A UNIQUENESS THEOREM FOR THE INVERSE BVP WITH THE JUMP CONDITION AND TURNING POINT

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In this paper, we study a second-order differential equation on the half-line having a turning point and jump condition. We establish properties of the spectrum, obtain the formulation of the inverse problem and prove the uniqueness theorem for the solution of the inverse problem.

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1. Introduction

This paper deal with the BVP (L) for a differential equation

$$y''(x) + \left(\rho^2 p_2(x) + i\rho p_1(x) + p_0(x)\right)y(x) = 0, \qquad x \geq 0, \tag{1}$$

on the half-line with nonlinear dependence on the spectral parameter ρ, with the boundary condition

$$U(y) := y'(0) + (\alpha_1\rho + \alpha_0)y(0) = 0, \tag{2}$$

and with the following jump condition for $a_1 \in (0,1)$,

$$y^{(m)}(d+0,\rho) = {a_1}^m\, y^{(m)}(d-0,\rho), \quad m = 0,1, \tag{3}$$

in an interior point $x = d$. Let

$$p_2(x) = \begin{cases} -1, & 0 \leq x < a, \\ 1, & x \geq a, \end{cases} \qquad 0 < a < d, \tag{4}$$

i.e., the weight-function $p_2(x)$ changes the sign in an interior point $x = a$, which is called the turning point. The functions $p_k(x),\ k = 0,1$ are complex-valued, $p_1(x)$ is absolutely continuous and $(1+x){p_k}^{(\eta)}(x) \in \mathcal{L}(0,\infty)$ for $0 \leq \eta \leq k \leq 1$. Also the coefficients α_1 and α_0 are complex numbers and $\alpha_1 \neq \pm 1$.

Differential equations with a turning point and jump condition arise in various problems of mathematics, physics, mechanics, geophysics, electronics and other branches of natural sciences (see [1, 5, 9, 11, 19]). Boundary value problems with discontinuities appear in geophysical models for oscillations of the earth [1, 10]. Here the main discontinuity is cased by reflection of the shear waves at the

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base of the crust. Differential equations with turning points occur in radio engineering problems to design directional couplers for non-uniform electronic lines [11, 13]. Turning points connect with physical situations in which zeros correspond to the limit of motion of a wave mechanical particle bounded by a potential field. These equations also arise in electronic engineering in designing heterogeneous transmission channels. Moreover, inverse problems appear in electronics for constructing parameters of heterogeneous electronic lines with desirable technical characteristics [9, 18]. After reducing the corresponding mathematical model, we obtain a BVP (L) with discontinuities in an interior point where the weight function $p_2(x)$ reflects a priori known parameters, and the potential function $q(x) = -\big(i\rho p_1(x) + p_0(x)\big)$ must be constructed from the given spectral information which describes desirable amplitude. Spectral information can be also used to reconstruct the conductivity profiles of a one-dimensional discontinuous medium.

Inverse problems for differential pencils without turning point were studied in [2, 17]. In [2] two methods the so-called half inverse and nodal points have been used for survey of the inverse problem. Also boundary value problems on the half-line without discontinuities have been studied in [15]. The presence of the turning point and discontinuity produces essential qualitative modifications in the investigation of the inverse problem. In [7, 8] the inverse problem was investigated for differential equations with turning points. Also boundary value problems with discontinuity in various formulations have been studied in [3, 6, 14]. Some aspects of the inverse spectral problem for differential pencils with the turning point and jump condition, simultaneous, are studied in [16]. But an inverse spectral problem is not studied for differential pencils with these conditions so far.

To prove the uniqueness solution of the inverse problem in this paper, we use a function the so-called Weyl function. This function is obtained by the special fundamental system of solutions for equation (1). The rest of this paper is organized as follows: In Section 2, we investigate the asymptotic form of the solutions, obtain the characteristic function and calculate eigenvalues. In Section 3, we get the asymptotic form of the Weyl function. In Section 4, we establish a formulation of the inverse problem and prove a uniqueness theorem for solution of the inverse problem. Finally, Section 5 contains some conclusions.

Notations. Throughout the paper, we denote by $\mathcal{L}(0,\infty)$ and $\mathcal{L}_2(0,\infty)$ the space of integrable and square integrable complex-valued functions on the half-line $\mathrm{x} \geq 0$, respectively. Next, $\mathrm{AC}[0,\mathrm{d})$ and $\mathrm{AC}_{\mathrm{loc}}(\mathrm{d},\infty)$ will be space of absolutely continuous functions on $[0,\mathrm{d})$ and absolutely continuous functions on each compact subset of (d,∞), respectively. Also in the sequel, *O* and *o* denote the Landau symbols.

