A NOTE ON THE CAUCHY PROBLEM FOR A HIGHER-ORDER μ-CAMASSA-HOLM EQUATION

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In this note, we consider the Cauchy problem for a higher-order μ-Camassa-Holm equation. By constructing two sequences of peakon solutions whose distance initially goes to zero but later becomes large, we prove that the Cauchy problem is not locally well-posed in the Sobolev space $H^s(S^1)$ for any $s < \frac{7}{2}$ in the sense that its solutions do not depend uniformly continuously on the initial data.

Keywords: Peakon solutions; higher-order μ-Camassa-Holm equation; locally well-posed; non-uniform dependence on initial data

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1. Introduction

In this article we focus on the following Cauchy problem for a higher-order μ-Camassa-Holm equation [1]

$$\begin{cases} m_t + 2mu_x + m_x u = 0, \ \ m = (\mu - \partial_x^2 + \partial_x^4)u, \\ u(0,x) = u_0(x), \end{cases} \tag{1}$$

where $u(t,x)$ is a time-dependent spatially periodic function on the unit-circle $S^1 = \mathbb{R}/\mathbb{Z}$ and $\mu(u) = \int_{S^1} u dx$ denotes its mean.

The system (1) is deeply related to the well-known μ-version of Camassa-Holm equation with its form as follows [2]

$$m_t + 2mu_x + m_x u = 0, \ \ m = (\mu - \partial_x^2)u. \tag{2}$$

The μ-Camassa-Holm equation (2) can describe the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal with external magnetic filed and self-interaction. It also arises geometrically as equations for geodesic flow in the context of the diffeomorphism group of the circle $Diff(S^1)$ endowed with a right-invariant Riemannian metric induced by the μ inner product $< u,v >= \mu(u)\mu(v) + \int_{S^1} u_x v_x dx$. Furthermore, the μ-Camassa-Holm equation (2) can be viewed as a natural generalization of the famous Camassa-Holm equation

$$m_t + 2mu_x + m_x u = 0, \ \ m = (1 - \partial_x^2)u. \tag{3}$$

The μ-Camassa-Holm equation (2) has recently been intensely studied from mathematical view. It was proved in [2, 3] that Eq.(2) is bihamiltonian and admits both cusped solitons as well as smooth traveling-wave solutions. Also, the authors proved that it is locally well-posed and established some results on the lifespan of its solutions. In particular, it was shown in [3] that the μ-Camassa-Holm equation (2) also admits parabola-shaped periodic peakons. Chen, Lenells, and

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Liu [4] showed that the above periodic peakons are orbitally stable. Following closely the ideas used in [5, 6], Tiğlay [7] studied the periodic Cauchy problem for Eq.(2) and proved its existence and uniqueness of conservative weak solutions. By employing a technique of change of variables on the Lagrangian variable [8, 9], Deng and Chen [10] also constructed the global weak conservative solutions of the μ-Camassa-Holm equation (2) in much simpler way.

More recently, Wang, Li and Qiao [1] studied the Cauchy problem of the higher-order μ-Camassa-Holm equation (1). They not only obtained the global existence results for strong solutions and weak solutions of Eq.(1), but also proved that the solution map is non-uniformly continuous in $H^s(S^1), s \geq 4$. Moreover, they proved that Eq.(1) still admits single peakon solutions.

Besides, Coclitea and Karlsen [12] studied the Cauchy problem for a generalized Camassa-Holm equation and also gave a note on it. They established the existence of global weak solutions for this generalized Camassa-Holm equation in the energy space H^1. However, different from their works, our research is mainly focused on the well-posedness problem of μ-version of the so-called well-known Camassa-Holm equation in the sense that whether its solutions depend uniformly continuously on the initial data.

More precisely, in this note we further consider the Cauchy problem of the higher-order μ-Camassa-Holm equation (1). Our aim is to prove that the data-to-solution map for the solutions to the Cauchy problem (1) is not uniformly continuous. Our method is motivated by the works of Himonas and Misiołek [11]. The main result is summarized as follows.

Theorem 1.1. The Cauchy problem (1) is not locally well-posed in the Sobolev space $H^s(S^1)$ for any $s < \frac{7}{2}$ in the sense that its solutions do not depend uniformly continuously on the initial data.

