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TWO BOUNDARY ELEMENT APPROACHES FOR THE COMPRESSIBLE FLUID FLOW AROUND A NON-LIFTING BODY

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În lucrare sunt prezentate două variante de aplicare a metodei elementelor pe frontieră pentru rezolvarea problemei curgerii unui fluid compresibil în jurul corpurilor, una utilizând o distribuție de surse pe frontiera corpului şi cealaltă o distribuție de vârtejuri. Se obțin rezultate numerice pentru un caz particular - obstacolul eliptic- şi de asemenea se face un studiu comparativ între aceste rezultate numerice şi soluția exactă a problemei. Soluțiile numerice sunt în concordanță cu soluția exactă a problemei în ambele cazuri, dar pentru acest caz particular soluția numerică ce utilizează vârtejuri este mai adecvată.

In the paper there are presented two variants of application of the boundary element method for solving the compressible fluid flow around bodies, one using a distribution of sources on the boundary and the other a vortex distribution. There are obtained numerical results for a particular case- the elliptical obstacle-and a comparison study between these numerical results and the exact solution of the problem is also done. The numerical solutions are in good agreement with the exact solution of the problem in both cases, but, for this particular case, the solution that uses the vortex distribution better fits.

Keywords: indirect boundary element method, integral equation, sources distribution, vortex distribution, compressible fluid flow, elliptical obstacle.

1. Introduction

The boundary integral method (BEM) is a modern numerical technique used to solve boundary value problems for systems of partial differential equations.

There exist two principal variants of applying this method: the direct method and the indirect one. Both of them offer the principal advantage of the BEM over the other numerical method - the ability to reduce the problem

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dimension by one. This property is advantageous as it reduces the size of the system the problem is equivalent with and so improves computational efficiency. To achieve this reduction of dimension it is necessary to formulate the governing equation as a boundary integral equation, which is usually a singular one ([1],[2]), and for this there can be used both techniques the indirect technique and the direct one. This paper is focused on a comparison between two variants of the first technique applied to solve the bivariate problem of an inviscid, compressive subsonic fluid flow around bodies, considering the case of a non-lifting obstacle.

The first variant uses a distribution of sources on the obstacle's boundary, and the second uses a vortex distribution.

For a better understanding we make a short presentation of the problem to solve, as in [3]. A uniform, steady, potential motion of an ideal inviscid fluid of subsonic velocity $U_\infty \bar{i}$, pressure p_∞ and density ρ_∞ is perturbed by the presence of a fixed body of a known boundary, noted *C*, assumed to be smooth and closed. We want to find out the perturbed motion, and the fluid action on the body.

Denoting by $\bar{v}$ the perturbation velocity (u,v its components along the axes) and by $\bar{V}$ the velocity field for the perturbed motion, and using dimensionless variables we have the following mathematical model:

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \\ \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} = 0 \end{cases} \tag{1}$$

with the boundary condition:

$$(\beta + u) n_x + \beta^2 v n_y = 0 \text{ on } C, \tag{2}$$

where $\bar{n}$ is the normal unit vector outward the fluid, β has the usual signification, $\beta = \sqrt{1 - M^2}$, and *M* the Mach number for the unperturbed motion. It is also required that the perturbation velocity vanishes at infinity: $\lim_{\infty} \bar{v} = 0$.

2. The boundary integral equation - sources distribution

The indirect method is applied to obtain the boundary integral equation. First, the boundary, so C, is assimilated with a sources distribution, having the unknown intensity $f(\bar{x})$, assumed to satisfy a *Hölder* condition on C. The perturbation velocity in a point $M(\bar{\xi})$ situated in the fluid domain is:

$$\bar{v}(\bar{\xi}) = -\frac{1}{2\pi}\oint_C f(\bar{x})\frac{\bar{x}-\bar{\xi}}{|\bar{x}-\bar{\xi}|^2}ds\,. \tag{3}$$

For getting the boundary integral equation we use relation (2) and the expression for the velocity on the boundary deduced using a limit process in (3).