2. Preliminary results

Denote $\Pi_+ \coloneqq \{\rho\colon\ Im\rho > 0\}$ and $\Pi_0 \coloneqq \{\rho\colon\ Im\rho = 0\}$. From [12, 20], we know that Eq. (1) has a unique solution $y = e(x,\rho)$ with the following properties:
1) For each fixed x, the functions $e^{(m)}(x,\rho), m = 0,1$ are holomorphic for $\rho \in \Pi_+$ and continuous for $\rho \in \overline{\Pi_+}$. These functions are also continuously differentiable for $\rho \in \overline{\Pi_+} \setminus \{0\}$.

2) For $|\rho| \to \infty,\ \rho \in \overline{\Pi_+},$ uniformly in $x > d,$

$$e^{(m)}(x,\rho) = (i\rho)^m\, exp\big(i\rho x - Q(x)\big)\,[1],\ m = 0,1, \tag{5}$$

where $Q(x) = \frac{1}{2}\int_0^x p_1(t)dt$ and $[1] = 1 + O(\rho^{-1})$.

3) For $|\rho| \to \infty,\ \rho \in \overline{\Pi_+},$ uniformly in $a \le x < d,$

$$e^{(m)}(x,\rho) = \frac{(i\rho)^m}{a_1}\Big(exp\big(i\rho x - Q(x)\big)\,[b_+]$$
$$+(-1)^{m+1}\,exp\Big(i\rho(2d - x) - \big(2Q(d) - Q(x)\big)\Big)\,[b_-]\Big),\ m = 0,1, \tag{6}$$

where $[b_\pm] = b_\pm + O(\rho^{-1})$ and $b_\pm = \frac{1 \pm a_1}{2}$.

4) For real $\rho \neq 0,\ x \in [0,\infty) \setminus \{d\},$ the functions $e_+(x,\rho)$ and $e_-(x,\rho)$ form a fundamental system of solutions of Eq. (1), where for $x > d$

$e_+(x,\rho) = \lim_{z\to\rho,\ z\in\Pi_+} e(x,z),\quad e_-(x,\rho) = exp\,(2Q(x))\lim_{z\to\rho,\ z\in\Pi_+} e(x,-\bar{z}).$
Also, we have

$$\langle e_+(x,\rho), e_-(x,\rho)\rangle = \begin{cases} -2i\rho {a_1}^{-1}, & 0 \le x < d, \\ -2i\rho, & x > d, \end{cases}$$

where $\langle y,z\rangle \coloneqq yz' - y'z$, and is called the Wronskian of the functions y and z.

Denote

$$\Delta(\rho) \coloneqq U\big(e(x,\rho)\big). \tag{7}$$

The function $\Delta(\rho)$ is called the characteristic function of BVP (L) and is entire function in ρ.

Let $\varphi_j(x,\rho),\ j = 1,2$ be the discontinuous solutions of Eq. (1) under the jump condition (3) and the initial conditions ${\varphi_j}^{(n-1)}(0,\rho) = \delta_{jn},\ n = 1,2$ (δ_{jn} is the Kronecker delta). For each fixed $x \ge 0$, the functions ${\varphi_j}^{(n-1)}(x,\rho)$ are entire functions in ρ and by virtue of Liouville's formula for the Wronskian, we have

$$\langle \varphi_1(x,\rho), \varphi_2(x,\rho)\rangle = \begin{cases} 1, & 0 \le x < d, \\ a_1, & x > d. \end{cases}$$

Theorem 2.1. The functions ${\varphi_j}^{(m)}(x,\rho),\ j = 1,2,\ m = 0,1$ have the following asymptotic forms for $|\rho| \to \infty$

i_1) Uniformly in $x \in [0,a],$

$$\begin{cases} \varphi_1{}^{(m)}(x,\rho) = \frac{\rho^m}{2}\left(exp\left(\rho x - iQ(x)\right)[1] + (-1)^m exp\left(-\rho x + iQ(x)\right)[1]\right), \\ \varphi_2{}^{(m)}(x,\rho) = \rho^m \left(exp\left(\rho x - iQ(x)\right)O(\rho^{-1}) + (-1)^m exp\left(-\rho x + iQ(x)\right)O(\rho^{-1})\right). \end{cases}$$

i_2) Uniformly in $x \in [a,d)$,

$$\begin{cases} \varphi_1{}^{(m)}(x,\rho) = \frac{(i\rho)^m}{4}\Big(V_1(\rho)exp\left(i\rho(x-a) - \left(Q(x)-Q(a)\right)\right)[1] \\ \qquad + (-1)^m V_2(\rho)exp\left(-i\rho(x-a) + \left(Q(x)-Q(a)\right)\right)[1]\Big), \\ \varphi_2{}^{(m)}(x,\rho) = (i\rho)^m \Big(V_1(\rho)exp\left(i\rho(x-a) - \left(Q(x)-Q(a)\right)\right)O(\rho^{-1}) \\ \qquad + (-1)^m V_2(\rho)exp\left(-i\rho(x-a) + \left(Q(x)-Q(a)\right)\right)O(\rho^{-1})\Big), \end{cases}$$

