C_j) \leq \sup_{k,j \in \mathbf{N}} m(A_i \cap C_j).$$

iii) Since $B \doteq C$, by (i) we have

$$diam B = diam C.$$

Also since $B \doteq C$, by (ii) we have

$$diam(A \vee B) = diam(A \vee C).$$

So we can write

$$\begin{aligned} H(A|B) &= -\ log \sup_{i\in\mathbf{N}} \frac{diam(A_i \bigtriangledown B)}{diam B} \\ &= -\ log \frac{diam(A \vee B)}{diam B} \\ &= -\ log \frac{diam(A \vee C)}{diam C} \\ &= -\ log \sup_{i\in\mathbf{N}} \frac{diam(A_i \bigtriangledown C)}{diam C}. \\ &= H(A|C). \end{aligned}$$

■

Proposition 2.2. *Let $A = \{A_i : i \in \mathbf{N}\}$ and $B = \{C_j : j \in \mathbf{N}\}$ be two countable partitions of (X, β, m) with $B \subset^o A$. Then $H(A|B) = 0$.*

Proof.
Since $B \subset^o A$, for every $j \in \mathbf{N}$ there is $i \in \mathbf{N}$, such that

$$m(A_i \cap C_j) = m(C_j).$$

So

$$\{m(C_j) : j \in \mathbf{N}\} \subseteq \{m(A_i \cap C_j) : i, j \in \mathbf{N}\}.$$

It follows that $\sup_{j\in\mathbf{N}} m(C_j) \leq \sup_{i,j\in\mathbf{N}} m(A_i \cap C_j)$ and it means

$$diam\eta \leq diam(A \vee B).$$

On the other hand $diam(A \vee B) \leq diam\eta$. Hence $H(A|B) = -\ log \frac{diam(A\vee B)}{diam\eta} = 0$. ■

Remark. If A and B are two finite partitions of (X, β, m) such that $H(A|B) = 0$ then $A \subset^o B$. (see [8]). But if A and B be two countable partitions of (X, β, m) such that $H(A|B) = 0$ then it does not imply $B \subset^o A$ or $A \subset^o B$ necessarily. In order to show this, we present the following example.

Example 2.1. *Let $X = (0, 1]$ and m be Lebesgue measure. If $A = \{A_i : i \in \mathbf{N}\}$ and $B = \{C_j : j \in \mathbf{N}\}$ such that*

$$\begin{aligned} A_1 &= (1/2, 1], \\ A_2 &= (1/3, 1/2], \\ A_3 &= (5/12, 1/3], \\ A_4 &= (1/4, 5/12], \\ A_5 &= (1/5, 1/4], \end{aligned}$$

and for $i \geq 6$,

$$A_i = (\frac{1}{i}, \frac{1}{i-1}].$$

And

$$\begin{aligned} C_1 &= (1/2, 1], \\ C_2 &= (5/12, 1/2], \\ C_3 &= (1/3, 5/12], \end{aligned}$$

and for $j \geq 4$,

$$C_j = (1/j, \frac{1}{j-1}].$$

A and B are countable partitions of probability space (X, β, m) *because* $\bigcup_{i=1}^{\infty} A_i = \bigcup_{j=1}^{\infty} C_j = X$ *and* $\{A_i : i \in \mathbf{N}\}$, $\{C_j : j \in \mathbf{N}\}$ *are disjoint separately. Now we have*

$$diam(A \vee B) = \sup_{i,j \in \mathbf{N}} m(A_i \cap C_j) = m(A_1 \cap C_1) = m(1/2, 1] = 1/2.$$

It follows that $H(A|B) = 0$.
But

$$m(A_2 \cap C_2) = 1/12 \neq 2/12 = m(A_2), m(A_2 \cap C_2) \neq 0.$$

So $B \not\subset^o A$. *Also*

$$m(A_4 \cap C_4) = 1/12 \neq 2/12 = m(C_4), m(A_4 \cap C_4) \neq 0.$$

So $A \not\subset^o B$.

Definition 2.8. *Let* $T : X \longrightarrow X$ *be a measure-preserving transformation of probability space* (X, β, m) *and A be a countable partition of* (X, β, m). *The entropy of T relative to A is defined by*

$$h(T, A) = H(A| \vee_{i=0}^{\infty} T^{-i} A).$$

Definition 2.9. *Let* $T : X \longrightarrow X$ *be a measure-preserving transformation of probability space* (X, β, m). *The entropy of T is defined by*

$$h(T) = \sup_{A} h(T, A),$$

where the supremum is taken over all countable partitions of (X, β, m).

Corollary 2.1. *Let T be a measure-preserving transformation of probability space* (X, β, m). *Let A be a countable partition of* (X, β, m). *If* $\vee_{i=0}^{\infty} T^{-i} A \subset^o A$, *then* $h(T, A) = 0$.

Proof.
It holds from Proposition 2.2.