If $\bar{x}^0$ denotes a regular point on the boundary we deduce:

$$\bar{v}(\bar{x}_0) = -\frac{1}{2}f(\bar{x}_0)\bar{n}^0 - \frac{1}{2\pi}\grave{\oint_C} f(\bar{x})\frac{\bar{x}-\bar{\xi}}{|\bar{x}-\bar{\xi}|^2}ds\,. \tag{4}$$

where the sign\` denotes the Cauchy principal value of the integral (see[6]).

We obtain the following singular boundary integral equation (see [3]):

$$\left({n_x^0}^2 + \beta^2 {n_y^0}^2\right)f(\bar{x}_0) + \frac{1}{\pi}\grave{\int_C} f(\bar{x})\frac{(x-x_0)n_x^0 + \beta^2(y-y_0)n_y^0}{|\bar{x}-\bar{x}_0|^2}ds = 2\beta n_x^0 \tag{5}$$

where n_x^0, n_y^0 are the components of the normal unit vector outward the fluid evaluated at $\bar{x}^0$.

In order to solve the singular boundary integral equation we use constant boundary elements. We approximate the boundary by a polygonal line $\{L_j\}$, $j=\overline{1,N}$ with the nodes on the real boundary and we consider that the unknown is constant on each segment.

We consider that on each L_i the unknown is equal with the value taken in the midpoint of the segment, noted

$$\bar{x}_i^0 = \frac{\bar{x}^i + \bar{x}^{i+1}}{2},\ i \in \{1,2,...,N\},\ \bar{x}^{N+1} = \bar{x}^1\,. \tag{6}$$

In (5) we consider than $\bar{x}_0 = \bar{x}_i^0$ and we deduce the discrete form of the boundary equation:

$$\left({n_x^0}^2 + \beta^2 {n_y^0}^2\right)f(\bar{x}_0) + \frac{1}{\pi}\sum_{j=1}^{N} f_j \grave{\int_{L_j}} \frac{(x-x_0)n_x^0 + \beta^2(y-y_0)n_y^0}{|\bar{x}-\bar{x}_0|^2}ds = 2\beta n_x^0 \tag{7}$$

Imposing relation (7) to be satisfied on every midpoint, we get (see [5]) the following linear algebraic system which unknowns are the values of the sources intensity for the middle points of the segments:

$$a_i f_i + \sum_{j=1}^{N} A_{ij} f_j = A_i \ , \ i = \overline{1, N} \ , \tag{8}$$

where

$$\begin{cases} 2a_i = n_x^2\left(\bar{x}_i^0\right) + \beta^2 \bar{n}_y^2\left(\bar{x}_i^0\right) \\ A_{ij} = n_x\left(\bar{x}_i^0\right)U_{ij} + \beta^2 n_y\left(\bar{x}_i^0\right)V_{ij} \ , \\ A_i = \beta n_x\left(\bar{x}_i^0\right) \end{cases} \tag{9}$$

with the same notation as in [5].

All the coefficients in (8) can be analytically evaluated, so even if they represent some integrals these don't have to be numerically evaluated and no errors can appear at this step. Their expressions can also be found in [5], and they depend only on the coordinates of the nodes chosen for the boundary discretization.

3. The boundary integral equation - vortex distribution

In the second approach the boundary is approximated with a vortex distribution having the unknown intensity $g(\bar{x})$, a $H\ddot{o}lder$ function on C too. The components of the perturbation velocity for a regular point on the boundary are:

$$\begin{aligned} u(\bar{x}_0) &= \frac{1}{2} g(\bar{x}_0) n_y^0 + \frac{1}{2\pi} \oint_C^{`} g(\bar{x}) \frac{y - y_0}{\left|\bar{x} - \bar{x}_0\right|^2} ds \\ v(\bar{x}_0) &= -\frac{1}{2} g(\bar{x}_0) n_x^0 - \frac{1}{2\pi} \oint_C^{`} g(\bar{x}) \frac{x - x_0}{\left|\bar{x} - \bar{x}_0\right|^2} ds \end{aligned} \tag{10}$$

where, as in (4) the sign\` denotes the Cauchy principal value of the integral.

