MULTIPLE (n,m)-HYBRID LAPLACE TRANSFORM AND APPLICATIONS TO MULTIDIMENSIONAL HYBRID SYSTEMS. PART II: DETERMINING THE ORIGINAL

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Acest articol completează studiul transformării (n,m) *- Laplace hibride din [11] cu teoremele de integrare şi sumă a originalului, integrare a imaginii, convoluție, produs al originalelor, valoare inițială şi valoare finală.*

Sunt prezentate metode de determinare a originalului. Se obține o formulă de inversiune de tip Mellin-Fourier şi se dă o teoremă de dezvoltare generalizată.

This paper completes the study of the (n,m)*-Hybrid Laplace Transform from [11] with the theorems regarding integration and sum of the original, integration of the image, convolution, product of originals, initial and final values.*

Some methods and formulas for determining the original of a given (n,m) *- Hybrid Laplace Transform are provided. A generalized Mellin-Fourier type inversion formula is established and two other formulas are derived using respectively the Residue Theorem and multivariable Laurent series expansions. A generalized Expansion Theorem is also given.*

Keywords: Hybrid Laplace Transform, Mellin-Fourier type inversion formula, expansion theorem

1. Introduction

In Part I [11] the multiple hybrid Laplace-z transform was defined and its main properties were proved, including linearity, homothety, two time-delay theorems, translation, differentiation and difference of the original and differentiation of the image. Such a transformation is necessary for the study of the continuous-discrete multidimensional systems [4], [5], [6], [9], [10], which appear as models in many problems, for instance in the study of linear repetitive processes [2], [3], [12] or in the iterative learning control synthesis [7].

This paper completes the study of the (n,m)-Hybrid Laplace Transform from [11] with the theorems given in Section 2 regarding integration and sum of

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the original, integration of the image, convolution, product of originals, initial and final values.

Section 3 provides some methods and formulas for determining the original of a given (n,m)-hybrid Laplace transform. A generalized Mellin-Fourier type inversion formula is established and two other formulas are derived using respectively the Residue Theorem and multivariable Laurent series expansions. A generalized Expansion Theorem is also given.

2. Multiple (n,m)-hybrid Laplace transform

We denote by $\langle n\rangle$ the set $\{1,2,\ldots,n\}$.

Definition 2.1. A function $f:\mathbf{R}^n\times\mathbf{Z}^m\to\mathbf{C}$ is said to be a *continuous-discrete original function* (or simply an *original*) if f has the following properties:

(i) $f(t_1,\ldots,t_n;k_1,\ldots,k_m)=0$ if $t_i<0$ or $k_j<0$ for some $i\in\langle n\rangle$ or $j\in\langle m\rangle$.

(ii) $f(\cdot,\ldots,\cdot;k_1,\ldots,k_m)$ is piecewise smooth on $\mathbf{R}_+^n$ for any $(k_1,\ldots,k_m)\in\mathbf{Z}_+^m$.

(iii) $\exists\, M_j>0$, $\sigma_{fi}\ge 0$, $i\in\langle n\rangle$, $R_{fj}>0$, $j\in\langle m\rangle$ such that

$$|f(t_1,\ldots,t_n;k_1,\ldots,k_m)|\le M_f\exp\left(\sum_{i=1}^{n}\sigma_{fi}t_i\right)\prod_{j=1}^{m}R_{fj}^{k_j} \qquad (2.1)$$

$\forall\, t_i>0$, $i\in\langle n\rangle$, $\forall\, k_j\ge 0$, $j\in\langle m\rangle$.

Definition 2.2. For any original f, the function

$$F(s_1,\ldots,s_n;z_1,\ldots,z_m)=\int_0^\infty\cdots\int_0^\infty\sum_{k_1=0}^{\infty}\cdots\sum_{k_m=0}^{\infty}f(t_1,\ldots,t_n;k_1,\ldots,k_m)\cdot$$
$$\cdot e^{-s_1t_1}\cdots e^{-s_nt_n}z_1^{k_1}\cdots z_m^{-k_m}dt_1\ldots dt_n \qquad (2.2)$$

is called the (n,m)-*hybrid Laplace transform* ((n,m)-HLT) or the *image* of f.

Sometimes we shall use the notations $f(t;k)=$ $f(t_1,\ldots,t_n;k_1,\ldots,k_m)=0$ and $F(s;z)=F(s_1,\ldots,s_n;z_1,\ldots,z_m)$.

Definition 2.3. For $\beta=\{j_1,\ldots,j_q\}\subset\langle m\rangle$, the β-sum of the original function $f(t;k)$ is the function

$$S_{\beta}f(t;k)=\begin{cases}0 \quad \text{if} \quad \exists k_j \le 0,\, j\in\beta \quad \text{or} \quad \exists k_j<0,\, j\in\overline{\beta}\\ \sum_{l_{j_1}=0}^{k_{j_1}-1}\cdots\sum_{l_{j_q}=0}^{k_{j_q}-1} f(t;k_{\overline{\beta}},l_{\beta}) \quad \text{otherwise,}\end{cases} \tag{2.3}$$

where $f(t;k_{\overline{\beta}},l_{\beta})=f(t_1,\ldots,t_n;\widetilde{k}_1,\ldots,\widetilde{k}_m)$ with $\widetilde{k}_j=\begin{cases}l_j & \text{if} \quad j\in\beta\\ k_j & \text{if} \quad j\in\overline{\beta}.\end{cases}$