Corollary 2.2. *Let T be a measure-preserving transformation of probability space* (X, β, m). *If for every countable partition A of* (X, β, m) *we have* $\vee_{i=0}^{\infty} T^{-i} A \subset^o A$, *then* $h(T) = 0$.

Proof.
By using of corollary 2.1, it is clear.

3. The Entropy on an Algebraic Structure with Countable Partitions

All definitions in this section are from [2, 3]. Let F be a non-empty totally ordered set. Also let $\oplus$, $\odot$ be two binary operations on F and 1 be a constant element of F such that

$$1 \odot a = a \geq a \odot b,$$

for any $b \in F$.

Definition 3.1. *A function* $m : F \longrightarrow [0, 1]$ *is called F-measure when for any* $a, b, c \in F$

i) $m(a \oplus b) = m(b \oplus a)$, $m(a \odot b) = m(b \odot a)$;
ii) $m(a \oplus (b \oplus c)) = m((a \oplus b) \oplus c)$, $m(a \odot (b \odot c)) = m((a \odot b) \odot c)$;
iii) $m(a \odot (b \oplus c)) = m((a \odot b) \oplus (a \odot c))$, $m(a \oplus (b \odot c)) = m((a \oplus b) \odot (a \oplus c))$;
iv) $m(\oplus_{i=1}^{n} a_i) = \sum_{i=1}^{n} m(a_i)$, *for any* $n \in \mathbb{N}$;
v) *If* $a \leq b$ *then* $m(a) \leq m(b)$;
vi) $m(a \odot b) \leq m(a)$;
vii) *If* $m(a) = m(1)$ *then* $m(a \odot b) = m(b)$;
viii) *If* $m(a) \leq m(b)$ *then* $m(a \odot c) \leq m(b \odot c)$.

Definition 3.2. *A countable partition in* F *is a sequence* $A = \{a_i\}_{i \in \mathbb{N}} \subseteq F$ *such that*
i) $m(1) = \sum_{i=1}^{\infty} m(a_i)$;
ii) $\sum_{i=1}^{\infty} m(a_i \odot b) = m(b)$, *for any* $b \in F$.

Definition 3.3. *Let* $A = \{a_i\}_{i \in \mathbb{N}}$ *and* $B = \{b_j\}_{j \in \mathbb{N}}$ *be two countable partitions in* F. *Their join is*

$$A \nabla B = \{a_i \odot b_j : a_i \in A, b_j \in B, i, j \in \mathbb{N}\},$$

if $A \neq B$, *and*

$$A \nabla A = A.$$

Definition 3.4. *Let* $A = \{a_i\}_{i \in \mathbb{N}}$ *be a countable partition in* F. *The entropy of* A *is defined in [3] by*

$$H(A) = -\log \sup_{i \in \mathbb{N}} m(a_i).$$

Definition 3.5. *Let* $A = \{a_i : i \in \mathbf{N}\}$, $B = \{b_j : j \in \mathbf{N}\}$ *be two countable partitions in* F. *The conditional entropy of* A *given* B *is defined in* [3] *by*

$$H(A|B) = -\log \frac{\sup_{i,j \in \mathbf{N}} m(a_i \odot b_j)}{\sup_{j \in \mathbf{N}} m(b_j)}.$$

Definition 3.6. *Let* $A = \{a_i\}_{i \in \mathbb{N}}$ *be a countable partition in* F. *The diameter of* A *is defined by*

$$diam(A) = \sup_{i \in \mathbb{N}} m(a_i).$$

Definition 3.7. *Let* G *be a non-empty subset of* F. *We say* G *is* $m-$*set when there exists* $k \in [0, 1]$ *such that* $m(a) = k$, *for any* $a \in G$. *In this case we set* $m(G) = k$.

Definition 3.8. *A function* $u : F \longrightarrow F$ *is m- preserving transformation when*
i) $u^{-1}(a)$ *is an m-set with* $k = m(a)$, *for any* $a \in F$;
ii) $u^{-1}(a \oplus b)$ *is an* $m-$*set and*

$$m(u^{-1}(a \oplus b)) = m(u^{-1}(a) \oplus u^{-1}(b)),$$

for any $a, b \in F$;
iii) $u^{-1}(a \odot b)$ *and* $u^{-1}(a) \odot u^{-1}(b)$ *are* $m-$*sets and*

$$m(u^{-1}(a \odot b)) = m(u^{-1}(a) \odot u^{-1}(b)),$$

for any $a, b \in F$.

Definition 3.9. *Let* A *and* B *be countable partitions in* F. *We write* $A \subset^o B$ *if for every* $a_i \in A$, $c_j \in B$ *we have* $m(a_i \odot c_j) = m(c_j)$ *or* $m(a_i \odot c_j) = 0$. *We write* $A \doteq B$ *if* $A \subset^o B$ *and* $B \subset^o A$.