The singular boundary equation for this case, deduced in an analogous manner (see [3]) is:

$$-M^2 g(\bar{x}_0) n_x^0 n_y^0 + \frac{1}{\pi} \int_C^{`} g(\bar{x}) \frac{\beta^2 (x - x_0) n_y^0 - (y - y_0) n_x^0}{\left|\bar{x} - \bar{x}_0\right|^2} ds = 2\beta n_x^0 \tag{11}$$

with the same notations as before.

For solving this singular boundary integral equation we use linear isoparametric boundary elements too and we follow the same steps as in the case of the sources distribution.

We deduce the discrete form of equation (11), and the system the problem is reduced at in this case:

$$b_i g_i + \sum_{j=1}^{N} B_{ij} g_j = A_i \ , i = \overline{1, N} \ , \tag{12}$$

where

$$\begin{cases} 2b_i = -M^2 n_x\left(\bar{x}_i^0\right) n_y\left(\bar{x}_i^0\right) \\ B_{ij} = \beta^2 n_y\left(\bar{x}_i^0\right) U_{ij} - n_x\left(\bar{x}_i^0\right) V_{ij} \end{cases} \tag{13}$$

the notations are those used in [5], where the analytical expressions are given too.

It is important to specify that all the coefficients have analytical expressions and for this reason no error is introduced for their evaluations. All these coefficients depend only on the coordinates of the nodes used for the boundary discretization, and so for a great number of nodes, it can be use a computer code to solve the problem.

As we can see from the above paragraphs both methods used in this paper to solve the problem of the compressible fluid flow around a non-lifting body lead to linear systems of equations with coefficients depending on the coordinates of the nodes used for the boundary discretization. For the first case the unknowns are the nodal values of the sources intensities, noted $f_i, i = \overline{1, N}$, and for the second case the nodal values of the vortices intensities, noted $g_i, i = \overline{1, N}$.

After solving each system we can compute for each case the fluid velocity for different points of the fluid domain and the fluid action over the body, evaluating the local pressure coefficient, noted c_p, using the relation:

$$c_p = -u^2 - v^2 - 2u \, .$$

4. Numerical solutions for the elliptical obstacle

For solving the systems and for evaluating the fluid velocity there are developed two computer codes in C programming language, one for each method, and there are compared the numerical solutions obtained.

These numerical solutions are also compared with the exact solution that exists for the particular case of an elliptical obstacle and an incompressible fluid (M=0).

In [4] the bivariate problem of the incompressible fluid flow around an elliptical object is exactly solved. The expression of the perturbed fluid velocity is obtained using the complex potential, given by the following expression:

$$f(z)=\frac{U_\infty}{2}\left[\left(z+\sqrt{z^2-c^2}\right)e^{-i\alpha}+\left(z-\sqrt{z^2-c^2}\right)\frac{(a+b)e^{i\alpha}}{a-b}\right]$$

where $x=a\cos t,\ y=b\sin t$, $z=x+iy$, and $c^2=a^2-b^2$.

We have for this case:

$$u(z)=\frac{1}{U_\infty}\operatorname{Re}\left(\frac{df}{dz}\right), v(z)=-\frac{1}{U_\infty}\operatorname{Im}\left(\frac{df}{dz}\right)$$

We consider an elliptical profile with a=2 and b=1.

Another computer code in MATHCAD gives us the solution for this case. These computer codes can be run for any number of nodes used for the boundary discretization. We can study using them the errors that appear for each of the two types of methods presented above, and how big are these errors for different numbers of boundary elements, so these computer codes can be used to make a comparison study between the two methods presented in this paper.

For the case when we use 10, 20 and 30 nodes for the discretization the solutions obtained are represented in the following graphics.

In fig 1, 2, 3 there are represented the values obtained for the velocity component along the Ox axis for 10, 20 and 30 nodes and in each of these graphics there are performed three situations: the exact solution, the sources solution and the vortex solution.

In fig 4, 5, 6 there are represented the values obtained for the velocity component along the Oy axis for each of the three mentioned situations.

The pressure coefficient is represented in fig. 7, 8,9 for the same cases.

