ANALYSIS OF TRIPARTITE GAME IN POWER SYSTEMS BASED ON UNCERTAIN NASH EQUILIBRIUM THEORY

Liang Wu[1], Xiaolin Yan[2], Zhangsong Yao[3], Teng Zhao[4]

Power systems serve as the fundamental infrastructure for the socioeconomic development of modern societies. Researching power systems can stimulate the growth of the power industry and contribute to the sustainable development of society. This paper aims to address uncertainties prevalent in power systems, including extreme weather events, disruptions in renewable energy supply, and adjustments in economic policies. To achieve this objective, an uncertain Nash equilibrium model is established. Subsequently, a game theoretic analysis is conducted to maximize the interests of users, grid companies, and proxy purchasers. Under the condition of not necessarily continuous differentiability, the Riemann-Stieltjes discrete approximation method has been proposed. The difficulty of calculating the expectation of the optimal revenue function has been resolved. Weak first-order equilibrium condition based on Clarke's generalized gradient, the convergence and convergence rate of uncertain Nash equilibrium solutions are proved.

Keywords: uncertain Nash equilibrium, power system, tripartite game, uncertainty theory.

1. Introduction

Power systems represent essential and intricate engineering systems within modern society. The development of the power industry is closely tied to economic progress, as electrical energy plays an indispensable role in socio-economic production. With the transformation of the electricity market, the decision objectives of participating entities have shifted from complying with uniform and centralized dispatch to pursuing individual profit maximization. Consequently, scholars have recognized the theoretical and practical value of applying game theory to study the competitive behavior of stakeholders in the electrical markets.

Game theory, originally proposed by Von Neumann in the early 20th century, is a discipline that primarily explores human decision-making behavior and interactions. It enables the investigation of optimal strategies and expected payoffs of decision-makers in different scenarios. In a game, each decision-maker faces a problem influenced by the decisions of others, leading to different outcomes and payoffs. Nash equilibrium, a central concept in non-cooperative games, provides a framework for understanding and analyzing various game situations. Studying Nash equilibrium helps us comprehend and explain social and economic phenomena such as market competition, political games, and decision-making in warfare. Game theory has been successfully applied to electrical markets related to economic

[1]School of Management and Ningxia Key Laboratory of intelligent information and Big Data Processing, North Minzu University, Yinchuan, 750021, Ningxia, China, e-mail: `bitwunmu@163.com`

[2]Corresponding author. School of Mathematical Sciences, Tiangong University, Tianjin 300387, China, e-mail: `yanxiaolin@tiangong.edu.cn`

[3]Corresponding author. School of Information Engineering, Nanjing Xiaozhuang University, Nanjing 211171, China, e-mail: `yaozhsong@163.com`

[4]School of Mathematics and Information Science, North Minzu University, Yinchuan, 750021, Ningxia, China, e-mail: `18392334843@163.com`

interests, yielding significant achievements in power systems research [1, 2, 8, 13]. Qin et al. [16] established a game theory model to determine the market equilibrium by minimizing the operating costs of the power system and maximizing the profits of mobile energy storage, using the sequential dispatch algorithm to obtain the optimal strategy. Moafi et al. [14] evaluated the cooperative strategies of participants in the power system based on a three-level game-playing framework, proposing particle swarm optimization and fuzzy logic algorithms to obtain effective cooperative strategies, thereby reducing electrical costs and increasing profits.

Addressing large-scale complex electrical market equilibrium problems presents scholars with new challenges. In such cases, Nash equilibrium has become a crucial theoretical tool for studying the behavior and decision-making of power system participants. Wu et al. [24] established a multi-player non-cooperative differential game model based on optimal control theory. They successfully proved the existence and uniqueness of Nash equilibrium prices by utilizing a distributed algorithm that incorporates neural dynamics and consensus theory. In the electricity market, the pricing strategy of the participants directly affect the competitiveness and efficiency of the market. By analyzing the strategic interactions of the participants, the equilibrium prices and supply-demand relationship in the market can be predicted, which can be used to optimize the market design and regulatory policies. Moreover, the study of Nash equilibrium contributes to risk management and coordination in the power system, promoting its stability and reliable operation [3, 4, 9, 12, 17–22, 26, 27, 29, 30, 32–34].

However, due to the complex multi-stakeholder nature of the power systems, encompassing power plants, transmission lines, distribution networks, and end-users, interdependency, and mutual influence among these participants, and their decisions have significant impacts on the entire system. The power system is also subject to various uncertainty factors affecting its operation and planning, including extreme weather events leading to load uncertainty, interruptions or reductions in renewable energy supply causing energy uncertainty, and uncertainties in electricity market prices due to economic impacts, policy changes, or energy storage technologies. Addressing these uncertainties is crucial for ensuring the resilience and efficiency of the power system [23, 25, 28]. Gao and Sheble [5] demonstrated the high sensitivity of equilibrium solutions to the level of asymmetric information faced by two-stage renewable energy in the presence of uncertainty. Qiao et al. [15] constructed a game model for real-time markets, taking into account the uncertainty of electrical supply and analyzing its impact on hourly electricity prices. They solved the optimal electricity pricing problem by maximizing expected revenue under risk conditions, obtaining the optimal electricity price under risk conditions.

While the existing literature has made significant contributions to power system research, there has been limited attention given to the phenomenon of uncertainty, characterized by a lack of historical sample data. To address this issue, Liu [11] proposed the uncertainty theory. In this paper, we aim to maximize the profits of users, power companies, and agent power purchasers by thinking over the uncertainties present in power systems. To achieve these objectives, an uncertain Nash equilibrium model is established. In cases where the optimal revenue function expectation cannot be computed and the objective function is non-differentiable, the Riemann-Stieltjes discrete approximation method is employed. Finally, the convergence of uncertain Nash equilibrium solutions is analyzed.

The remainder of the paper is structured as follows: Section 2 provides some necessary definitions. Section 3 considers user-based uncertain loads and establishes an uncertain Nash equilibrium model. Section 4 formulates the equilibrium solution within a three-layer game aiming to maximize participants' profits. Section 5 presents the convergence and convergence rate of uncertain Nash equilibrium solutions based on the weak clarke first-order equilibrium conditions.