For $\alpha=\{i_1,\ldots,i_p\}\subset\langle n\rangle$ and $t=\{t_1,\ldots,t_n\}\in(\mathbf{R}^+)^n$ with $t_i>0$, $\forall i\in\alpha$ we denote by $D_{\alpha,t}$ the cartesian product $D_{\alpha,t}=\prod_{i\in\alpha}[0,t_i]$ and by $\int_{D_{\alpha,t}} fd\tau$ the multiple integral $\int_0^{t_{i_1}}\ldots\int_0^{t_{i_p}} f(\tau_1^*,\ldots,\tau_n^*;k_1,\ldots,k_m)d\tau_{i_1}\ldots d\tau_{i_p}$ where $\tau_i^*=\begin{cases}\tau_i & \text{if} \quad i\in\alpha\\ t_i & \text{if} \quad i\in\overline{\alpha}.\end{cases}$

Theorem 2.4 (Integration and sum of the original). *For any* $\alpha=\{i_1,\ldots,i_p\}\subset\langle n\rangle$ *and* $\beta=\{j_1,\ldots,j_q\}\subset\langle m\rangle$,

$$\mathcal{L}_{n,m}\left[\int_{D_{\alpha,t}} S_{\beta}f(\tau;k)d\tau\right]=\left(\prod_{i\in\alpha} s_i^{-1}\right)\left(\prod_{j\in\beta}(z_j-1)^{-1}\right)F(s;z). \tag{2.4}$$

Proof. Let us denote by $g(t;k)$ the function $g(t;k)=\int_{D_{\alpha,t}} S_{\beta}f(\tau;k)d\tau$. By Definition 2.3 $g(t;k)=0$ if $k_j=0$ for some $j\in\beta$ or $t_i=0$ for some $i\in\alpha$. The $(\beta,1)$-difference of $S_{\beta}f(\tau;k)$ (see [11, Definition 2.19]) is $\Delta_{\beta}S_{\beta}f(\tau;k)=f(\tau;k)$, hence $\Delta_{\beta}g(t;k)=\int_{D_{\alpha,t}} f(\tau;k)d\tau$. By deriving with respect $t_{i_1},\ldots,t_{i_p}$ we obtain $\frac{\partial^p}{\partial t_{i_1}\cdots\partial t_{i_p}}\Delta_{\beta}g(t;k)=f(t;k)$. By applying the operator $\mathcal{L}_{n,m}$ and by [11, Theorem 2.21] this equality becomes

$$\left(\prod_{i\in\alpha} s_i\right)\left(\prod_{j\in\beta}(z_j-1)\right)G(s;z)=F(s;z)$$

hence

$$\mathcal{L}_{n,m}[g(t;k)] = G(s;z) = \left(\prod_{i\in\alpha} s_i^{-1}\right)\left(\prod_{j\in\beta} (z_j-1)^{-1}\right)F(s;z).$$

□

Theorem 2.5 (Integration of the image). *If the following multiple improper integral converges, then*

$$\mathcal{L}_{n,m}\left[\frac{f(t_1,\ldots,t_n;k_1,\ldots,k_m)}{t_1\cdots t_n k_1\cdots k_m}\right] = \\ = \int_{s_1}^{\infty}\cdots\int_{s_n}^{\infty}\int_{z_1}^{\infty}\cdots\int_{z_m}^{\infty}\frac{F(\tau_1,\ldots,\tau_n;\zeta_1,\ldots,\zeta_m)}{\zeta_1\cdots\zeta_m}d\tau_1\ldots d\tau_n d\zeta_1\ldots d\zeta_m. \quad (2.5)$$

Proof. Let us denote by $G(s;z) = G(s_1,\ldots,s_n;z_1,\ldots,z_m)$ the function defined by the multiple improper integral in (2.5) and by $g(t;k)$ its original. By deriving it, one obtains

$$\frac{\partial^{n+m}G(s_1,\ldots,s_n;z_1,\ldots,z_m)}{\partial s_1\cdots\partial s_n\partial z_1\cdots\partial z_m} = (-1)^{n+m}\frac{F(s_1,\ldots,s_n;z_1,\ldots,z_m)}{z_1\cdots z_m}.$$

By applying [11, Th. 2.22] with $p=n$, $q=m$, $\gamma_{i_1}=\cdots=\gamma_{i_n}=1$, $b_{j_1}=\cdots=b_{j_m}=1$ we get

$$\mathcal{L}_{n,m}\left[(-1)^{n+m}\left(\prod_{i=1}^{n} t_i\right)\left(\prod_{j=1}^{m} k_j\right)g(t;k)\right] = \left(\prod_{j=1}^{m} z_j\right)\frac{\partial^{n+m}G(s;z)}{\partial s_1\cdots\partial s_n\partial z_1\cdots\partial z_m} = \\ = (-1)^{n+m}F(s;z) = \mathcal{L}_{n,m}[(-1)^{n+m}f(t;k)].$$

Therefore $g(t;k) = \dfrac{f(t;k)}{\left(\prod_{i=1}^{n} t_i\right)\left(\prod_{j=1}^{m} k_j\right)}$ and (2.5) results from $\mathcal{L}_{n,m}[g(t;k)] = G(s;z)$.