where

$$V_1(\rho) = (1+i)exp\left(-\rho a + iQ(a)\right) + (1-i)exp\left(\rho a - iQ(a)\right),$$
$$V_2(\rho) = (1-i)exp\left(-\rho a + iQ(a)\right) + (1+i)exp\left(\rho a - iQ(a)\right).$$

i_3) Uniformly in $x \in (d,\infty)$,

$$\begin{cases} \varphi_1{}^{(m)}(x,\rho) = \frac{(i\rho)^m}{4}\Big(W_1(\rho)exp\left(i\rho(x-d) - \left(Q(x)-Q(d)\right)\right)[1] \\ \qquad + (-1)^m W_2(\rho)exp\left(-i\rho(x-d) + \left(Q(x)-Q(d)\right)\right)[1]\Big), \\ \varphi_2{}^{(m)}(x,\rho) = (i\rho)^m \Big(W_1(\rho)exp\left(i\rho(x-d) - \left(Q(x)-Q(d)\right)\right)O(\rho^{-1}) \\ \qquad + (-1)^m W_2(\rho)exp\left(-i\rho(x-d) + \left(Q(x)-Q(d)\right)\right)O(\rho^{-1})\Big), \end{cases}$$

where

$$\begin{aligned} W_1(\rho) &= b_+ exp\left(i\rho(d-a) - \left(Q(d)-Q(a)\right)\right)V_1(\rho) \\ &\quad + b_- exp\left(-i\rho(d-a) + \left(Q(d)-Q(a)\right)\right)V_2(\rho), \\ W_2(\rho) &= b_- exp\left(i\rho(d-a) - \left(Q(d)-Q(a)\right)\right)V_1(\rho) \\ &\quad + b_+ exp\left(-i\rho(d-a) + \left(Q(d)-Q(a)\right)\right)V_2(\rho). \end{aligned}$$

Proof. Denote $\Pi_\pm^1 := \{\rho\colon \pm Re\rho > 0\}$. Let $\{y_j(x,\rho)\}_{j=1,2}$ and $\{Y_j(x,\rho)\}_{j=1,2}$ be the Birkhoff-type smooth fundamental system of solutions of Eq. (1) with the following asymptotic forms on the intervals $[0,a]$ and $[a,\infty)$ respectively, for $|\rho| \to \infty$, $\rho \in \overline{\Pi_\pm^1}$, $m = 0,1$,

$$y_j{}^{(m)}(x,\rho) = ((-1)^j\rho)^m exp\left((-1)^j\left(\rho x - iQ(x)\right)\right)[1], \tag{8}$$

and

$$Y_j{}^{(m)}(x,\rho) = ((-1)^{j-1}i\rho)^m exp\left((-1)^{j-1}\left(i\rho x - Q(x)\right)\right)[1], \tag{9}$$

(see [12, 21]). Then

$$\varphi_j{}^{(m)}(x,\rho) = A_{1j}(\rho)y_1{}^{(m)}(x,\rho) + A_{2j}(\rho)y_2{}^{(m)}(x,\rho),\ j = 1,2,\ \ x \in [0,a]. \quad (10)$$

Using (8), (10) and initial conditions $\varphi_j(x,\rho)$ in $x = 0$, we calculate

$$A_{1j}(\rho) = \frac{1}{2}[1], \qquad A_{2j}(\rho) = O(\rho^{-1}). \quad (11)$$

Substituting (8) and (11) into (10), we obtain $\varphi_j{}^{(m)}(x,\rho)$ in $[0,a]$.

Analogously, taking the smooth condition

$$\varphi_j{}^{(m)}(a-0,\rho) = \varphi_j{}^{(m)}(a+0,\rho),\ \ m = 0,1, \quad (12)$$

and the jump condition (3), we arrive at $\varphi_j{}^{(m)}(x,\rho)$ in $[a,d)$ and (d,∞).

Theorem 2.2. The characteristic function $\Delta(\rho)$ has simple zeros of the form

$$\rho_k = \frac{1}{a}(k\pi i + iQ(a) + \kappa_1 + \kappa_2) + O(k^{-1}), \quad |k| \to \infty, \quad (13)$$

where

$$\kappa_1 = \frac{1}{2} ln\frac{\alpha_1+1}{\alpha_1-1}, \qquad \kappa_2 = \frac{1}{2} ln\frac{i+1}{i-1}.$$

Proof. By Birkhoff-type fundamental system of solutions of Eq. (1) on the interval $[0,a]$, we have

$$e^{(m)}(x,\rho) = h_1(\rho)y_1{}^{(m)}(x,\rho) + h_2(\rho)y_2{}^{(m)}(x,\rho), \quad x \in [0,a]. \quad (14)$$