2. Preliminaries

Definition 2.1 ([31])**.** *If* $\limsup_{h\to\bar{h}} H(h) \subset H(\bar{h})$*, it is equivalent to* $\limsup_{h\to\bar{h}} H(h) = H(\bar{h})$*. In this case, a set-valued mapping* $H : \mathcal{R}^n \to \mathcal{R}^m$ *is said to be upper semi-continuous at* $\bar{h}$*. Here,* τ *is a non-negative integer, and*

$$\begin{aligned}\limsup_{h\to\bar{h}} H(h) :&= \bigcup_{h^\tau\to\bar{h}} \limsup_{\tau\to\infty} H(h^\tau)\\ &= \left\{ j \mid \exists h^\tau \to \bar{h}, \exists j^\tau \to j \text{ with } j^\tau \in H(h^\tau) \right\}.\end{aligned}$$

Definition 2.2 ([7])**.** *Suppose we have a sequence of functions* $\{_n(x)\}$ *that satisfies the following two conditions: (1) For all n, the function* $_n(\S)$ *is bounded and integrable, i.e.,* $\int |_n(\S)| d\S < \infty$*, and there exist constants* μ_n *such that* $|_n(\S)| \leq \mu_n$ *holds for almost all* $\S$*. (2) The function sequence* $\{_n(\S)\}$ *converges to the function* $(\S)$*, i.e.,* $\lim_{n\to\infty} {}_n(\S) = (\S)$ *holds for almost all* $\S$*. Under these conditions, the function* $(\S)$ *is uniformly integrable.*

Definition 2.3 ([7])**.** *Consider a sequence of integrable and bounded random closed sets in a separable Banach space, denoted as* $\mathfrak{T}n$*. If* $\|\mathfrak{T}_n\|$ *is uniformly bounded and, as* $n \to \infty$*, the expectation of* $\mathfrak{T}n$ *denoted by* $E(\mathfrak{T}n)$*, converges to the expectation of* $\mathfrak{T}$ *denoted by* $E(\mathfrak{T})$*, we have* $\lim_{n\to\infty} E(\mathfrak{T}_n) = E(\mathfrak{T})$*.*

Definition 2.4 ([6])**.** *Let* (Ω, Γ, Θ) *be a measure space, and let* ϑ_n *be a sequence of measurable functions defined on* (Ω, Γ, Θ) *such that* ϑ_n *converges almost everywhere to* ϑ*. Furthermore, assume there exists an integrable function* ζ *such that* $|\vartheta_n(\theta)| \leq \zeta(\theta)$ *for all n and* $\theta \in \Omega$*. Then, the following equalities hold:*

$$\int_\Omega \vartheta(\theta) d\Theta(\theta) = \lim_{n\to\infty} \int_\Omega \vartheta_n(\theta) d\Theta(\theta) = \int_\Omega \lim_{n\to\infty} \vartheta_n(\theta) d\Theta(\theta).$$

Definition 2.5 ([6])**.** *Let* $\|\mathcal{P}\| := \underset{\S\in\mathcal{P}}{max} \|\S\|$*, then the deviation between the sets* $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *is defined as* $\mathcal{D}(\mathcal{P}_1, \mathcal{P}_2) := \sup_{\S\in\mathcal{P}_1} dist(\S, \mathcal{P}_2)$*, where* $\mathcal{P}$ *represents a compact vector set.* $dist(\S, \mathcal{P}) := \underset{\S'\in\mathcal{P}}{inf} \|\S - \S'\|$ *represents the distance from a point* $\S$ *to* $\mathcal{P}$*.*

The deviation function $\mathcal{D}$ satisfies the following properties for sets $\mathcal{Z}_1$,$\mathcal{Z}_2$,$\mathcal{Z}_3$:

$$\mathcal{D}(\mathcal{Z}_1, \mathcal{Z}_2) \leq \mathcal{D}(\mathcal{Z}_1, \mathcal{Z}_3) + \mathcal{D}(\mathcal{Z}_3, \mathcal{Z}_2), \quad (1)$$

$$\mathcal{D}(\mathcal{Z}_1 + \mathcal{Z}_3, \mathcal{Z}_2 + \mathcal{Z}_3) \leq \mathcal{D}(\mathcal{Z}_1, \mathcal{Z}_2), \quad (2)$$

$$\mathcal{D}(\mathcal{Z}_1 + \mathcal{Z}_2, \mathcal{Z}_3 + \mathcal{Z}_4) \leq \mathcal{D}(\mathcal{Z}_1, \mathcal{Z}_3) + \mathcal{D}(\mathcal{Z}_2, \mathcal{Z}_4). \quad (3)$$

This paper encompasses several definitions related to the theory of uncertainty. For detailed information, please refer to [10].

3. Uncertain Nash Equilibrium Model in Power Systems

3.1. User Game Model D

The user model D comprises d load-consuming users. Each user possesses multiple electrical devices, and we represent the user set as $D = \{1, 2, \cdots, d\}$, with corresponding geographic locations $S_d = [S_1, S_2, \cdots, S_d]$. Throughout the game process, based on the different load attributes of the electrical devices, we classify the user loads into four categories based on their load attributes: quantitatively adjustable but time-dependent loads (A_1), time-dependent but quantitatively adjustable loads (A_2), fixed loads (A_3), and uncertain loads (A_4).

Assuming that the electricity consumption for each day is divided into time intervals, denoted as $\Delta m = m - m_{-1}$, we represent the set of time points as $M = (m_1, m_2, \cdots, m)$.

The total electricity consumption of all electrical devices owned by user d at time m is determined by the following expression: $(d, , \theta) = (d, A_1,) + (d, A_2,) + (d, A_3,) + (d, A_4,)$.

In the context of this academic paper, the electricity consumption of user d for devices A_1, A_2, A_3, and A_4 during time interval m is represented by $(d, A_1,)$, $(d, A_2,)$, $(d, A_3,)$, and $(d, A_4,)$, respectively. These quantities indicate the amount of electricity consumed by user d for each specific device during the given time interval. The distribution matrix of electricity consumption for all users is defined as follows:

$$\mathscr{Q}(d, , \theta) = \begin{bmatrix} (1,1,\theta) & (1,2,\theta) & \cdots & (1, ,\theta) \\ (2,1,\theta) & (2,2,\theta) & \cdots & (2, ,\theta) \\ \vdots & \vdots & & \vdots \\ (d,1,\theta) & (d,2,\theta) & \cdots & (d, ,\theta) \end{bmatrix}.$$

Unit electricity price chosen by user d in time interval is represented by the vector:

$$(d) = [(d,1), (d,2), \cdots, (d,)],$$

$$(d,) = \begin{cases} f(), & f() = \min\{f(n,) - \Delta f(n), f()\}; \\ f(n,), & f(n,) - \Delta f(n) = \min\{f(n,) - \Delta f(n), f()\}, \end{cases}$$

where $f = [f(1), f(2), \cdots, f()]$ represents the price vector corresponding to the time vector published by the power company. There are n proxy power retailers, and the price vector for retailer D_n is denoted as $f(n) = [f(n,1), f(n,2), \cdots, f(n,)]$. $\Delta f(n)$ represents the unit electricity subsidy obtained when a user chooses proxy retailer D_n.