As we can see from these graphics the elliptical profile is a non-lifting profile because there is no difference between the values of the local pressure coefficient on the two sides of the profile, and as it is known this is a consequence of the fact that it has a smooth boundary.

As we can see from the graphics below both numerical solutions are in good agreement with the exact solution for the cases when more than 10 nodes are used for the boundary discretization, , but better solutions are obtained with the second method, so for the case when the obstacle's boundary is assimilated with a vortex distribution.

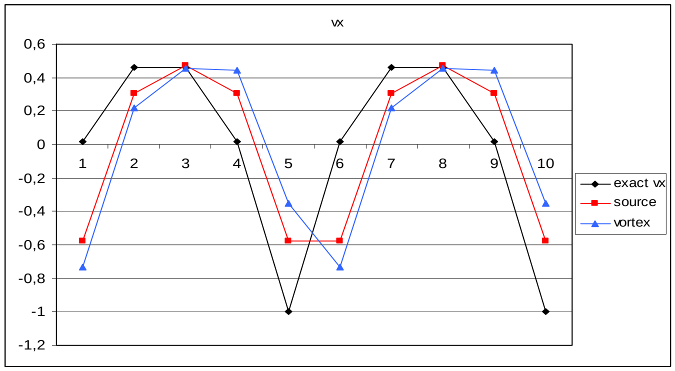

Fig. 1. The velocity along the Ox axis for 10 nodes

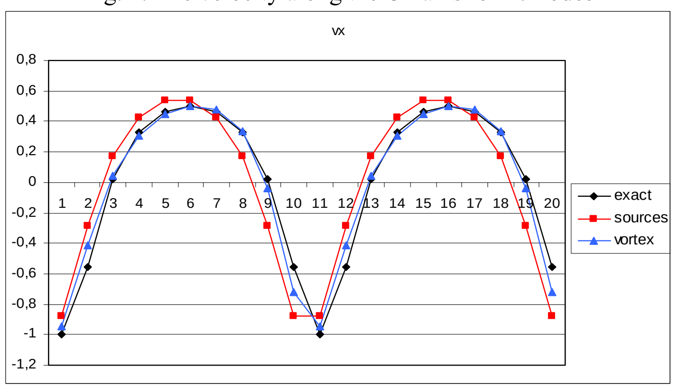

Fig. 2. The velocity along the Ox axis for 20nodes