Using (6), (8) and the smooth condition $e(x,\rho)$ in $x = a$, i.e., $e^{(m)}(a-0,\rho) = e^{(m)}(a+0,\rho)$, $m = 0,1$, we obtain

$$h_1(\rho) = \frac{1}{2a_1} exp\big(\rho a - iQ(a)\big)\Big((1-i)exp\big(i\rho a - Q(a)\big)[b_+]$$
$$-(1+i)exp\Big(i\rho(2d-a) - \big(2Q(d) - Q(a)\big)\Big)[b_-]\Big),$$
$$h_2(\rho) = \frac{1}{2a_1} exp\big(-\rho a + iQ(a)\big)\Big((1+i)exp\big(i\rho a - Q(a)\big)[b_+]$$
$$-(1-i)exp\Big(i\rho(2d-a) - \big(2Q(d) - Q(a)\big)\Big)[b_-]\Big).$$

Now, substituting (8) and the coefficients $h_1(\rho)$ and $h_2(\rho)$ into (14), we have as $|\rho| \to \infty$, $\rho \in \overline{\Pi_+}$, $m = 0,1$, uniformly in $x \in [0,a]$

$$e^{(m)}(x,\rho) = \frac{\rho^m}{2a_1}\Big(E_1(\rho)exp\Big(\rho(x-a) - i\big(Q(x) - Q(a)\big)\Big)$$
$$+ E_2(\rho)exp\Big(-\rho(x-a) + i\big(Q(x) - Q(a)\big)\Big)\Big),$$

where

$$E_1(\rho) = (1+i)exp\big(i\rho a - Q(a)\big)[b_+]$$
$$-(1-i)exp\Big(i\rho(2d-a) - \big(2Q(d) - Q(a)\big)\Big)[b_-],$$
$$E_2(\rho) = (-1)^m\Big((1-i)exp\big(i\rho a - Q(a)\big)[b_+]$$
$$-(1+i)exp\Big(i\rho(2d-a) - \big(2Q(d) - Q(a)\big)\Big)[b_-]\Big).$$

Taking (2) and (7), this yields the following characteristic function for $|\rho| \to \infty$, $\rho \in \overline{\Pi_+}$,

$$\Delta(\rho) = \frac{\rho}{2a_1}\Big((\alpha_1 - 1)N_1(\rho)exp\big(\rho a - iQ(a)\big) + (\alpha_1 + 1)N_2(\rho)exp\big(-\rho a + iQ(a)\big)\Big),$$

where

$$N_1(\rho) = (1-i)exp\big(i\rho a - Q(a)\big)[b_+] - (1+i)exp\Big(i\rho(2d-a) - \big(2Q(d) - Q(a)\big)\Big)[b_-],$$
$$N_2(\rho) = (1+i)exp\big(i\rho a - Q(a)\big)[b_+] - (1-i)exp\Big(i\rho(2d-a) - \big(2Q(d) - Q(a)\big)\Big)[b_-].$$

Using Rouche's Theorem (see [4]), we arrive at the zeros of the form (13). Now we prove that these zeros are simple. Since $e(x,\rho)$ and $\varphi(x,\rho)$ are solutions of Eq. (1), we have

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$$\begin{cases} e''(x,\rho) + \big(\rho^2 p_2(x) + i\rho p_1(x) + p_0(x)\big)e(x,\rho) = 0, \\ \varphi''(x,\rho_k) + \big({\rho_k}^2 p_2(x) + i\rho_k p_1(x) + p_0(x)\big)\varphi(x,\rho_k) = 0. \end{cases}$$

We get

$$\frac{d}{dx}\langle\varphi(x,\rho_k), e(x,\rho)\rangle + ip_1(x)\varphi(x,\rho_k)e(x,\rho)(\rho-\rho_k) = -(\rho^2 - {\rho_k}^2)p_2(x)\varphi(x,\rho_k)e(x,\rho).$$

Thus

$$-(\rho^2 - {\rho_k}^2)\int_0^\infty p_2(t)\varphi(t,\rho_k)e(t,\rho)dt = \langle\varphi(x,\rho_k), e(x,\rho)\rangle|_0^\infty + (\rho-\rho_k)\int_0^\infty ip_1(t)\varphi(t,\rho_k)e(t,\rho)dt. \quad (15)$$

since

$$\Delta(\rho) = \langle\varphi(x,\rho), e(x,\rho)\rangle,$$

we have

$$e(x,\rho_k) = \beta_k\varphi(x,\rho_k), \quad (16)$$

for $\beta_k \neq 0$ (see [16]). Also from [15], we know that for $x \longrightarrow \infty$,

$$e^{(m)}(x,\rho) = (i\rho)^m\, exp\big(i\rho x - Q(x)\big)\,(1 + o(1)),\; m = 0,1. \quad (17)$$

Therefore from (16) and (17), we have

$$\lim_{x\longrightarrow\infty}\langle\varphi(x,\rho_k), e(x,\rho)\rangle = 0. \quad (18)$$

Now, taking (15) and (18), we get

$$-\int_0^\infty p_2(t)\varphi(t,\rho_k)e(t,\rho)dt = \frac{0 - \Delta(\rho)}{\rho^2 - {\rho_k}^2} + \frac{\rho - \rho_k}{\rho^2 - {\rho_k}^2}\int_0^\infty ip_1(t)\varphi(t,\rho_k)e(t,\rho)dt.$$

For sufficiently large ρ_k, if $\rho \longrightarrow \rho_k$, then

$$\int_0^\infty p_2(t)\varphi(t,\rho_k)e(t,\rho_k)dt = \Delta_1(\rho_k),$$

where $\Delta_1(\rho_k) = \frac{d\Delta(\rho)}{2\rho d\rho}$. Using (16), this yields

$$\Delta_1(\rho_k) = \beta_k \int_0^\infty p_2(t)\varphi^2(t,\rho_k)dt \neq 0,$$

i.e., $\Delta(\rho)$ has simple zeros. The proof is completed.