The payment distribution matrix for user d is given by:

$$\Pi = \begin{bmatrix} \Pi(1,1) & \Pi(1,2) & \cdots & \Pi(1,) \\ \Pi(2,1) & \Pi(2,2) & \cdots & \Pi(2,) \\ \vdots & \vdots & & \vdots \\ \Pi(d,1) & \Pi(d,2) & \cdots & \Pi(d,) \end{bmatrix},$$

where $\Pi(d,)$ represents the payment made by user d at time m, defined as: $\Pi(d,) = (d,) \cdot (d, , \theta)$. The payment matrix for all users is defined as follows:

$$F = \begin{bmatrix} F(1) \\ F(2) \\ \vdots \\ F(d) \end{bmatrix} = \begin{bmatrix} \Pi(1,1) + \Pi(1,2) + \cdots + \Pi(1,) \\ \Pi(2,1) + \Pi(2,2) + \cdots + \Pi(2,) \\ \vdots \\ \Pi(d,1) + \Pi(d,2) + \cdots + \Pi(d,) \end{bmatrix},$$

where $F(d)$ represents the total payment made by user d in one cycle, given by:

$$F(d) = \sum_{m \in M} \Pi(d,) = \Pi(d,1) + \Pi(d,2) + \cdots + \Pi(d,).$$

The electricity consumed by various types of devices within m brings the utility value to user d as follows: $B_1 = a \cdot (d, A_1,), B_2 = b \cdot (d, A_2,), B_3 = c \cdot (d, A_3,)$ and $B_4 = w_\theta \cdot (d, A_4, , \theta)$, here, a and c are constants representing the proportional relationship between the utility of A_1 and A_3 devices, respectively, and the amount of energy consumed. b is a function of m that represents user d's requirements for the operating time of A_2 devices. w_θ is a function of θ that reflects user d's requirement regarding the uncertain energy consumption of A_4 devices.

Utility obtained from the energy consumption of all devices for user d within m is given by: $B = B_1 + B_2 + B_3 + B_4$, the total utility obtained by user d within one time period is defined as $B(d) = \sum_{m \in M}(B_1 + B_2 + B_3 + B_4)$, and thus the utility matrix for the users is:

$$B = \begin{bmatrix} B(1) \\ B(2) \\ \vdots \\ B(d) \end{bmatrix} = \begin{bmatrix} a \cdot (1, A_1,) + b \cdot (1, A_2,) + c \cdot (1, A_3,) + w_\theta \cdot (1, A_4, , \theta) \\ a \cdot (2, A_1,) + b \cdot (2, A_2,) + c \cdot (2, A_3,) + w_\theta \cdot (2, A_4, , \theta) \\ \vdots \\ a \cdot (d, A_1,) + b \cdot (d, A_2,) + c \cdot (d, A_3,) + w_\theta \cdot (d, A_4, , \theta) \end{bmatrix}.$$

User revenue $C(D) = B - F$.

3.2. Grid company model D

The income distribution matrix of the power grid company is given by:

$$I_0 = \Phi_0 \cdot \Pi \cdot \Phi_0' + \mu \cdot \overline{\Phi}_0 \cdot \Pi \cdot \overline{\Phi}_0',$$

where μ represents the transaction fee coefficient for interregional transactions. Φ_0 and Φ_0' are the selection matrices of users, while $\overline{\Phi}_0$ and $\overline{\Phi}_0'$ are used to calculate the supply volume of the power purchasing agent. Power grid company's income is $I = I_0(1) + I_0(2) + \cdots + I_0(d)$. C represents the cost of the power grid company. The revenue of the power grid company is denoted as: $C(D) = I - C$.

3.3. Agent Power Purchasing Model D_n

There are n agent power purchasing entities, represented by the collective set $D_n = \{1, 2, \cdots, n\}$, corresponding to geographical locations $S_n = [S_1, S_2, \cdots, S_n]$. The income distribution matrix for the agent power purchasing entities is given by:

$$I_n' = \Phi_n \cdot \Pi \cdot \Phi_n' = \begin{bmatrix} I_1'(1) & I_1'(2) & \cdots & I_1'() \\ I_2'(1) & I_2'(2) & \cdots & I_2'() \\ \vdots & \vdots & & \vdots \\ I_n'(1) & I_n'(2) & \cdots & I_n'() \end{bmatrix},$$

here, Φ_n and Φ_n' represent the user selection matrices. $I_n = I_n'(1) + I_n'(2) + \cdots + I_n'()$ said agent power purchase business income. C_n represents the cost of the purchasing agent for electricity. This income matrix of the agent power purchasing entities is represented as:

$$\begin{bmatrix} I_1 & I_2 & \vdots & I_n \end{bmatrix}^T.$$

Profit of agent power purchasing entities $C(D_n) = I_n - C_n$.

4. Equilibrium Solution in the Tripartite Game of Users, Power Grid Companies and Power Purchasing Agents

For users, their optimal strategy is determined by maximizing the profit of the agent power purchasing entities:

$$\mathscr{Q}^* = \arg\max_{(d,)} C(D),$$

$$\text{such that} \quad \begin{cases} \sum_{m \in M} (d, A_1,) = \mathscr{Q}(d, A_1), \\ \sum_{m \in M} (d, A_2,) = [\mathscr{Q}_{\min}(d, A_2), \mathscr{Q}_{\max}(d, A_2)], \\ \sum_{m \in M} (d, A_3,) = \mathscr{Q}(d, A_3), \\ \sum_{m \in M} (d, A_4, , \theta) = [\mathscr{Q}_{\min}(d, A_4, \theta), \mathscr{Q}_{\max}(d, A_4, \theta)]. \end{cases}$$

For both the power grid company and the agent power purchasing entities, their optimal strategy is determined by maximizing the profit of the power purchasing operation:

$$f^* = \arg\max_{f()} C(D), \ f^*(n) = \arg\max_{f(n,)} C(D_n),$$

$$s.t.\quad \begin{cases} f() \in [f_{\min}, f_{\max}], \\ f(n,) \in [f_{\min}(n), f_{\max}(n)], \\ \mathcal{Q}(n,) \in [\mathcal{Q}_{\min}(n), \mathcal{Q}_{\max}(n)]. \end{cases}$$