2. **Proof of Theorem 1.1**

Before we begin the proof of Theorem 1.1, we recall that the higher-order μ-Camassa-Holm equation admits the following periodic peakon solutions [1].

Lemma 2.1. For any $c > 0$, Eq.(1) admits the peaked periodic-one traveling wave solutions

$$u(x,t) = u(\xi) = \gamma c \left[\frac{1}{2}(\xi - [\xi] - \frac{1}{2})^2 - \frac{\cosh(\xi - [\xi] - \frac{1}{2})}{2\sinh(\frac{1}{2})} + \frac{47}{24} \right],$$

where $\xi = x - ct, \gamma = \frac{12\sinh(\frac{1}{2})}{25\sinh(\frac{1}{2}) - 6\cosh(\frac{1}{2})}$.

By using these periodic peakons, we can construct sequences which show that the data-to-solution map for the Cauchy problem (1) is not uniformly continuous on $H^s(S^1)$ when $s < \frac{7}{2}$. Theorem 1.1 is a consequence of the following proposition.

Proposition 2.1. If $s < \frac{7}{2}$, then there exist two sequences of solutions v_n^1 and v_n^2 in $H^s(S^1)$ of the Cauchy problem (1) such that for any $t > 0$ we have

$$\|v_n^2(0) - v_n^1(0)\|_{H^s} \leq C_1(s)\frac{1}{nt}, \tag{4}$$

and

$$\|v_n^2(t) - v_n^1(t)\|_{H^s} \geq C_2(s) n^{s+|s|+\frac{1}{2}}, \tag{5}$$

where $C_j(s), j = 1, 2$, are positive constants defined by

$$C_1^2(s) \doteq \frac{\gamma^2}{64\pi^6} \sum_{m\neq 0, m\in\mathbb{Z}} (1 + 4\pi^2 m^2)^{s-2} m^{-4}, \tag{6}$$

and

$$C_2^2(s) \doteq 8^{s-5}\pi^{2s-12}\gamma^2(1 - \cos 1). \tag{7}$$

Proof. In fact, we must determine positive constants $c_1 = c_1(n)$ and $c_2 = c_2(n)$ such that the sequences of periodic peakon solutions given by

$$\begin{aligned} u_{c_j}(x,t) =& \gamma c_j[\frac{1}{2}(x - c_j t - [x - c_j t] - \frac{1}{2})^2 \\ & - \frac{\cosh(x - c_j t - [x - c_j t] - \frac{1}{2})}{2\sinh(\frac{1}{2})} + \frac{47}{24}], j = 1,2, \end{aligned}$$

satisfy the above conditions (4) and (5). We begin by computing the partial Fourier transform of u_c with respect to x. First, at $t = 0$, we have

$$\begin{aligned} \hat{u}_c(m,0) &= \gamma c \int_0^1 e^{-2\pi imx}\left[\frac{1}{2}(x - [x] - \frac{1}{2})^2 - \frac{\cosh(x - [x] - \frac{1}{2})}{2\sinh(\frac{1}{2})} + \frac{47}{24}\right] dx \\ &= \gamma c(\frac{1}{4\pi^2 m^2} - \frac{1}{1 + 4\pi^2 m^2}) \\ &= \frac{\gamma c}{4\pi^2 m^2(1 + 4\pi^2 m^2)}. \end{aligned}$$

Thus, it follows that for any $t \geq 0$, we have

$$\hat{u}_c(m,t) = \frac{\gamma c}{4\pi^2 m^2(1 + 4\pi^2 m^2)} e^{-2\pi imct}.$$

Next, computing the H^s-distance between the two peakon sequences at $t = 0$, we get

$$\begin{aligned} \|u_{c_2}(0) - u_{c_1}(0)\|_{H^s}^2 &= \sum_{m\neq 0}(1 + 4\pi^2 m^2)^s \mid \frac{\gamma}{4\pi^2 m^2(1 + 4\pi^2 m^2)}(c_2 - c_1) \mid^2 \\ &= \frac{\gamma^2}{16\pi^4}(c_2 - c_1)^2 \sum_{m\neq 0}(1 + 4\pi^2 m^2)^{s-2} m^{-4}. \end{aligned}$$