Denote

$$\Lambda' = \{\rho \in \Pi_+ : \ \Delta(\rho) = 0\}, \ \Lambda'' = \{\rho \in \mathbb{R} : \ \Delta(\rho) = 0\}, \ \Lambda = \Lambda' \cup \Lambda''.$$

3. The Weyl function

We put

$$\phi(x,\rho) = \frac{e(x,\rho)}{\Delta(\rho)}. \tag{19}$$

The function $\phi(x,\rho)$ is a solution of Eq. (1) that is called the Weyl solution for BVP (L). Denote

$$M(\rho) = \phi(0,\rho). \tag{20}$$

We will call it the Weyl function for BVP (L). It follows from (19) and (20) that

$$M(\rho) = \frac{e(0,\rho)}{\Delta(\rho)}. \tag{21}$$

Using the initial conditions $\varphi_j(x,\rho)$ at the point $x = 0$, we get

$$\phi(x,\rho) = \varphi_2(x,\rho) + M(\rho)\varphi(x,\rho), \tag{22}$$

where

$$\varphi(x,\rho) = \varphi_1(x,\rho) - (\alpha_1\rho + \alpha_0)\varphi_2(x,\rho). \tag{23}$$

By virtue of Liouville's formula for the Wronskian (see [16]) and (22), (23) , we obtain

$$\langle \varphi(x,\rho), \phi(x,\rho) \rangle = \begin{cases} 1, & 0 \le x < d, \\ a_1, & x > d. \end{cases} \tag{24}$$

Moreover, taking Theorem 2.1 and (23), we have for positive constant C and $m = 0,1$,

i_1) For $x \in [0,a]$,

$$\begin{cases} \left|\varphi_1^{(m)}(x,\rho)\right| \le C|\rho|^m \exp(|Re\rho|x), \\ \left|\varphi_2^{(m)}(x,\rho)\right| \le C|\rho|^{m-1} \exp(|Re\rho|x), \\ \left|\varphi^{(m)}(x,\rho)\right| \le C|\rho|^m \exp(|Re\rho|x). \end{cases} \tag{25}$$

i_2) For $x \in [a,d)$,

$$\begin{cases} \left|\varphi_1^{(m)}(x,\rho)\right| \le C|\rho|^m\, exp(|Re\rho|a)\, exp(|Im\rho|(x-a)), \\ \left|\varphi_2^{(m)}(x,\rho)\right| \le C|\rho|^{m-1}\, exp(|Re\rho|a)\, exp(|Im\rho|(x-a))\,, \\ \left|\varphi^{(m)}(x,\rho)\right| \le C|\rho|^m\, exp(|Re\rho|a)\, exp(|Im\rho|(x-a)). \end{cases} \qquad (26)$$

i_3) For $x \in (d,\infty)$,

$$\begin{cases} \left|\varphi_1^{(m)}(x,\rho)\right| \le C|\rho|^m\, exp(|Re\rho|a)\, exp(|Im\rho|(x-a)), \\ \left|\varphi_2^{(m)}(x,\rho)\right| \le C|\rho|^{m-1}\, exp(|Re\rho|a)\, exp(|Im\rho|(x-a))\,, \\ \left|\varphi^{(m)}(x,\rho)\right| \le C|\rho|^m\, exp(|Re\rho|a)\, exp(|Im\rho|(x-a)). \end{cases} \qquad (27)$$

Definition 3.1. The set of singularities of the Weyl function $M(\rho)$ is called the spectrum of BVP (L). The values of the parameter ρ for which the Eq. (1) has nontrivial solutions satisfying the conditions $U(y)=0,\ y(\infty)=0$ (i.e., $\lim_{x\to\infty} y(x)=0$), are called eigenvalues of BVP (L), and the corresponding solutions are called eigenfunctions.

Theorem 3.1. The Weyl function $M(\rho)$ is holomorphic in $\Pi_+\backslash\Lambda'$ and continuously differentiable in $\overline{\Pi_+}\backslash\Lambda$. The set of singularities of $M(\rho)$ (as an analytic function) coincides with the set $\Lambda_0=\mathbb{R}\cup\Lambda$. For $|\rho|\to\infty,\ \rho\in\Pi_\pm^1$,

$$M(\rho)=\frac{1}{\rho(\alpha_1\mp 1)}\,[1].$$

Proof. Theorem 3.1 follows from (21) and properties of the functions $e(0,\rho)$ and $\Delta(\rho)$.