Assuming the strategy combination of the three-party game is denoted as $k' = [f, f_1, f_2, \cdots, f_n, _1, _2, \cdots, _d]$, we can define the problem of uncertain Nash equilibrium from a mathematical perspective as follows:

Definition 4.1. *To find a point k' that satifies the inequality $C_g(k'_g, k'_{-g}, \theta) \geq C_g(k_g, k'_{-g}, \theta)$, where k_g represents the decision variable of the g-th participant with its strategy selection as k'_g; k_{-g} represents the decision variables of all other participants except the g-th participant with their strategy selection as k'_{-g}. $_d$ represents the electricity allocation vector for user d, f represents the electricity pricing arrangement vector for the power grid company, and f_n represents the electricity pricing arrangement vector for the agent power purchasing entities.*

Assuming that participant g achieves the optimal value $\mathcal{C}_g(k'_g, k'_{-g}, \theta)$ at the equilibrium point k', we can further define the uncertain Nash equilibrium problem as follows:

Definition 4.2. *Consider an uncertain Nash equilibrium problem: the objective is to find a point k' that satisfies the following condition:*

$$\mathcal{C}_g(k'_g, k'_{-g}, \theta) := \min_{k_g \in K_g} E[C_g, C'_{-g}, \theta], \tag{4}$$

where K_g represents the decision set for participant g, $C_g(\cdot, k_{-g}, \theta) : \mathcal{R}^{n_m} \to \mathcal{R}$ is Lipschitz continuous function, $\theta \in \Theta$ is an uncertain variable defined in the uncertain space (Υ, Θ, U), and E denotes the mathematical expectation with respect to the distribution of the uncertain vector θ.

Let $C_g(k_g, k_{-g}, \theta)$ represent the optimal profit function, when it is not possible to express $E\left[C_g(k_g, k_{-g}, \theta)\right]$ in a closed form, the objective function in equation (4) is approximated by discretizing it using Riemann-Stieltjes integration. This approximation can be expressed as: $\mathcal{C}_g^L(k_g, k_{-g}, \theta) := \frac{1}{L}\sum_{l=1}^{L} C_g(k_g, k_{-g}, \theta^l)$. Therefore, we consider the Nash equilibrium problem with a discrete approximation using Riemann-Stieltjes integration. The gogal is to find a point $k^L := (k_1^L, k_2^L, \cdots, k_g^L) \in K_1 \times K_2 \times \cdots \times K_g$ that satisfies:

$$\mathcal{C}_g^L(k_g^L, k_{-g}^L, \theta^l) := \min_{k_g \in K_g} \mathcal{C}_g^L(k_g, k_{-g}^L, \theta^l). \tag{5}$$

In practice, equation (4) is referred to as the true problem, while equation (5) is known as the discrete approximation problem.

5. Convergence Analysis of Uncertain Nash Equilibrium Solutions

For uncertain optimization problems involving multiple participant g, when the optimal profit function $C_g(k_g, k_{-g}, \theta)$ is not necessarily continuously differentiable, this paper consider a weak Clarke first-order equilibrium conditions for (4) and (5) based on the Clarke generalized gradient. These conditions can be formulated as follows:

$$0 \in E\left[\partial_{k_g} C_g(k_g, k_{-g}, \theta)\right] + \mathcal{O}_{K_g}(k_g), \tag{6}$$

$$0 \in \frac{1}{L}\sum_{l=1}^{L} \partial_{k_g} C_g(k_g, k_{-g}, \theta^l) + \mathcal{O}_{K_g}(k_g), \tag{7}$$

where $\partial_{k_g} C_g(k_g, k_{-g}, \theta)$ represents the Clarke generalized gradient of $C_g(k_g, k_{-g}, \theta)$ with respect to k_g. We refer to the points k' that satisfy (6) as weak uncertain Clarke Nash equilibrium points. The points k^L that satisfy (7) are referred to as discrete approximate weak uncertain Clarke Nash equilibrium points.

To simplify the notation, let's denote

$$\prod_g \partial_{k_g} C_g(k_g, k_{-g}, \theta) := \partial_{k_1} C_1(k_1, k_{-1}, \theta) \times \cdots \times \partial_{k_g} C_g(k_g, k_{-g}, \theta)$$

and

$$\prod_g \mathcal{O}_{K_g}(k_g) = \mathcal{O}_{K_1}(k_1) \times \cdots \times \mathcal{O}_{K_g}(k_g).$$

Let

$$E\big[\prod_g \partial_{k_g} C_g(k_g, k_{-g}, \theta)\big] := E\big[\partial_{k_1} C_1(k_1, k_{-1}, \theta)\big] \times \cdots \times E\big[\partial_{k_g} C_g(k_g, k_{-g}, \theta)\big],$$

then equation (6) can be formulated as follows:

$$0 \in E\big[\prod_g \partial_{k_g} C_g(k_g, k_{-g}, \theta)\big] + \prod_g \mathcal{O}_{K_g}(k_g). \tag{8}$$

If $\{k^L\}$ is a sequence of approximate clarke Nash equilibrium points for problem (5), then its accumulation point represent a weak uncertain clarke Nash equilibrium point for the true problem (4). In some cases, it is possible to obtain an uncertain Nash equilibrium when solving the discrete approximation problem, i.e., $\{k^L\}$ serves as an uncertain Nash equilibrium for (5). To substantiate this claim, we present the following theorem for proof.