□

Theorem 2.6 (Convolution). *For any original functions* f *and* g,

$$\mathcal{L}_{n,m}[(f*g)(t;k)] = F(s;z)G(s;z). \quad (2.6)$$

Proof. Since the originals f and g are null if one of their arguments t_i or k_i is negative, then the definition of the (n,m)-hybrid convolution [11, Definition 2.3] can be written by replacing $\int_0^{t_i}$ and $\sum_{l_j=0}^{k_j}$ by $\int_0^{\infty}$ and $\sum_{l_j=0}^{\infty}$. Then

$$\mathcal{L}_{n,m}[(f*g)(t_1,\dots,t_n;k_1,\dots,k_m)]=$$
$$=\int_0^\infty\dots\int_0^\infty\sum_{k_1=0}^\infty\dots\sum_{k_m=0}^\infty(\int_0^\infty\dots\int_0^\infty\sum_{l_1=0}^\infty\dots\sum_{l_m=0}^\infty f(u_1,\dots,u_n;l_1,\dots,l_m)\cdot$$
$$\cdot g(t_1-u_1,\dots,t_n-u_n;k_1-l_1,\dots,k_m-l_m)du_1\dots du_n)\exp\left(-\prod_{i=1}^n s_i t_i\right)\left(\prod_{j=1}^m z_j^{-k_j}\right)dt_1\dots dt_n.$$

By changing the order of integration and of summation and then by the change of variables of integration $t_i-u_i=\tau_i$, $i\in\langle n\rangle$ and the change of the indices of summation $k_j-l_j=\lambda_j$, $j\in\langle m\rangle$, we get

$$\mathcal{L}_{n,m}[(f*g)(t;k)]=\int_0^\infty\dots\int_0^\infty\sum_{l_1=0}^\infty\dots\sum_{l_m=0}^\infty f(u_1,\dots,u_n;l_1,\dots,l_m)\cdot$$
$$\cdot(\int_{-u_1}^\infty\dots\int_{-u_n}^\infty\sum_{\lambda_1=-l_1}^\infty\dots\sum_{\lambda_m=-l_m}^\infty\exp\left(-\sum_{i=1}^n s_i(\tau_i+u_i)\right)\left(\prod_{j=1}^m z_j^{-(\lambda_j+l_j)}\right)\cdot$$
$$\cdot g(\tau_1,\dots,\tau_n;\lambda_1,\dots,\lambda_m)d\tau_1\dots d\tau_n),$$

formula which becomes, since f and g are 0 for negative arguments:

$$\mathcal{L}_{n,m}[(f*g)(t;k)]=\left(\int_0^\infty\dots\int_0^\infty\sum_{l_1=0}^\infty\dots\sum_{l_m=0}^\infty f(u_1,\dots,u_n;l_1,\dots,l_m)\cdot\right.$$
$$\left.\cdot\exp\left(-\sum_{i=1}^n s_i u_i\right)\left(\prod_{j=1}^m z_j^{-l_j}\right)du_1\dots du_n\right)\left(\int_0^\infty\dots\int_0^\infty\sum_{\lambda_1=0}^\infty\dots\sum_{\lambda_m=0}^\infty g(\tau_1,\dots,\tau_n;\lambda_1,\dots,\lambda_m)\cdot\right.$$
$$\left.\cdot\exp\left(-\sum_{i=1}^n s_i\tau_i\right)\left(\prod_{j=1}^m z_j^{-\lambda_j}\right)d\tau_1\dots d\tau_n\right)=\mathcal{L}_{n,m}[f(t;k)]\mathcal{L}_{n,m}[g(t;k)].$$

□

Theorem 2.7 (Product of originals). *For any original functions f , g and* $\sigma_{fi}<a_i<\operatorname{Re}s_i-\sigma_{gi}$, $i\in\langle n\rangle$, $R_{fj}<r_j<\dfrac{|z_j|}{R_{gj}}$ *we have*

$$\mathcal{L}_{n,m}[f(t;k)g(t;k)]=\frac{1}{(2\pi i)^{n+m}}\int_{a_1-i\infty}^{a_1+i\infty}\dots\int_{a_n-i\infty}^{a_n+i\infty}$$
$$\left(\oint_{|z_1|=r_1}\dots\oint_{|z_m|=r_m}F(q;\zeta)G\left(s-q;\frac{z}{\zeta}\right)\frac{d\zeta}{\zeta}\right)dq \qquad (2.7)$$

where we used the following notations: $q=(q_1,\dots,q_n)$, $\zeta=(\zeta_1,\dots,\zeta_m)$,

$$\frac{z}{\zeta}=\prod_{j=1}^{m}\left(\frac{z_j}{\zeta_j}\right),\ \frac{d\zeta}{\zeta}=\frac{d\zeta_1\ldots d\zeta_m}{\zeta_1\cdots\zeta_m},\ dq=dq_1\ldots dq_n.$$