Thus, according to Theorem 3.1, the spectrum of BVP (L) is equal to Λ_0.

Remark 3.1. On can introduce the operator

$$\ell: D(\ell)\to\mathcal{L}_2(0,\infty),\quad y\to\frac{-1}{p_2(x)}\big(y''+\big(i\rho p_1(x)+p_0(x)\big)y\big),$$

with the domain of definition $D(\ell)=\{y\colon\ y\in\mathcal{L}_2(0,\infty)\cap AC[0,d)\cap AC_{loc}(d,\infty),$ $y'\in AC[0,d)\cap AC_{loc}(d,\infty),\ \ell y\in\mathcal{L}_2(0,\infty),\ U(y)=0,\ and\ y(x)$ satisfies (3)$\}$. It is easy to verify that the spectrum of ℓ coincides with Λ_0. There is no difference between working either with the operator ℓ or with the BVP (L).

Theorem 3.2. BVP (L) has no eigenvalue for real $\rho\neq 0$.

Proof. Suppose that $\rho_0\neq 0$ is an eigenvalue, and let $y(x,\rho_0)$ be a corresponding eigenfunction. Since the functions $\{e_+(x,\rho_0),e_-(x,\rho_0)\}$ form a fundamental system of solutions for Eq. (1), we have $y(x,\rho_0)=C_1e_+(x,\rho_0)+C_2e_-(x,\rho_0)$. As $y(x,\rho_0)\cong 0$ for $x\to\infty$, this is possible iff $C_1=C_2=0$. Therefore $\rho_0\neq 0$ is not an eigenvalue. Theorem 3.2 is proved.

4. The uniqueness theorem

We are going to study the inverse problem for BVP (L). The inverse problem is formulated as follows:

Inverse Problem 4.1. Given the Weyl function $M(\rho)$, construct the coefficients of the pencil (1)-(2).

At first, we have presented the following lemma that it is used in the uniqueness theorem.

Lemma 4.1. Let $f(x,\rho)$ be an entire function.

i_1) If $|f(x,\rho)| \le C$ for any $x \ge 0$ and sufficiently large ρ, then $f(x,\rho) = F(x)$ for $x \ge 0$.

i_2) If $|f(x,\rho)| \le C|\rho|^{-1}$ for any $x \ge 0$ and sufficiently large ρ, then $f(x,\rho) = 0$.

Proof. Let $f(x,\rho)$ depends on x and ρ. Therefore by the assumption of lemma, we have $|f(x,\rho)| \le C|\rho|$ for $x \ge 0$ and sufficiently large ρ. This means that the function $f(x,\rho)$ is unbounded which contradicts the assumption. To prove part 2, by using the assumption and Pressure Theorem for $|\rho| \to \infty$, we arrive at $|f(x,\rho)| = 0$. Therefore $f(x,\rho) = 0$.

In this section, we prove the uniqueness theorem for the solution of the inverse problem. For this purpose, together with $L = L(p_1(x), p_0(x), \alpha_1, \alpha_0)$, we will consider a BVP $\tilde{L} = L(\tilde{p}_1(x), \tilde{p}_0(x), \tilde{\alpha}_1, \tilde{\alpha}_0)$ of the same form (1)-(2) but with different coefficients. If a certain symbol denotes an object related to L, then the same symbol with tilde will denote the analogs object related to $\tilde{L}$.

Theorem 4.1. If $M(\rho) = \widetilde{M}(\rho)$ then $p_1(x) = \tilde{p}_1(x)$, $p_0(x) = \tilde{p}_0(x)$ for $x \ge 0$, and $\alpha_1 = \tilde{\alpha}_1$, $\alpha_0 = \tilde{\alpha}_0$. Thus the specification of the Weyl function $M(\rho)$ uniquely determines the coefficients of the pencil (1)-(2).

Proof. At first, for brevity, we assume that a, d and a_1 are known a priori. We consider the matrix $P(x,\rho) = \left[P_{jk}(x,\rho)\right]_{j,k=1,2}$ defined by

$$P(x,\rho)\begin{bmatrix} \tilde{\varphi}(x,\rho) & \tilde{\phi}(x,\rho) \\ \tilde{\varphi}'(x,\rho) & \tilde{\phi}'(x,\rho) \end{bmatrix} = \begin{bmatrix} \varphi(x,\rho) & \phi(x,\rho) \\ \varphi'(x,\rho) & \phi'(x,\rho) \end{bmatrix}. \tag{28}$$