Proposition 3.1. *If A,B,C are countable partitions in F. Then*
i) *If $A \doteq B$ then $H(A) = H(B)$;*
ii) *If $A \doteq C$ then $H(A|B) = H(C|B)$;*
iii) *If $B \doteq C$ then $H(A|B) = H(A|C)$.*

Proof.

i) Let $A = \{a_i : i \in \mathbf{N}\}$, $B = \{c_j : j \in \mathbf{N}\}$. Since $A \doteq B$, for every $a_i \in A$, $c_j \in B$ we have $m(a_i \odot c_j) = m(c_j) = m(a_i)$ or $m(a_i \odot c_j) = 0$. So

$$\sup_{i\in\mathbf{N}} m(a_i) = \sup_{j\in\mathbf{N}} m(c_j).$$

ii) Since $A \doteq C$, then $A \vee B \doteq C \vee B$. So by (i),

$$\begin{aligned} H(A|B) &= H(A \vee B) - H(B) \\ &= H(C \vee B) - H(B) \\ &= H(C|B). \end{aligned}$$

iii) Since $B \doteq C$, then $A \vee B \doteq A \vee C$. Hence by (i),

$$\begin{aligned} H(A|B) &= H(A \vee B) - H(B) \\ &= H(A \vee C) - H(C) \\ &= H(A|C). \end{aligned}$$

■

Proposition 3.2. *Let $A = \{a_i : i \in \mathbf{N}\}$ and $B = \{c_j : j \in \mathbf{N}\}$ be two countable partitions in F with $B \subset^o A$. Then $H(A|B) = 0$.*

Proof.

Since $B \subset^o A$, for every $j \in \mathbf{N}$ there is $i \in \mathbf{N}$, such that

$$m(a_i \odot c_j) = m(c_j).$$

So

$$\{m(c_j) : j \in \mathbf{N}\} \subseteq \{m(a_i \odot c_j) : i, j \in \mathbf{N}\}.$$

It follows that $\sup_{j\in\mathbf{N}} m(c_j) \leq \sup_{i,j\in\mathbf{N}} m(a_i \odot c_j)$ and it means

$$diam B \leq diam(A \vee B).$$

On the other hand $diam(A \vee B) \leq diam\eta$. Hence $H(A|B) = -\ log\frac{diam(A\vee B)}{diam B} = 0$ ■

Definition 3.10. *Let u be an m- preserving transformation of probability and A be a countable partition in F, The entropy of u relative to A is defined by*

$$h(u, A) = H(A| \vee_{i=1}^{\infty} u^{-i} A).$$

Definition 3.11. *Let u be an m- preserving transformation in F. The entropy of u is defined by*

$$h(u) = \sup_{A} h(u, A),$$

where the supremum is taken over all countable partitions in F.

Corollary 3.1. *Let u be an m- preserving transformation in F. Let A be a countable partition in F. If $\vee_{i=1}^{\infty} u^{-i} A \subset^o A$, then $h(u, A) = 0$.*

Proof.
It holds from Proposition 3.2.

Corollary 3.2. *Let u be an m- preserving transformation in F. If for every countable partition A in F we have $\vee_{i=1}^{\infty} u^i A \subset^o A$, then $h(u) = 0$.*

Proof.
By corollary 3.1, it is clear.

4. **Conclusion**

This paper has defined $A \doteq B$ for two countable partitions A, B in a probability space (X, β, m) and in an algebraic structure separately. We proved some properties about entropy and conditional entropy. It is shown by an example that if A and B are two countable partitions of (X, β, m) such that $H(A|B) = 0$, then necessarily it does not imply $B \subset^o A$ or $A \subset^o B$. At the end we expressed some conditions under which the entropy of dynamical system is zero.

REFERENCES

[1] D. Dumitrescu, Entropy of a fuzzy dynamical system, Fuzzy Sets and Systems 70 (1995), 45-57.
[2] M. Ebrahimi, Generators of probability dynamical systems, Geom.Dyn. Syst. 8(2006), 90-97.
[3] M. Ebrahimi and N. Mohamadi, The entropy function on an algebraic structure with infinite partitions and m-preserving transformation, Applied Science, 12 (2010), 48-63.
[4] Alan B. Poritz, Jonathan A. Poritz, On entropy-preserving stochastic averages, Linear Algebra and its Applications, 434 (2011), 1425-1443.
[5] B. Riecan, D. Markechova, The Entropy of fuzzy dynamical systems, general scheme and generators, Fuzzy Sets and Systems 96 (1993), 191-199.
[6] Ya. G. Sinai, Introduction to Ergodic theory, Princeton Univ. Press, Princeton, N. J., 1976.
[7] Ya. G. Sinai, Ergodic theory with applications to dynamical systems and statistical mechanics, Springer Verlag, 1990.
[8] P. Walters, An introduction to ergodic theory, Springer Verlag, 1982.