Proof. Let us denote $F_1(s;k)=F_1(s_1,\ldots,s_n;k_1,\ldots,k_m)$ the multiple Laplace transform (see [11, Definition 2.14])

$$F_1(s;k)=\mathcal{L}_n[f(t;k)]=\mathcal{L}_{n,m}^{\langle n\rangle,\phi}[f(t;k)]=$$
$$=\int_0^{\infty}\ldots\int_0^{\infty}f(t_1,\ldots,t_n;k_1,\ldots,k_m)\exp\left(-\sum s_it_i\right)dt_1\ldots dt_n.$$

Then (2.2) can be rewritten as

$$F(s;z)=\sum_{k_1=0}^{\infty}\ldots\sum_{k_m=0}^{\infty}F_1(s;k)\prod_{j=1}^{m}z_j^{-k_j} \tag{2.8}$$

By [11, Theorem 2.16] and (2.8) one obtains

$$\mathcal{L}_{n,m}\left[\exp\left(\sum_{i=1}^{n}\ q_it_i\right)\left(\sum_{j=1}^{m}\ \zeta_j^{l_j}\right)g(t_1,\ldots,t_n;l_1,\ldots,l_m)\right]=$$
$$=G(s_1-q_1,\ldots,s_n-q_n;z_1\zeta_1^{-1},\ldots,z_m\zeta_m^{-1})= \tag{2.9}$$
$$=\sum_{l_1=0}^{\infty}\ldots\sum_{l_m=0}^{\infty}G_1(s_1-q_1,\ldots,s_n-q_n;l_1,\ldots,l_m)\prod_{j=1}^{m}(z_j\zeta_j^{-1})^{l_j}$$

where $G(s;z)=\mathcal{L}_{n,m}[g(t;k)]$ and $G_1(s;l)=\mathcal{L}_n[g(t;l)]$.

By Mellin-Fourier formula [1, §2 (25)], [8, Ch. III, §7.2 (10)] and its generalization for multiple Laplace transform, we get for $a_i>\sigma_{fi}$ and $b_i>\sigma_{fi}+\sigma_{gi}$, $i\in\langle n\rangle$

$$f(t;k)=\frac{1}{(2\pi i)^n}\int_{a_1-i\infty}^{a_1+i\infty}\ldots\int_{a_n-i\infty}^{a_n+i\infty}F_1(q;k)\exp\left(\sum_{i=1}^{n}q_it_i\right)dq_1\ldots dq_n \tag{2.10}$$

and

$$\exp\left(\sum_{i=1}^{n}q_it_i\right)g(t;k)=\frac{1}{(2\pi i)^n}\int_{b_1-i\infty}^{b_1+i\infty}\ldots\int_{b_n-i\infty}^{b_n+i\infty}G_1(s-q;k)\exp\left(\sum_{i=1}^{n}s_it_i\right)ds_1\ldots ds_n.$$

We shall use the notations $\int_{a-i\infty}^{a+i\infty}$ and $\oint_{|\zeta|=r}$ for the multiple integrals $\int_{a_1-i\infty}^{a_1+i\infty}\ldots\int_{a_n-i\infty}^{a_n+i\infty}$ and $\oint_{|\zeta_1|=r_1}\ldots\oint_{|\zeta_m|=r_m}$ respectively and $\exp\left(\sum_{i=1}^{n}\ s_it_i\right)=e^{st}$, $z_1^{k_1}\cdots z_m^{k_m}=z^k$. Then, the multiplication of (2.10) by $g(t;k)$ and the generalized Mellin-Fourier formula give

$$f(t;k)g(t;k) = \frac{1}{(2\pi i)^n}\int_{a-i\infty}^{a+i\infty} F_1(q;k)e^{qt}g(t,k)dq =$$
$$= \frac{1}{(2\pi i)^n}\int_{a-i\infty}^{a+i\infty} F_1(q;k)\left(\frac{1}{(2\pi i)^n}\int_{b-i\infty}^{b+i\infty} G_1(s-q;k)e^{st}ds\right)dq =$$
$$= \frac{1}{(2\pi i)^n}\int_{b-i\infty}^{b+i\infty} e^{st}\left(\frac{1}{(2\pi i)^n}\int_{a-i\infty}^{a+i\infty} F_1(q;k)G_1(s-q;k)dq\right)ds.$$

Again, by the generalized Mellin-Fourier formula we obtain

$$\mathcal{L}_n[f(t,k)g(t,k)] = \frac{1}{(2\pi i)^n}\int_{a-i\infty}^{a+i\infty} F_1(q;k)G_1(s-q;k)dq \tag{2.11}$$