By virtue of (24), this yields

$$\begin{cases} P_{j1}(x,\rho) = \left(\xi(x)\right)^{-1}\left(\varphi^{(j-1)}(x,\rho)\tilde{\phi}'(x,\rho) - \phi^{(j-1)}(x,\rho)\tilde{\varphi}'(x,\rho)\right), \\ P_{j2}(x,\rho) = \left(\xi(x)\right)^{-1}\left(\phi^{(j-1)}(x,\rho)\tilde{\varphi}(x,\rho) - \varphi^{(j-1)}(x,\rho)\tilde{\phi}(x,\rho)\right), \end{cases} \tag{29}$$

where $\xi(x) = 1$ for $x < d$ and $\xi(x) = a_1$ for $x > d$. Also we have

$$\begin{cases} \varphi(x,\rho) = P_{11}(x,\rho)\tilde{\varphi}(x,\rho) + P_{12}(x,\rho)\tilde{\varphi}'(x,\rho), \\ \phi(x,\rho) = P_{11}(x,\rho)\tilde{\phi}(x,\rho) + P_{12}(x,\rho)\tilde{\phi}'(x,\rho). \end{cases} \qquad 30)$$

Using (22) and (29), we get

$$\begin{cases} P_{j1}(x,\rho) = \left(\xi(x)\right)^{-1}\left(\varphi^{(j-1)}(x,\rho)\tilde{\varphi}_2'(x,\rho) - \varphi_2{}^{(j-1)}(x,\rho)\tilde{\varphi}'(x,\rho) \right. \\ \qquad\qquad \left. +\widehat{M}(\rho)\varphi^{(j-1)}(x,\rho)\tilde{\varphi}'(x,\rho)\right), \\ P_{j2}(x,\rho) = \left(\xi(x)\right)^{-1}\left(\varphi_2{}^{(j-1)}(x,\rho)\tilde{\varphi}(x,\rho) - \varphi^{(j-1)}(x,\rho)\tilde{\varphi}_2(x,\rho) \right. \\ \qquad\qquad \left. -\widehat{M}(\rho)\varphi^{(j-1)}(x,\rho)\tilde{\varphi}(x,\rho)\right), \end{cases}$$

where $\widehat{M}(\rho) = \widetilde{M}(\rho) - M(\rho)$. Since $M(\rho) = \widetilde{M}(\rho)$, deduce $\widehat{M}(\rho) = 0$ and consequently, the functions $P_{jk}(x,\rho)$ are entire in ρ for each fixed $x \ge 0$.

Fix $\delta > 0$. Denote $G_\delta = \{\rho \in \overline{\Pi_+} : |\rho - \rho_k| \ge \delta,\ \rho_k \in \Lambda\}$. It follows from (5), (6), (19) and the functions $e(x,\rho)$ and $\Delta(\rho)$ in Theorem 2.2 that

$$\begin{cases} \left|e^{(m)}(x,\rho)\right| \le C|\rho|^m \exp(-|Im\rho|x),\ x > d,\quad \rho \in \overline{\Pi_+}, \\ \left|e^{(m)}(x,\rho)\right| \le C|\rho|^m \exp(-|Im\rho|x),\ x \in [a,d),\quad \rho \in \overline{\Pi_+}, \\ \left|e^{(m)}(x,\rho)\right| \le C|\rho|^m \exp(-|Im\rho|a)\exp(|Re\rho|(a-x)),\ x \in [0,a],\ \rho \in \overline{\Pi_+}, \end{cases} \tag{31}$$

$$|\Delta(\rho)| \ge C|\rho| \exp(-|Im\rho|a)\exp(|Re\rho|a),\ \rho \in G_\delta, \tag{32}$$

$$\begin{cases} \left|\phi^{(m)}(x,\rho)\right| \le C|\rho|^{m-1} \exp(-|Re\rho|a)\exp(-|Im\rho|(x-a)),\ x > d,\quad \rho \in G_\delta, \\ \left|\phi^{(m)}(x,\rho)\right| \le C|\rho|^{m-1} \exp(-|Re\rho|a)\exp(-|Im\rho|(x-a)),\ x \in [a,d),\quad \rho \in G_\delta, \\ \left|\phi^{(m)}(x,\rho)\right| \le C|\rho|^{m-1} \exp(-|Re\rho|x),\ x \in [0,a],\ \rho \in G_\delta. \end{cases} \tag{33}$$

It follows from (25), (26), (27), (29) and (33) that for $x \ge 0$, $\rho \in G_\delta$,

$$|P_{11}(x,\rho)| \le C, \quad |P_{12}(x,\rho)| \le C|\rho|^{-1}.$$

Using Lemma 4.1, we have $P_{11}(x,\rho) = \mathcal{F}_1(x)$ and $P_{12}(x,\rho) = 0$ for each $x \ge 0$. Together with (30), we have for all x, ρ that

$$\mathcal{F}_1(x)\tilde{\varphi}(x,\rho) = \varphi(x,\rho),\ \mathcal{F}_1(x)\tilde{\phi}(x,\rho) = \phi(x,\rho). \tag{34}$$

By the assumption of Theorem 4.1 and the Weyl function in Theorem 3.1, we infer $\alpha_1 = \tilde{\alpha}_1$.