where $\sum_{m\neq 0}(1 + 4\pi^2 m^2)^{s-2} m^{-4} < \infty$, provided that $2s - 8 < -1$(namely, $s < \frac{7}{2}$). On the other hand, for any $t > 0$ we get

$$\begin{aligned} & \|u_{c_2}(t) - u_{c_1}(t)\|_{H^s}^2 \\ =& \frac{\gamma^2}{16\pi^4}\sum_{m\neq 0}(1 + 4\pi^2 m^2)^{s-2} m^{-4} \mid c_2 e^{-2\pi imc_2 t} - c_1 e^{-2\pi imc_1 t} \mid^2 \\ =& \frac{\gamma^2}{16\pi^4}\sum_{m\neq 0}(1 + 4\pi^2 m^2)^{s-2} m^{-4}[c_1^2 + c_2^2 - 2c_1 c_2 \cos 2\pi m(c_2 - c_1)t] \\ =& \frac{\gamma^2}{16\pi^4}(c_2 - c_1)^2 \sum_{m\neq 0}(1 + 4\pi^2 m^2)^{s-2} m^{-4} \\ & + \frac{\gamma^2}{8\pi^4}\sum_{m\neq 0} c_1 c_2 (1 + 4\pi^2 m^2)^{s-2} m^{-4}[1 - \cos 2\pi(c_2 - c_1)mt]. \end{aligned}$$

Given $n \in \mathbb{N}$, choose the constant c_2 so that $c_2 = c_1 + \frac{1}{2\pi nt}$, so we have

$$\begin{aligned} \|u_{c_2}(t) - u_{c_1}(t)\|_{H^s}^2 &\geq \|u_{c_2}(0) - u_{c_1}(0)\|_{H^s}^2 + \frac{\gamma^2}{8\pi^4}(1 - \cos 1)c_1^2(1 + 4\pi^2 n^2)^{s-4} \\ &\geq 8^{s-5}\pi^{2s-12}\gamma^2(1 - \cos 1)c_1^2 n^{2s-8}. \end{aligned}$$

Choosing the constant $c_1 = n^{\frac{9}{2}+|s|}$ yields

$$\|u_{c_2}(t) - u_{c_1}(t)\|_{H^s}^2 \geq 8^{s-5}\pi^{2s-12}\gamma^2(1 - \cos 1)n^{1+2s+2|s|}. \tag{8}$$

However, at $t=0$, we have

$$\begin{aligned}\|u_{c_2}(0)-u_{c_1}(0)\|_{H^s}^2 &= \frac{\gamma^2}{16\pi^4}(c_2-c_1)^2\sum_{m\neq 0}(1+4\pi^2m^2)^{s-2}m^{-4}\\ &\leq \frac{\gamma^2}{64\pi^6}\frac{1}{n^2t^2}\sum_{m\neq 0}(1+4\pi^2m^2)^{s-2}m^{-4}. \qquad (9)\end{aligned}$$

Now we set $v_n^1(x,t)=u_{c_1}(x,t)$ and $v_n^2(x,t)=u_{c_2}(x,t)$, and therefore Theorem 1.1 follows immediately from (8) and (9) with

$$C_1^2(s)=\frac{\gamma^2}{64\pi^6}\sum_{m\neq 0}(1+4\pi^2m^2)^{s-2}m^{-4},$$

and

$$C_2^2(s)=8^{s-5}\pi^{2s-12}\gamma^2(1-\cos 1).$$

□

Remark 2.1. Proposition 2.1 indicates that the data-to-solution map is not globally uniformly continuous. However, it should be noted that more desirable result would be that the data-to-solution map is not uniformly continuous on bounded subsets of $H^s(S^1), s<\frac{7}{2}$.

Remark 2.2. In [1], the authors have shown that the Cauchy problem (1) is locally well-posed in $H^s(S^1)$ in the sense of Hadamard for $s>\frac{7}{2}$. Therefore, combining this result with our results stated in this note suggests that $s=\frac{7}{2}$ is the critical Sobolev index for well-posedness.

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