Theorem 5.1. *Consider k^L as a solution to (7). Assuming the following conditions hold: (1) $C(\cdot, k_{-g}, \theta)$ is Lipschitz continuous on K_g with modulus $\hbar_g(\theta)$, where $\hbar : \Omega \to \mathcal{R}^+$ is a measurable and integrable function satisfying $E[\hbar_g(\theta)] < \infty$. (2) $\partial_{k_g} C_g(k_g, k_{-g}, \theta)$ is closed for (k_g, k_{-g}) in the space $K_g \times K_{-g}$, then the sequence $\{k^L\}$ possesses a bounded subsequence that lies within a compact subset $\mathscr{K}$ of K. A limit point from this subsequence satisfies (6).*

Proof. Taking into account the case where $\{k^L\}$ is contained in $\mathscr{K}$. Since $\partial_{k_g} C_g(k_g, k_{-g}, \theta)$ is closed for (k_g, k_{-g}) in the space $K_g \times K_{-g}$, it following that $\partial_{\hat{k}_g} C_g(\hat{k}_g, \hat{k}_{-g}, \theta)$ contains all its limit points. In other words,

$$\limsup_{k \to \hat{k}} \partial_{k_g} C_g(k_g, k_{-g}, \theta) \subset \partial_{\hat{k}_g} C_g(\hat{k}_g, \hat{k}_{-g}, \theta).$$

By utilizing the Definition 2.1 of outer semi-continuity, we conclude that $\partial_{k_g} C_g(k_g, k_{-g}, \theta)$ is outer semi-continuous. Consequently, $\prod_g \partial_{k_g} C_g(k_g, k_{-g}, \theta)$ is also outer semi-continuous on K. Based on assumption (2), we have

$$\big\| \prod_g \partial_{k_g} C_g(k_g, k_{-g}, \theta) \big\| \le \sum_g \hbar_g(\theta),$$

where $E\big[\sum_g \hbar_g(\theta)\big] < \infty$.

For $\forall k \in \mathscr{K}$, $\|C_g(k_g, k_{-g}, \theta)\| \le \hbar_g(\theta)$, which implies $\|\text{clconv} C(k_g, k_{-g}, \theta)\| \le \hbar_g(\theta)$. In other words, $E[\|\text{clconv} C(k_g, k_{-g}, \theta)\|] \le E[\hbar_g(\theta)] < \infty$. For $\forall k, \hat{k} \in \mathscr{K}$ and $\theta \in \Omega$, we can obtain from (3) that

$$\begin{aligned}
&\mathcal{D}\left(\text{clconv} C(\hat{k}_g, \hat{k}_{-g}, \theta), \text{clconv} C(k_g, k_{-g}, \theta)\right) \\
&\le \mathcal{D}\left(\text{clconv} C(\hat{k}_g, \hat{k}_{-g}, \theta) + \text{clconv} C(k_g, k_{-g}, \theta), N(\hat{k}_g, \hat{k}_{-g}, \theta) + C(k_g, k_{-g}, \theta)\right) \\
&\le \mathcal{D}\left(\text{clconv} C(\hat{k}_g, \hat{k}_{-g}, \theta), C(\hat{k}_g, \hat{k}_{-g}, \theta)\right) + \mathcal{D}\left(\text{clconv} C(k_g, k_{-g}, \theta), C(k_g, k_{-g}, \theta)\right) \\
&\le \hbar_g(\theta) + \hbar_g(\theta) \\
&\le 2\hbar_g(\theta).
\end{aligned}$$

Consider a monotonically decreasing sequence $_x \to 0$. Let $\mathscr{S}(k, _x)$ denote a ball centered at k with a radius of $_x$. Based on Definition 2.4, we have

$$\begin{aligned}&\lim_{x\to\infty}\int_\Omega \sup_{\hat{k}\in\mathscr{S}(k,x)} \mathcal{D}\left(\text{clconv}C(\hat{k}_g,\hat{k}_{-g},\theta),\text{clconv}C(k_g,k_{-g},\theta)\right)d\Phi(\theta)\\&=\int_\Omega \lim_{x\to\infty}\sup_{\hat{k}\in\mathscr{S}(k,x)} \mathcal{D}\left(\text{clconv}C(\hat{k}_g,\hat{k}_{-g},\theta),\text{clconv}C(k_g,k_{-g},\theta)\right)d\Phi(\theta).\end{aligned}$$

Since $C(\cdot,k_{-g},\theta)$ and $\text{clconv}C(\cdot,k_{-g},\theta)$ is outer semi-continuous at K_g and point k. For all $\theta\in\Omega$, $\sup_{\hat{k}\in\mathscr{S}(k,x)} \mathcal{D}\left(\text{clconv}C(\hat{k}_g,\hat{k}_{-g},\theta),\text{clconv}C(k_g,k_{-g},\theta)\right)\to 0$, we have

$$\lim_{x\to\infty}\int_\Omega \sup_{\hat{k}\in\mathscr{S}(k,x)} \mathcal{D}\left(\text{clconv}C(\hat{k}_g,\hat{k}_{-g},\theta),\text{clconv}C(k_g,k_{-g},\theta)\right)d\Phi(\theta)=0. \tag{9}$$

According to (3), we can conclude the following for any $k,\hat{k}\in\mathscr{K}$:

$$\begin{aligned}&\mathcal{D}\big(\frac{1}{L}\sum_{l=1}^{L}C_g(\hat{k}_g,\hat{k}_{-g},\theta^l),\frac{1}{L}\sum_{l=1}^{L}C_g(k_g,k_{-g},\theta^l)\big)\\&=\frac{1}{L}\big[\mathcal{D}\big(C_g(\hat{k}_g,\hat{k}_{-g},\theta^1)+\cdots+C_g(\hat{k}_g,\hat{k}_{-g},\theta^L),C_g(k_g,k_{-g},\theta^1)+\cdots+C_g(k_g,k_{-g},\theta^L)\big)\big]\\&\le\frac{1}{L}\big[\mathcal{D}\big(C_g(\hat{k}_g,\hat{k}_{-g},\theta^1),C_g(k_g,,k_{-g},\theta^1)\big)+\cdots+\mathcal{D}\big(C_g(\hat{k}_g,\hat{k}_{-g},\theta^L),C_g(k_g,k_{-g},\theta^L)\big)\big]\\&=\frac{1}{L}\sum_{l=1}^{L}\mathcal{D}\big(C_g(\hat{k}_g,\hat{k}_{-g},\theta^l),C_g(k_g,k_{-g},\theta^l)\big),\end{aligned}$$

therefore, we have

$$\begin{aligned}&\sup_{\hat{k}\in\mathscr{S}(k,x)}\mathcal{D}\left(\frac{1}{L}\sum_{l=1}^{L}C_g(\hat{k}_g,\hat{k}_{-g},\theta^l),\frac{1}{L}\sum_{l=1}^{L}C_g(k_g,k_{-g},\theta^l)\right)\\&\le\frac{1}{L}\sum_{l=1}^{L}\sup_{\hat{k}\in\mathscr{S}(k,x)}\mathcal{D}\left(C_g(\hat{k}_g,\hat{k}_{-g},\theta^l),C_g(k_g,k_{-g},\theta^l)\right).\end{aligned}\tag{10}$$