By employing (2.8) and (2.10) one obtains

$$\frac{1}{(2\pi i)^m}\oint_{|z|=r} F(q;\zeta)G\left(s-q;\frac{z}{\zeta}\right)\frac{d\zeta}{\zeta} =$$
$$= \frac{1}{(2\pi i)^m}\oint_{|z|=r}\left(\sum_{k=0}^{\infty} F_1(q;k)\zeta^{-k}\right)\left(\sum_{l=0}^{\infty} G_1(s-q;l)\left(\frac{z}{\zeta}\right)^{-l}\frac{d\zeta}{\zeta}\right) =$$
$$= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} F_1(q;k)G_1(s-q;l)z^{-l}\frac{1}{(2\pi i)^m}\oint_{|z|=r}\zeta^{-k+l-1}d\zeta = \sum_{k=0}^{\infty} F_1(q;k)G_1(s-q;k)z^{-k}$$

since $\oint_{|\zeta_j|=r_j}\zeta_j^{-k_j+l_j-1}d\zeta_j$ equals 0 if $-k_j+l_j-1\neq 0$, i.e. if $l_j\neq k_j$ and it equals $2\pi i$ if $l_j=k_j$, $\forall j\in\langle m\rangle$. Then (2.10) implies

$$\mathcal{L}_{n,m}[f(t;k)g(t;k)] = \sum_{k=0}^{\infty}\mathcal{L}_n[f(t;k)g(t;k)]z^{-k} =$$
$$= \frac{1}{(2\pi i)^n}\int_{a-i\infty}^{a+i\infty}\left(\sum_{k=0}^{\infty} F_1(q;k)G_1(s-q;k)z^{-k}\right)dq =$$
$$= \frac{1}{(2\pi i)^{n+m}}\int_{a-i\infty}^{a+i\infty}\left(\oint_{|\zeta|=r} F(q;\zeta)G\left(s-q;\frac{z}{\zeta}\right)\frac{d\zeta}{\zeta}\right)dq.$$

□

We shall use the notation $\lim\limits_{s\to\infty} F(s;z)$ for

$$\lim_{s_1\to\infty}\ldots\lim_{s_n\to\infty} F(s_1,\ldots,s_n;z_1,\ldots,z_m)$$

and a similar notation for $z\to\infty$.

Let E' be the family of the unvoid subsets $\varepsilon = \{\varepsilon_1, \ldots, \varepsilon_l\}$ of $\langle n \rangle$, $\varepsilon \neq \langle n \rangle$.

Theorem 2.8 (Initial value). *For* $s_i \in \mathbf{C}$ *with* $\operatorname{Arg} s_i \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$, $\forall i \in \langle n \rangle$

$$f(0+,\ldots,0+;0,\ldots,0) = \lim_{s\to\infty} \lim_{z\to\infty} \Big((-1)^n [s_1 \cdots s_n] F(s;z) + \\ + \sum_{\varepsilon \in E'} (-1)^{|\varepsilon|} \Big(\prod_{j \in \bar{\varepsilon}} s_j \Big) \mathcal{L}_{n,m}^{\bar{\varepsilon},\langle m \rangle} \left[\frac{\partial^{|\varepsilon|}}{\partial t_{\varepsilon_1} \cdots \partial t_{\varepsilon_l}} f(0_\varepsilon^+;0) \right] \Big) \tag{2.12}$$

Proof. By [11, Definition 2.14 and Theorem 2.17]

$$\mathcal{L}_{n,m} \left[\frac{\partial^n}{\partial t_1 \cdots \partial t_n} f(t;k) \right] = s_1 \cdots s_n F(s;z) + \\ + \sum_{\varepsilon \in E'} (-1)^{|\varepsilon|} \Big(\prod_{j \in \bar{\varepsilon}} s_j \Big) \mathcal{L}_{n,m}^{\bar{\varepsilon},\langle m \rangle} \left[\frac{\partial^{|\varepsilon|}}{\partial t_{\varepsilon_1} \cdots \partial t_{\varepsilon_l}} f(0_\varepsilon^+;k) \right] + (-1)^n \mathcal{Z}_m f(0^+;k) \tag{2.13}$$

By [11, Remark 2.9] the left hand member of (2.13) tends to 0 as $s_1 \to \infty, \ldots, s_n \to \infty$. We can write

$$\mathcal{Z}_m[f(0+;k)] = f(0+;0) + \sum_{j=1}^{m} z_j^{-1} H_j(z)$$

where $H_j(z) = H_j(z_1, \ldots, z_m)$ are analytic functions on the domain $|z_j| > R_{fj}$, $j \in \langle m \rangle$; therefore $\lim_{z\to\infty} \mathcal{Z}_m[f(0+;k)] = f(0+;0)$. Similarly,

$$\lim_{z\to\infty} \mathcal{L}_{n,m}^{\bar{\varepsilon},\langle m \rangle} \left[\frac{\partial^{|\varepsilon|}}{\partial t_{\varepsilon_1} \cdots \partial t_{\varepsilon_l}} f(0_\varepsilon^+;k) \right] = \mathcal{L}_{n,m}^{\bar{\varepsilon},\langle m \rangle} \left[\frac{\partial^{|\varepsilon|}}{\partial t_{\varepsilon_1} \cdots \partial t_{\varepsilon_l}} f(0_\varepsilon^+;0) \right]$$

and (2.12) results by taking the limit in (2.13) as $s_i \to \infty$, $\forall i \in \langle n \rangle$ and $z_j \to \infty$, $\forall j \in \langle m \rangle$.

□

Theorem 2.9 (Final value). *If the following limits exist, then*

$$\lim_{t_1\to\infty} \ldots \lim_{t_n\to\infty} \lim_{k_1\to\infty} \ldots \lim_{k_m\to\infty} f(t;k) = \lim_{s_1\to 0} \ldots \lim_{s_n\to 0} \lim_{z_1\to 1+} \ldots \lim_{z_m\to 1+} s_1 \cdots s_n (z_1 - 1) \cdots (z_m - 1) F(s;z) \tag{2.14}$$

.