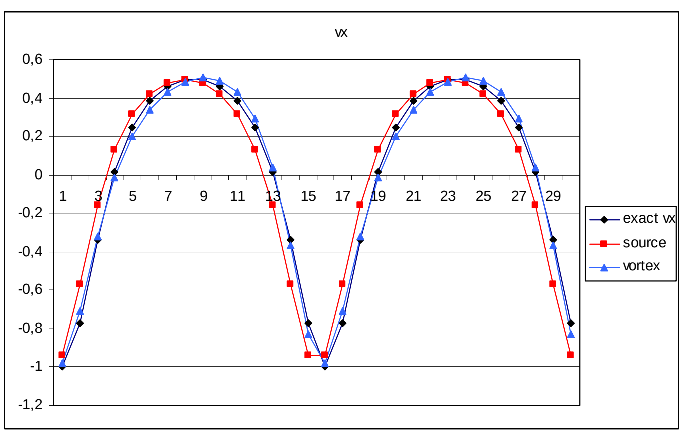

Fig. 3. The velocity along the Ox axis for 30 nodes

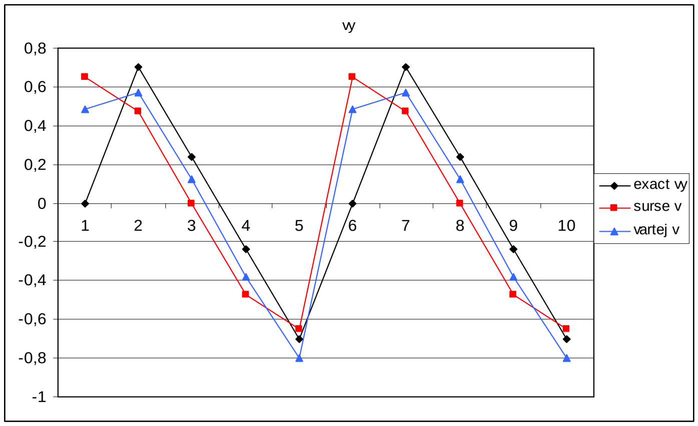

Fig. 4. The velocity along the Oy axis for 10 nodes

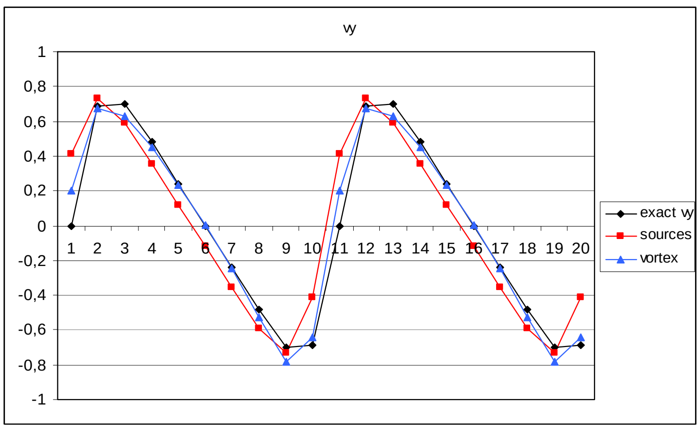

Fig. 5. The velocity along the Oy axis for 20 nodes

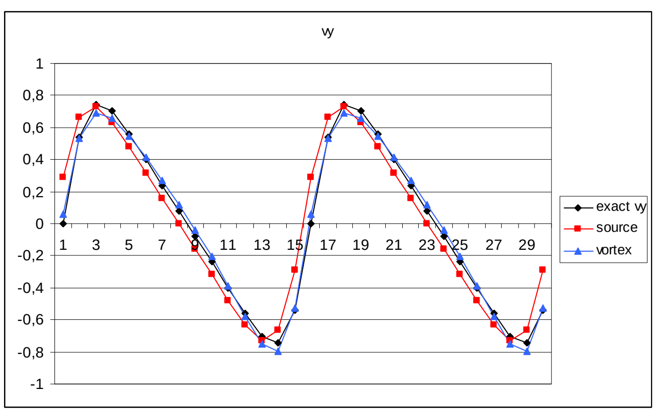

Fig. 6. The velocity along the Oy axis for 30 nodes

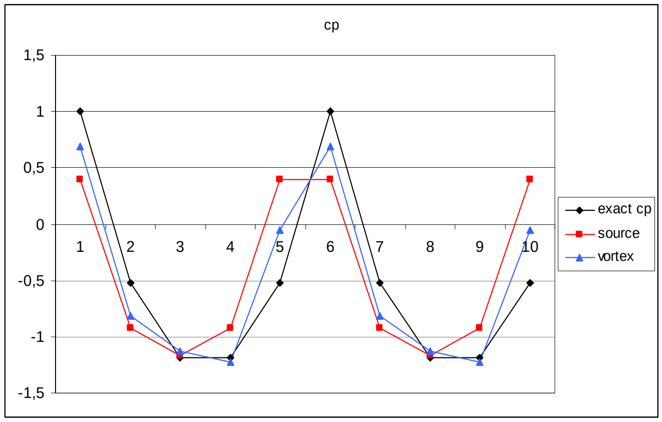

Fig. 7. The local pressure coefficient for 10 nodes

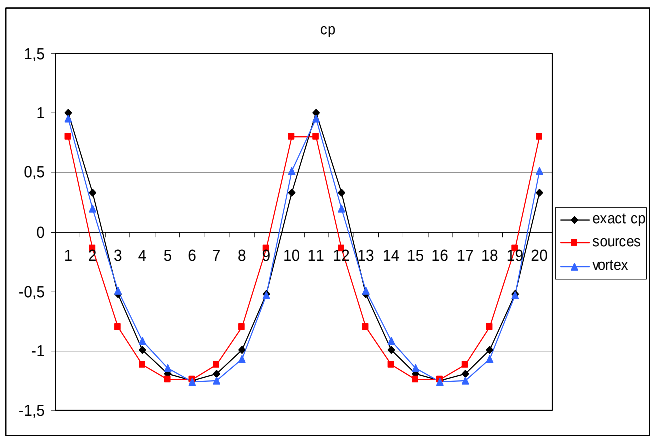

Fig. 8. The local pressure coefficient for 20 nodes

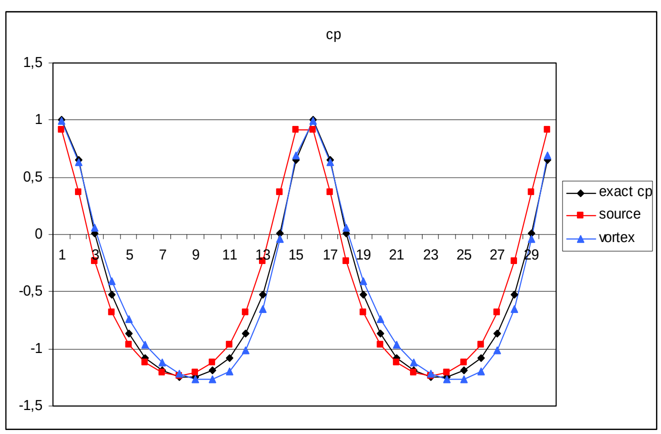

Fig. 9. The local pressure coefficient for 30 nodes

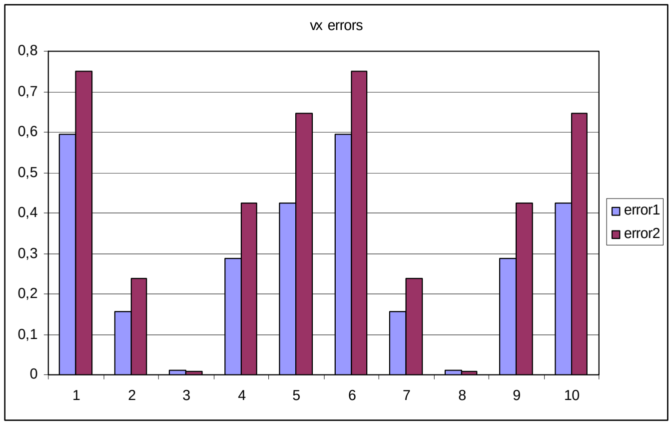

Fig.10. Errors for vx-case of 10 nodes

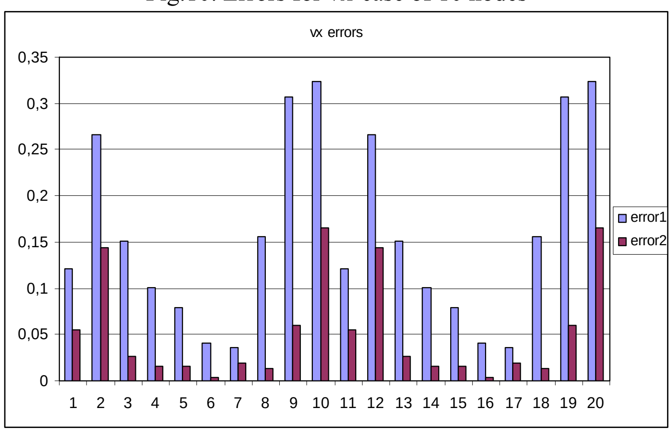