Using the method of discrete approximation, we can derive the following:

$$\begin{aligned}&\lim_{x\to\infty}\frac{1}{L}\sum_{l=1}^{L}\sup_{\hat{k}\in\mathscr{S}(k,x)}\mathcal{D}\left(C_g(\hat{k}_g,\hat{k}_{-g},\theta^l),C_g(k_g,k_{-g},\theta^l)\right)\\&=E\big[\sup_{\hat{k}\in\mathscr{S}(k,x)}\mathcal{D}\big(C_g(\hat{k}_g,\hat{k}_{-g},\theta),C_g(k_g,k_{-g},\theta)\big)\big].\end{aligned}\tag{11}$$

With an increased level of discretization, we can observe that for any given $\epsilon>0$, the following inequality holds based on equations (9), (10), and (11):

$$\sup_{\hat{k}\in\mathscr{S}(k,x)}\mathcal{D}\left(\frac{1}{L}\sum_{l=1}^{L}C_g(\hat{k}_g,\hat{k}_{-g},\theta^l),\frac{1}{L}\sum_{l=1}^{L}C_g(k_g,k_{-g},\theta^l)\right)\le\epsilon.$$

Since $\mathscr{K}$ is a compact set, there exist a finite set of points $k_g\in\mathscr{K}$, each with its own neighborhood $\mathscr{S}(k_g,{}'_x)$, such that $\mathscr{K}\subset\bigcup_g\mathscr{S}(k_g,{}'_x)$. For $\forall\theta\in\Omega$, there exists an integer L' such that when $L\ge L'$ and $r'\le r$, the following inequality holds:

$$\sup_{\hat{k}\in\mathscr{S}(k_g,{}'_x)}\mathcal{D}\left(\frac{1}{L}\sum_{l=1}^{L}C_g(\hat{k}_g,\hat{k}_{-g},\theta^l),\frac{1}{L}\sum_{l=1}^{L}C_g(k_g,k_{-g},\theta^l)\right)\le\epsilon.$$

Therefore, for each point k_g when $L \geq L'$, the following inequality holds:

$$\mathcal{D}\big(\frac{1}{L}\sum_{l=1}^{L} C_g(k_g, k_{-g}, \theta^l), \frac{1}{L}\sum_{l=1}^{L} C_g(k_g, k_{-g}, \theta^l)\big) \leq \epsilon.$$

For any point $k \in \mathscr{K}$, there exists $k \in \mathscr{S}(k_g, ')$ for some g, and we have

$$\begin{aligned}
&\mathcal{D}\big(\frac{1}{L}\sum_{l=1}^{L} C_g(k_g, k_{-g}, \theta^l), E\big[\bigcup_{\hat{k}\in\mathscr{S}(k,)} \mathrm{clconv} C(\hat{k}_g, \hat{k}_{-g}, \theta)\big]\big) \\
&\leq \mathcal{D}\big(\frac{1}{L}\sum_{l=1}^{L} C_g(k_g, k_{-g}, \theta^l), \frac{1}{L}\sum_{l=1}^{L} C_g(k_g, k_{-g}, \theta^l)\big) \\
&+ \mathcal{D}\big(\frac{1}{L}\sum_{l=1}^{L} C_g(k_g, k_{-g}, \theta^l), E\big[\mathrm{clconv} C(k_g, k_{-g}, \theta)\big]\big) \\
&+ \mathcal{D}\big(E\big[\mathrm{clconv} C(k_g, k_{-g}, \theta)\big], E\big[\bigcup_{\hat{k}\in\mathscr{S}(k,)} \mathrm{clconv} C(\hat{k}_g, \hat{k}_{-g}, \theta)\big]\big) \\
&\leq 2\epsilon.
\end{aligned}$$

Since ϵ can be arbitrarily small, we can conclude that:

$$\lim_{L\to\infty} \frac{1}{L}\sum_{l=1}^{L} C_g(k_g, k_{-g}, \theta^l) \subset E\big[\bigcup_{\hat{k}\in\mathscr{S}(k,)} \mathrm{clconv} C(\hat{k}_g, \hat{k}_{-g}, \theta)\big]. \tag{12}$$

According to (12), consider any fixed positive value for . For $k \in \mathscr{K}$, the product $\prod_g \left[\frac{1}{L}\sum_{l=1}^{L} \partial_{k_g} C_g(k_g, k_{-g}, \theta^l)\right]$ over the compact set $\mathscr{K}$ satisfies the following:

$$\lim_{L\to\infty} \prod_g \big[\frac{1}{L}\sum_{l=1}^{L} \partial_{k_g} C_g(k_g, k_{-g}, \theta^l)\big] \subset E\big[\bigcup_{\hat{k}\in\mathscr{S}(k,)} \mathrm{clconv} \prod_g \partial_{k_g} C_g(\hat{k}_g, \hat{k}_{-g}, \theta)\big].$$

Let k' be an accumulation point of $\{K^L\}$, and assume (if necessary, by considering a subsequence) that $\{K^L\} \to k'$. Utilizing the properties of $\mathcal{D}$ described in (1), we obtain:

$$\begin{aligned}
&\mathcal{D}\big(\prod_g \big[\frac{1}{L}\sum_{l=1}^{L} \partial_{k_g^L} C_g(k_g^L, k_{-g}^L, \theta^l)\big], E\big[\bigcup_{k'\in\mathscr{S}(k,)} \mathrm{clconv} \prod_g \partial_{k'_g} C_g(k'_g, k'_{-g}, \theta)\big]\big) \\
&\leq \mathcal{D}\big(\prod_g \big[\frac{1}{L}\sum_{l=1}^{L} \partial_{k_g^L} C_g(k_g^L, k_{-g}^L, \theta^l)\big], E\big[\bigcup_{k^L\in\mathscr{S}(k,)} \mathrm{clconv} \prod_g \partial_{k_g^L} C_g(k_g^L, k_{-g}^L, \theta)\big]\big) \\
&+ \mathcal{D}\big(E\big[\bigcup_{k^L\in\mathscr{S}(k,)} \mathrm{clconv} \prod_g \partial_{k_g^L} C_g(k_g^L, k_{-g}^L, \theta)\big], E\big[\bigcup_{k'\in\mathscr{S}(k,)} \mathrm{clconv} \prod_g \partial_{k'_g} C_g(k'_g, k'_{-g}, \theta)\big]\big).
\end{aligned}$$