First let $x \in [0,a]$. Taking the functions $\varphi_j(x,\rho)$ and $e(x,\rho)$ for $x \in [0,a]$, $\Delta(\rho)$, (19), (23) and equality $\alpha_1 = \tilde{\alpha}_1$, we get as $|\rho| \to \infty$, arg $\rho \in \left(0, \frac{\pi}{2}\right)$,

$$\begin{cases} \frac{\varphi(x,\rho)}{\tilde{\varphi}(x,\rho)} = \exp\left(-i\left(Q(x) - \tilde{Q}(x)\right)\right)[1], \\ \frac{\phi(x,\rho)}{\tilde{\phi}(x,\rho)} = \exp\left(i\left(Q(x) - \tilde{Q}(x)\right)\right)[1]. \end{cases} \tag{35}$$

One has from (34) and (35) that

$$\mathcal{F}_1(x) = \exp\left(-i\left(Q(x) - \tilde{Q}(x)\right)\right)[1],\ \mathcal{F}_1(x) = \exp\left(i\left(Q(x) - \tilde{Q}(x)\right)\right)[1], \tag{36}$$

and consequently, $Q(x) = \tilde{Q}(x)$ and $\mathcal{F}_1(x) = 1$ for $x \in [0,a]$.

Let $x \in [a,d)$. Taking the functions $\varphi_j(x,\rho)$ and $e(x,\rho)$ for $x \in [a,d)$, $\Delta(\rho)$, (19), (23) and equalities $\alpha_1 = \tilde{\alpha}_1$ and $Q(a) = \tilde{Q}(a)$ into accounts, we have as $|\rho| \to \infty$, $\arg \rho \in \left(0, \frac{\pi}{2}\right)$,

$$\begin{cases} \frac{\varphi(x,\rho)}{\tilde{\varphi}(x,\rho)} = exp\left(Q_a(x) - \tilde{Q}_a(x)\right)[1], \\ \frac{\phi(x,\rho)}{\tilde{\phi}(x,\rho)} = exp\left(-\left(Q_a(x) - \tilde{Q}_a(x)\right)\right)[1], \end{cases} \tag{37}$$

where $Q_a(x) = \frac{1}{2}\int_a^x p_1(t)dt$. It follows from (34) and (37) that

$$\mathcal{F}_1(x) = exp\left(Q_a(x) - \tilde{Q}_a(x)\right)[1], \ \mathcal{F}_1(x) = exp\left(-\left(Q_a(x) - \tilde{Q}_a(x)\right)\right)[1]. \tag{38}$$

Therefore $Q_a(x) = \tilde{Q}_a(x)$ and $\mathcal{F}_1(x) = 1$ for $x \in [a,d)$.

Now let $x > d$. Analogously taking the functions $\varphi_j(x,\rho)$ and $e(x,\rho)$ for $x > d$, $\Delta(\rho)$, (19), (23) and equalities $\alpha_1 = \tilde{\alpha}_1$, $Q(a) = \tilde{Q}(a)$ and $Q_a(d) = \tilde{Q}_a(d)$ into accounts, we get as $|\rho| \to \infty$, $\arg \rho \in \left(0, \frac{\pi}{2}\right)$,

$$\begin{cases} \frac{\varphi(x,\rho)}{\tilde{\varphi}(x,\rho)} = exp\left(Q_d(x) - \tilde{Q}_d(x)\right)[1], \\ \frac{\phi(x,\rho)}{\tilde{\phi}(x,\rho)} = exp\left(-\left(Q_d(x) - \tilde{Q}_d(x)\right)\right)[1], \end{cases} \tag{39}$$

where $Q_d(x) = \frac{1}{2}\int_d^x p_1(t)dt$. It follows from (34) and (39) that

$$\mathcal{F}_1(x) = exp\left(Q_d(x) - \tilde{Q}_d(x)\right)[1], \ \mathcal{F}_1(x) = exp\left(-\left(Q_d(x) - \tilde{Q}_d(x)\right)\right)[1], \tag{40}$$

and consequently, $Q_d(x) = \tilde{Q}_d(x)$ and $\mathcal{F}_1(x) = 1$ for $x > d$.

Thus $p_1(x) = \tilde{p}_1(x)$, $\mathcal{F}_1(x) = 1$ for all $x \geq 0$. According to (34), we have

$$\tilde{\varphi}(x,\rho) = \varphi(x,\rho), \quad \tilde{\phi}(x,\rho) = \phi(x,\rho). \tag{41}$$

Hence $p_0(x) = \tilde{p}_0(x)$ for all $x \geq 0$, and $\alpha_0 = \tilde{\alpha}_0$. Theorem 4.1 is proved.

5. Conclusions

Through a survey of previous works, it is revealed that discontinuous inverse problems for differential pencils with turning points are not investigated. In this paper, a uniqueness theorem for a solution of this inverse problem has been studied and the spectral mappings' method has been used that the Weyl function plays an important role in it.

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