Proof. By [11, Theorem 2.18, formula (2.15) with $\gamma = \beta = 1$] we get

$$\mathcal{L}_{1,1} \left[\frac{\partial}{\partial t} \Delta f(t;k) \right] = s(z-1)F(s,z) - sz\mathcal{L}[f(t,0)] - \\ - (z-1)\mathcal{Z}[f(0+;k)] + zf(0+,0) \tag{2.15}$$

where $\mathcal{L}$ and $\mathcal{Z}$ are the usual (1-dimensional) Laplace and z transformations. For $s \to 0$, $z \to 1+$, the left hand member of (2.15) can be written as a series whose sum can be expressed as the limit of the sequence of its partial sums; by reducing some terms we obtain

$$\lim_{s\to 0}\lim_{z\to 1+}\mathcal{L}_{1,1}\left[\frac{\partial}{\partial t}\Delta(t;k)\right]==\lim_{s\to 0}\lim_{z\to 1+}\int_0^{\infty}\sum_{k=0}^{\infty}\left(\frac{\partial f}{\partial t}(t,k+1)-\frac{\partial f}{\partial t}(t;k)\right)e^{-st}z^{-k}dt=$$

$$=\sum_{k=0}^{\infty}\int_0^{\infty}\left(\frac{\partial f}{\partial t}(t,k+1)-\frac{\partial f}{\partial t}(t;k)\right)dt=\lim_{k\to\infty}[(f(t,1)-f(t,0))\big|_0^{\infty}+$$

$$+[(f(t,2)-f(t,1))\big|_0^{\infty}+\cdots+(f(t,k-1)-f(t,k-2))\big|_0^{\infty}+$$

$$+(f(t,k)-f(t,k-1))\big|_0^{\infty}]=\lim_{k\to\infty}\left(f(t,k)\big|_0^{\infty}-f(t,0)\big|_0^{\infty}\right)=$$

$$=\lim_{k\to\infty}\lim_{t\to\infty}f(t,k)-\lim_{k\to\infty}f(0+,k)-\lim_{t\to\infty}f(t,0)+f(0+,0).$$

By the final value theorems for 1D Laplace and z-transformations we have $\lim_{t\to\infty}f(t;0)=\lim_{s\to\infty}s\mathcal{L}[f(t;0)]$ and $\lim_{z\to 1+}(z-1)\mathcal{Z}[0+;k]=\lim_{k\to\infty}f(0+;k)$.

By taking the limit as $s\to\infty$, $z\to 1+$ in (2.15) and by reducing these terms, one obtains (2.14).

□

3. Methods for determining the original

We consider the following problem: given a function $F(s;z)=F(s_1,\ldots,s_n;z_1,\ldots,z_m)$ which is analytic on a domain $D(f)$ as in [15, (2.4)], determine an original function $f(t;k)$ such that $\mathcal{L}_{n,m}[f(t;k)]=F(s;z)$. Firstly we shall establish an inversion formula for the (n,m)-hybrid Laplace transform.

Theorem 3.1. *If $f(t;k)$ is the original of $F(s;z)$, then*

$$f(t_1,\ldots,t_n;k_1,\ldots,k_m)=\frac{1}{(2\pi i)^{n+m}}\int_{a_1-i\infty}^{a_1+i\infty}\cdots\int_{a_n-i\infty}^{a_n+i\infty}\Big(\oint_{|z_1|=r_1}\cdots\oint_{|z_m|=r_m}$$
$$\cdot f(s_1,\ldots,s_n;z_1,\ldots,z_m)z_1^{k_1-1}\cdots z_m^{k_m-1}dz_1\ldots dz_n\Big)\exp\left(\sum_{i=1}^{n}s_it_i\right)ds_1\ldots ds_n \tag{3.1}$$

where $a_i>\sigma_{gi}$, $\forall i\in\langle n\rangle$ *and* $r_j>R_{fj}$, $\forall j\in\langle m\rangle$.

Proof. We shall define by induction the following 1D Laplace transforms and Z-transforms :

$$\widetilde{F}(t_2,\ldots,t_n;k_1,\ldots,k_m;s_1)=\mathcal{L}[f(t_1,t_2,\ldots,t_n;k_1,\ldots,k_m)](s_1)= \\ =\int_0^{\infty} f(t_1,t_2,\ldots,t_n;k_1,\ldots,k_m)\exp(-s_1t_1)dt_1 \tag{3.2}$$

$$\widetilde{F}(t_{i+1},\ldots,t_n;k_1,\ldots,k_m;s_1,\ldots,s_i)= \\ =\mathcal{L}[\widetilde{F}(t_i,t_{i+1},\ldots,t_n;k_1,\ldots,k_m;s_1,\ldots,s_{i-1})](s_i)= \\ =\int_0^{\infty}\widetilde{F}(t_i,t_{i+1},\ldots,t_n;k_1,\ldots,k_m;s_1,\ldots,s_{i-1})\exp(-s_it_i)dt_i,\ i\in\langle n\rangle \tag{3.3}$$