From equation (12), it is evident that as $L \to \infty$, the first term on the right-hand side of the inequality approaches 0. Furthermore, based on the convergence of $\{K^L\} \to k'$, we can conclude that the second term also tends to 0. In other words,

$$\lim_{L\to\infty} \prod_g \big[\frac{1}{L}\sum_{l=1}^{L} \partial_{k_g^L} C_g(k_g^L, k_{-g}^L, \theta^l)\big] \subset E\big[\bigcup_{k'\in\mathscr{S}(k,)} \mathrm{clconv} \prod_g \partial_{k'_g} C_g(k'_g, k'_{-g}, \theta)\big]. \tag{13}$$

Since k' is an uncertain clarke Nash stationary point, it satisfies (6). The clarke normal cone $\mathcal{O}_{K_g}(k_g)$ is outer semi-continuous, which implies that $\prod_g \mathcal{O}_{K_g}(k_g)$ is also outer semi-continuous. From (13), we can deduce that k' satisfies

$$0 \in E\big[\bigcup_{k' \in \mathscr{S}(k,)} \text{clconv} \prod_g \partial_{k'_g} C_g(k'_g, k'_{-g}, \theta)\big] + \prod_g \mathcal{O}_{K_g}(k'_g). \tag{14}$$

Considering that $\bigcup_{k' \in \mathscr{S}(k,)} \prod_g \partial_{k'_g} C_g(k'_g, k'_{-g}, \theta)$ is bounded and integrable by $\sum_g \hbar_g(\theta)$, and since

$$\lim_{r \to 0} \bigcup_{k' \in \mathscr{S}(k,)} \prod_g \partial_{k'_g} C_g(k'_g, k'_{-g}, \theta) = \prod_g \partial_{k'_g} C_g(k'_g, k'_{-g}, \theta),$$

as per the definition of uniform integrability, we have $\| \bigcup_{k' \in \mathscr{S}(k,)} \prod_g \partial_{k'_g} C_g(k'_g, k'_{-g}, \theta)\|$ is uniformly integrable.

Moreover, since $\| \bigcup_{k' \in \mathscr{S}(k,)} \prod_g \partial_{k'_g} C_g(k'_g, k'_{-g}, \theta)\|$ is closed, we can apply Definition 2.3, resulting in:

$$\lim_{r \to 0} E\big[\bigcup_{k' \in \mathscr{S}(k,)} \prod_g \partial_{k'_g} C_g(k'_g, k'_{-g}, \theta)\big] = E\big[\lim_{r \to 0} \bigcup_{k' \in \mathscr{S}(k,)} \prod_g \partial_{k'_g} C_g(k'_g, k'_{-g}, \theta)\big]$$
$$= E\big[\prod_g \partial_{k'_g} C_g(k'_g, k'_{-g}, \theta)\big].$$

According to (14), this implies that k' satisfies (8), thereby demonstrating the convergence of weakly uncertain clarke Nash equilibrium points. □

The following theorem establishes the exponential convergence of the discrete approximately stationary sequence $\{k^L\}$ to the solution K' of (4), without requiring the condition of the metric regularity type.

Assume the following hold:

(a) The mapping $C(\cdot, k_{-g}, \theta)$ is Lipschitz continuous on K_g with a modulus $\hbar_g(\theta)$, where $E[\hbar_g(\theta)] < \infty$.

(b) $E[(C_g)_{k_g}(k_g, k_{-g}, \theta, v_g)]$ is continuous on $\mathscr{K}$.

(c) Define $f_g(\theta) \equiv \hbar_g(\theta) + \ell_g(\theta)$, where $\hbar_g$ represent the moment generating function of $f_g(\theta)$, and $E[e^{tf_g(\theta)}]$ is finite for t close to 0.

(d) $(C_g)_{k_g}(k_g, k_{-g}, \theta, v_g)$ has a modulus $\ell_g(\theta)$ and order δ on $\mathscr{K}$, for $\hat{k} \in \mathscr{K}$ with $\|\hat{k} - k\| \leq$, satisfying $|\wp(\hat{k}, \theta) - \wp(k, \theta)| \leq \ell_g(\theta)\|\hat{k} - k\|^{\delta}$, where $\wp$ is a real-valued function.

Theorem 5.2. *Let $\mathscr{K} \subset K$ be a non-empty compact subset of K, and K' be a set of weak clarke Nash equilibria of the true problem (4) within $\mathscr{K}$. Assuming that for a sufficiently large $\hat{L}$, the sequence $\{k^L\}_{L > \hat{L}}$ lies within $\mathscr{K}$, under assumptions (a)-(d), the sequence $\{k^L\}$ converges to K' at an exponential rate. In other words, for any small positive $\epsilon > 0$, there exist positive integers ρ and σ independent of K, such that when L is sufficiently large, we have: $U\{dist(k^L, K') \geq \epsilon\} \leq \rho \cdot e^{-\mathscr{M}\sigma}$.*

Proof. See [10] for detailed proof. □

6. **Conclusion**

Based on an uncertain Nash equilibrium model, this paper investigates the game strategies of users, power grid companies, and proxy electricity purchasers in a power system. We develop a model based on the uncertain load of the users and derive the profit function for the three parties. Finally, the uncertain Nash equilibrium solutions are obtaine, and its convergence is analyzed. Theorem 5.1 provides proof of convergence for weakly uncertain Clarke Nash equilibrium points. Furthermore, Theorem 5.2 demonstrates the convergence rate of weakly uncertain Clarke Nash equilibrium points, indicating that the discrete approximation sequence $\{k^L\}$ converges exponentially to the solution K'. This research is critical in providing risk management and decision support tools to assist participants in power systems in making informed decisions, mitigating risks, and optimizing system operations in uncertain environments.

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REFERENCES

[1] *C. Alasseur, L. Campi, R. Dumitrescu and J. Zeng*, MFG model with a long-lived penalty at random jump times: application to demand side management for electricity contracts, Ann. Oper. Res., **336**(2024), 541-569.

[2] *M. Aryanezhad*, A robust game-theoretic optimization model for battery energy storage in multi-microgrids by considering of renewable based DGs uncertainty, Electr. Pow. Syst. Res., **204**(2022), Art. No. 107591.

[3] *M. Banaei, M. O. Buygi and H. R. Sheybani*, Supply function Nash equilibrium of joint day-ahead electricity markets and forward contracts, In. J. Elect. Power, **113**(2019), 104-116.

[4] *Z. Fu, Y. Lin, D. Yang and S. Yang*, Fractional fourier transforms meet Riesz potentials and image processing, SIAM J. Imaging Sci., **17**(2024), 476-500.