Fig.11. Errors for vx-case of 20 nodes

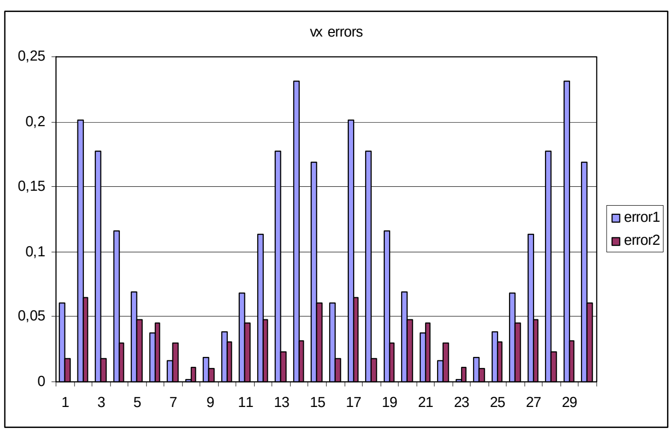

Fig.12. Errors for vx-case of 30 nodes

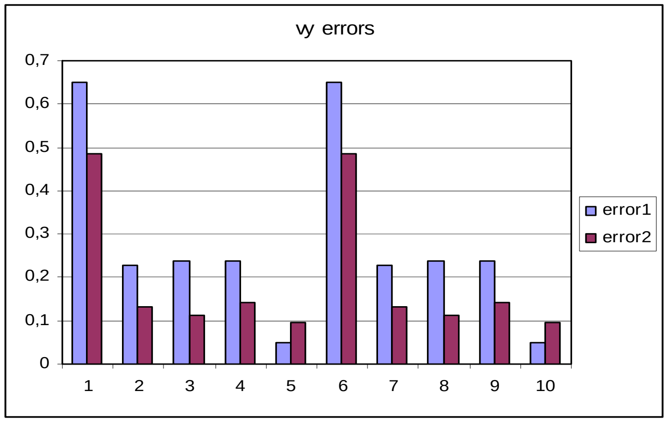

Fig.13. Errors for vy-case of 10 nodes

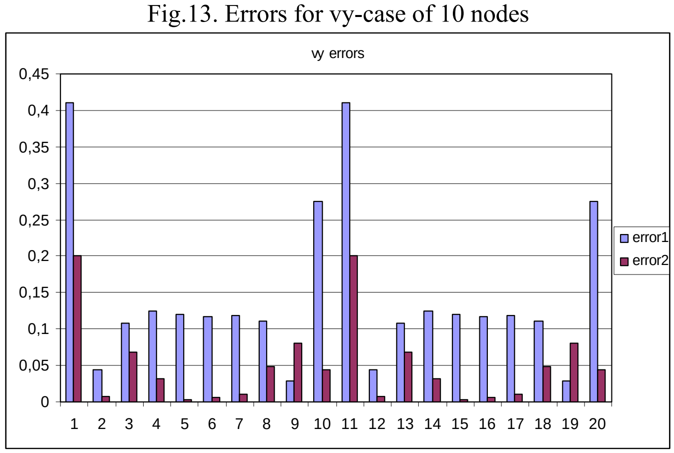

Fig.14. Errors for vy-case of 20 nodes

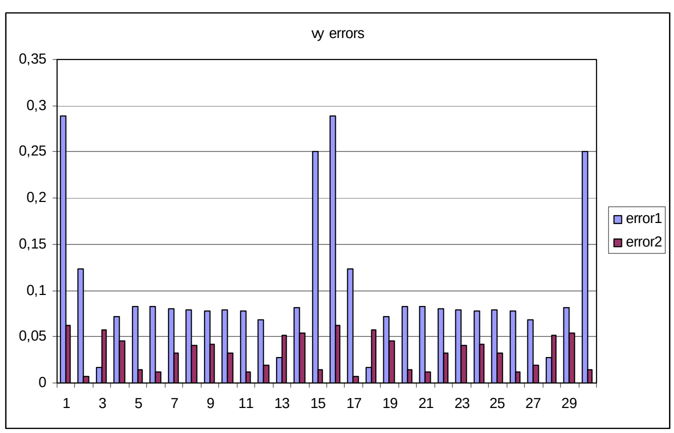

Fig.15. Errors for vy-case of 30 nodes

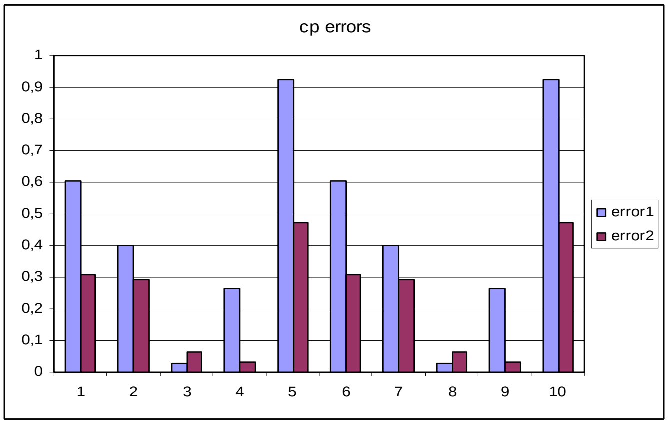

Fig.16. Errors for cp-case of 10 nodes

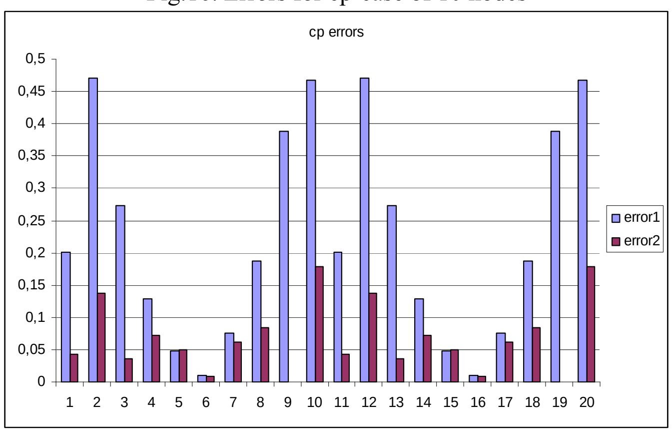

Fig.17. Errors for cp-case of 20 nodes

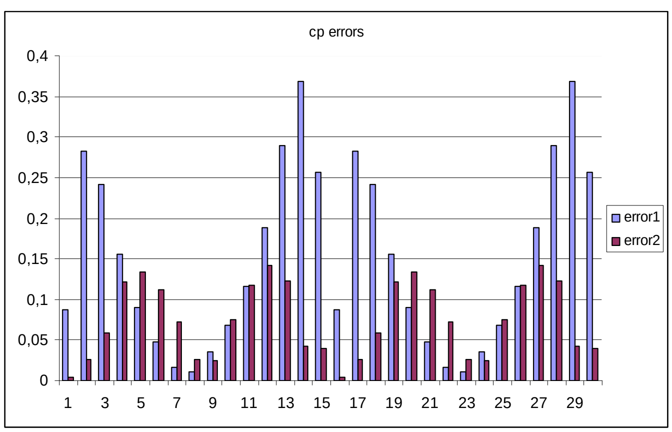

Fig.18. Errors for cp-case of 30 nodes

So both methods offer good numerical solutions but better results are obtained for the vortex distribution. This can be seen from the graphics where the errors are represented (Fig.10, 11, etc.) for 10, 20 and 30 nodes. Error1 is an absolute error for the numerical solution obtained for the case of sources distribution, and error2 for the case of vortex distribution. They are evaluated for the components of the velocity and also for the local pressure coefficient.

5. Conclusions

From the above graphics that show the comparison between the errors obtained when we use vortex and sources distributions for 10, 20 and 30 nodes we observe that the vortex distribution offers better results.

We notice that differences between the numerical solutions and the exact one appear in each situation, no matter if the components of the velocity or the local pressure coefficient are involved and haw many nodes are used for the boundary discretization.

As we can see the improvement achieve by using the vortex distribution is growing when more nodes are used for the boundary discretization. We can further study which is the critical number of nodes in case of this profile. This number of nodes represents the one after which even if we grow the number of the boundary elements the solution is not very much improved, and so the computational effort that appears is not justified.

With the same computer codes we can also study the influence of the compressibility on the motion and also how the ratio between the semi-axis of the ellipse influences the values of the velocities and of the local pressure coefficient.

Even these computer codes are realized for the case of an elliptical obstacle they can be easily modified for running for other smooth obstacles too.

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