$$\hat{F}(k_2,\ldots,k_m;s_1,\ldots,s_n;z_1)=\mathcal{Z}[\widetilde{F}(k_1,\ldots,k_m;s_1,\ldots,s_n)](z_1)= \\ =\sum_{k_1=0}^{\infty}\widetilde{F}(k_1,\ldots,k_m;s_1,\ldots,s_n)z_1^{-k_1} \tag{3.4}$$

$$\hat{F}(k_{j+1},\ldots,k_m;s_1,\ldots,s_n;z_1,\ldots,z_j)= \\ =\mathcal{Z}[\hat{F}(k_j,k_{j+1},\ldots,k_m;s_1,\ldots,s_n;z_1,\ldots,z_{j-1})](z_j)= \\ =\sum_{k_j=0}^{\infty}\hat{F}(k_j,k_{j+1},\ldots,k_m;s_1,\ldots,s_n;z_1,\ldots,z_{j-1})z_j^{-k_j},\ i\in\langle m\rangle. \tag{3.5}$$

Obviously

$$F(s_1,\ldots,s_n;z_1,\ldots,z_m)=\mathcal{Z}[\hat{F}(k_m;s_1,\ldots,s_n;z_1,\ldots,z_{m-1})](z_m)= \\ =\sum_{k_m=0}^{\infty}\hat{F}(k_m;s_1,\ldots,s_n;z_1,\ldots,z_{m-1})z_1^{-k_m}. \tag{3.6}$$

By using the formula $c_n=\frac{1}{2\pi i}\oint_{\Gamma}\frac{f(z)}{(z-a)^{n+1}}dz$ for the coefficients of a Laurent series $f(z)=\sum_{n=-\infty}^{\infty}c_n(z-a)^n$, by the Mellin-Fourier formula applied to the Laplace transforms (3.3) and (3.2) and by taking for the closed contour Γ the circles $|z_j|=r_j$, we obtain from the Laurent series (3.6) and (3.5)

$$f(t_1,\ldots,t_n;k_1,\ldots,k_m)=\frac{1}{(2\pi i)^n}\int_{a_1-i\infty}^{a_1+i\infty}\ldots\int_{a_n-i\infty}^{a_n+i\infty}\widetilde{F}(k_1,\ldots,k_m;s_1,\ldots,s_n)\cdot \\ \cdot\exp\left(\sum_{i=1}^{n}s_it_i\right)ds_1\ldots ds_n. \tag{3.7}$$

By replacing in (3.7) the recurrent $\mathcal{Z}$ -transforms one obtains the inversion formula (3.1).

□

Remark 3.2. Under certain conditions imposed by Jordan's Lemma, the multiple complex integrals in (3.1) can be calculated using Residue Theorem.

Theorem 3.3. *The original of* $F(s_1,\ldots,s_n;z_1,\ldots,z_n)$ *is given by*

$$f(t_1,\ldots,t_n;k_1,\ldots,k_m)= \\ =\frac{1}{(2\pi i)^n k_1!\cdots k_m!}\int_{a_1-i\infty}^{a_1+i\infty}\cdots\int_{a_n-i\infty}^{a_n+i\infty}\frac{\partial^{k_1+\cdots+k_m}}{\partial z_1^{k_1}\cdots\partial z_m^{k_m}}\cdot \\ \cdot F(s_1,\ldots,s_n;z_1^{-1},\ldots,z_m^{-1})|_{z=0}\exp\left(\sum_{i=1}^{n}s_i t_i\right)ds_1\ldots ds_n \tag{3.8}$$

where $z=0$ *indicates that the integrand is calculated at* $z_1=0,\ldots,z_m=0$.

Proof. If z_m is replaced by z_m^{-1}, (3.6) becomes

$$F(s_1,\ldots,s_n;z_1,\ldots,z_{m-1},z_m^{-1})= \\ =\sum_{k_m=0}^{\infty}\hat{F}(k_m;s_1,\ldots,s_n;z_1,\ldots,z_{m-1})z_1^{k_m},$$

hence $\hat{F}$ can be considered as the set of the coefficients of the Taylor series expansion of $F(s;z)$ about $z_m=0$, and the usual formula for the Taylor series coefficients of function f, namely $c_n=\dfrac{f^{(n)}(a)}{n!}$ gives

$$\hat{F}(k_n;s_1,\ldots,s_n;z_1,\ldots,z_{m-1})=\frac{1}{k_m!}\frac{\partial^{k_m}}{\partial z_m^{k_m}}F(s;z_1,\ldots,z_{m-1},z_m^{-1})|_{z_m=0}.$$

Similar formulas can be obtained by replacing z_j by z_j^{-1}, $j\in\langle m-1\rangle$ in formulas (3.4) and (3.5); finally we get

$$\widetilde{F}(k_1,\ldots,k_m;s_1,\ldots,s_n)= \\ =\frac{1}{k_1!\cdots k_m!}\frac{\partial^{k_1+\cdots+k_m}}{\partial z_1^{k_1}\cdots\partial z_m^{k_m}}F(s_1,\ldots,s_n;z_1^{-1},\ldots,z_m^{-1})|_{z=0} \tag{3.9}$$

and (3.2) can be obtained by replacing $\widetilde{F}$ given by (3.9) in (3.7).