[5] *F. Gao F and G. B. Sheble*, Electricity market equilibrium model with resource constraint and transmission congestion, Elect. Pow. Syst. Res., **80**(2010), 9-18.

[6] *O. Hernndez-Lerma and J. B. Lasserre*, Fatou's lemma and Lebesgue's convergence theorem for measures, Int. J. Stochastic Anal., **13**(2000), 137-146.

[7] *F. Hiai*, Convergence of conditional expectations and strong laws of large numbers for multivalued random variables, T. Am. Math. Soc., **291**(1985), 613-627.

[8] *Q. Huang, Y. Xu and C. A. Courcoubetis*, Strategic storage operation in wholesale electricity markets: A networked cournot game analysis, IEEE Trans. Netw. Sci. Eng., **8**(2021), 1789-1801.

[9] *Z. Jia*, Global boundedness of weak solutions for an attraction-repulsion chemotaxis system with p-Laplacian diffusion and nonlinear production, Discrete and Continuous Dynamical Systems-Series B, **28**(2023), 4847-4863.

[10] *C. Li, T. Zhao and H. M. Yee*, Convergence rate analysis of a class of uncertain variational inequalities, IEEE Access, **11**(2023), 57772-57782.

[11] *B. Liu*, Uncertainty theory, M. Springer Berlin Heidelberg, pages 131-175, 2010.

[12] *G. Liu, X. Chen, Z. Shen, Y. Liu and X. Jia*, Reachable set estimation for continuous delayed singularly perturbed systems with bounded disturbances, Appl. Math. Comput., **416**(2022), Article number 126751.

[13] *L. Mitridati, J. Kazempour and P. Pinson P*, Design and game-theoretic analysis of community-based market mechanisms in heat and electricity systems, Omega, **99**(2021), Art. No. 102177.

[14] *M. Moafi, R. R. Ardeshiri, M. W. Mudiyanselage, et al.*, Optimal coalition formation and maximum profit allocation for distributed energy resources in smart grids based on cooperative game theory, Int. J. Elect. Power, **144**(2023), Art. No. 108492.

[15] *H. Qiao, Z. Zhang and Q. Su*, The optimal hourly electricity price considering wind electricity uncertainty based on conditional value at risk, Int. J. Green Energy, **18**(2021), 512-524.

[16] *Z. Qin, Y. Mo, H. Liu and Y. Zhang*, Operational flexibility enhancements using mobile energy storage in day-ahead electricity market by game-theoretic approach, Energy, **232**(2021), Art. No. 121008.

[17] *H. Saeian, T. Niknam, M. Zare and J. Aghaei*, Coordinated optimal bidding strategies methods of aggregated microgrids: A game theory-based demand side management under an electricity market environment, Energy, **245**(2022), Art. No. 123205.

[18] *S. Shi, Z. Fu and Q. Wu*, On the average operators, oscillatory integrals, singulars, singular integrals and their applications, J. Appl. Anal. Comput., **14**(2024), 334-378.

[19] *S. Shi, Z. Zhai and L. Zhang*, Characterizations of the viscosity solution of a non-local and nonlinear equation induced by the fractional p-Laplace and the fractional p-convexity, Adv. Calc. Var., **17**(2024), 195-207.

[20] *S. Shi and L. Zhang*, Dual characterization of fractional capacity via solution of fractional p-Laplace equation, Math. Nach., **293**(2020), 2233-2247.

[21] *Q. Sun, J. Ren and F. Zhao*, Sliding mode control of discrete-time interval type-2 fuzzy Markov jump systems with the preview target signal, Appl. Math. Comput., **435**(2022), Article number 127479.

[22] *X. S. Wang, Z. Q. Wang and Z. Jia*, Global weak solutions for an attraction-repulsion chemotaxis system with p-Laplacian diffusion and logistic source, Acta Math. Sci., **44**(2024), 909-924.

[23] *X. Wang, H. Zhao, H. Lu, et al.*, Decentralized coordinated operation model of VPP and P2H systems based on stochastic-bargaining game considering multiple uncertainties and carbon cost, Appl. Energy, **312**(2022), Art. No. 118750.

[24] *C. Wu, W. Gu, Z. Yi, et al.*, Non-cooperative differential game and feedback Nash equilibrium analysis for real-time electricity markets, Int. J. Elect. Power, **144**(2023), Art. No. 108561.

[25] *Y. Xia, Q. Xu, S. Tao, et al.*, Preserving operation privacy of peer-to-peer energy transaction based on Enhanced Benders Decomposition considering uncertainty of renewable energy generations, Energy, **250**(2022), Art. No. 123567.

[26] *M. R. Xu, S. Liu and Y. Lou*, Persistence and extinction in the anti-symmetric Lotka-Volterra systems, J. Differ. Equations, **387**(2024), 299-323.

[27] *Y. M. Xue, J. K. Han, Z. Q. Tu and X. Y. Chen*, Stability analysis and design of cooperative control for linear delta operator system, AIMS Math., **8**(2023), 12671-12693.

[28] *N. Yang, T. Qin, L. Wu, et al.*, A multi-agent game based joint planning approach for electricity-gas integrated energy systems considering wind power uncertainty, Elect. Pow. Syst. Res., **204**(2022), Art. No. 107673.

[29] *Y. Yao, A. Adamu and Y. Shehu*, Strongly convergent inertial forward-backward-forward algorithm without on-line rule for variational inequalities, Acta Math. Sci., **44**(2024), 551-566.

[30] *Y. Yao, L.O. Jolaoso and Y. Shehu*, C-FISTA type projection algorithm for quasi-variational inequalities, Numerical Algorithms, https://doi.org/10.1007/s11075-024-01852-6.

[31] *D. Zhang and H. Xu*, Two-stage stochastic equilibrium problems with equilibrium constraints: modeling and numerical schemes, Optim., **62**(2013), 1627-1650.

[32] *L. J. Zhu, J. C. Yao and Y. Yao*, Approximating solutions of a split fixed point problem of demicontractive operators, Carpathian J. Math., **40**(2024), 195-206.

[33] *L. J. Zhu and Y. Yao*, Modified splitting algorithms for approximating solutions of split variational inclusions in Hilbert spaces, Fixed Point Theory, **25**(2024), 773-784.

[34] *Y. Zou, B. Huang, Z. Meng and W. Ren*, Continuous-time distributed Nash equilibrium seeking algorithms for non-cooperative constrained games, Automatica, **127**(2021), Art. No. 109535.