□

Now let us assume that the transform F is separable with respect to $s_1,\dots,s_n$, that is $F(s_1,\dots,s_n;z_1,\dots,z_n)=\prod_{i=1}^{n} F_i(s_i;z_1,\dots,z_m)$, where each F_i fulfil the conditions of Jordan's Lemma on a suitable domain and F_i has the singular points $s_{i,l}$, $l\in\langle n_i\rangle$, $n_i\in\mathbf{N}^*$ such that $\operatorname{Re}s_{i,l}<a_i$, $i\in\langle n\rangle$. By applying the Residue Theorem to the integrals in (3.8) we get:

Corollary 3.4. If $F(s_1,\dots,s_n;z_1,\dots,z_n)$ is separable with respect to $s_1,\dots,s_n$, then its original is

$$f(t_1,\dots,t_n;k_1,\dots,k_m)=\frac{1}{k_1!\cdots k_m!}\sum_{i=1}^{n}\sum_{l=1}^{n_i}\cdot$$

$$\cdot\operatorname{res}\left(\frac{\partial^{k_1+\cdots+k_m}}{\partial z_1^{k_1}\cdots\partial z_m^{k_m}}F(s_1,\dots,s_n;z_1^{-1},\dots,z_m^{-1})\exp\left(\sum_{i=1}^{n}s_it_i\right)\Bigg|_{z=0},s_{i,l}\right).$$

Let us denote by δ_j, $j\in\langle m\rangle$ the discrete impulse function $\delta_j(k_j)=\begin{cases}1 & \text{if } k_j=0\\ 0 & \text{if } k_j\neq 0\end{cases}$ and by $\delta(k)$, $k=(k_1,\dots,k_m)\in\mathbf{Z}^m$ the function $\delta(k)=\prod_{j=1}^{m}\delta_j(k_j)$.

Another method for determining the original is given by the following expansion theorem:

Theorem 3.5. *If the image* $F(s;z)$ *has the Laurent series expansion about infinity*

$$F(s;z)=\sum_{\alpha\ge 1}\sum_{\beta\ge 0}a_{\alpha\beta}s^{-\alpha}z^{-\beta}$$

then its original has the Taylor series expansion

$$f(t;k)=\sum_{\alpha\ge 1}\sum_{\beta\ge 0}\frac{a_{\alpha\beta}}{(\alpha_1-1)!\cdots(\alpha_n-1)!}t^{\alpha-1}\delta(k-\beta)$$

where $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbf{N}^n$, $\beta=(\beta_1,\dots,\beta_m)\in\mathbf{N}^m$, $a_{\alpha\beta}=a_{\alpha_1,\dots,\alpha_n;\beta_1,\dots,\beta_m}\in\mathbf{C}$, $s^{-\alpha}=s_1^{-\alpha_1}\cdots s_n^{-\alpha_n}$, $z^{-\beta}=z_1^{-\beta_1}\cdots z_m^{-\beta_m}$, $t^{\alpha-1}=t_1^{\alpha_1-1}\cdots t_n^{\alpha_n-1}$ *;* $\alpha\ge 1$ *means* $\alpha_i\ge 1$, $\forall i\in\langle n\rangle$ *and* $\beta\ge 0$ *means* $\beta_j\ge 0$, $\forall j\in\langle m\rangle$.

Proof. By [11, Definitions 2.6 and 2.10]

$$\mathcal{L}_{n,m}[t^{\alpha-1}\delta(k-\beta)] = \left((\int_0^\infty t_1^{\alpha_1-1} e^{-\alpha_1 t_1} dt_1) \cdots (\int_0^\infty t_n^{\alpha_n-1} e^{-\alpha_n t_n} dt_n) \right) \cdot$$
$$\cdot \left((\sum_{k_1=0}^{\infty} \delta_1(k_1-\beta_1) z_1^{-k_1}) \cdots (\sum_{k_m=0}^{\infty} \delta_m(k_m-\beta_m) z_m^{-k_m}) \right) =$$
$$= ((\alpha_1-1)! s_1^{-\alpha_1}) \cdots ((\alpha_n-1)! s_n^{-\alpha_n}) z_1^{-\beta_1} \cdots z_m^{-\beta_m}.$$

By linearity, one obtains

$$\mathcal{L}_{n,m}[f(t;k)] = \sum_{\alpha \geq \mathbf{1}} \sum_{\beta \geq \mathbf{0}} \frac{a_{\alpha\beta}}{(\alpha_1-1)! \cdots (\alpha_n-1)!} \mathcal{L}_{n,m}[t^{\alpha-1}\delta(k-\beta)] =$$
$$= \sum_{\alpha \geq \mathbf{1}} \sum_{\beta \geq \mathbf{0}} a_{\alpha\beta} s^{-\alpha} z^{-\beta} = F(s;z).$$

□

4. Conclusion

In this paper and in [11] a complete theory of a multiple (n,m)-Hybrid Laplace transformation has been developed. In a subsequent paper its applications will be provided, including solutions of differential-difference and integral equations, as well as the frequency-domain representation of multidimensional hybrid control systems.

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