RATES OF CONVERGENCE FOR A CLASS OF GENERALIZED QUASI CONTRACTIVE MAPPINGS IN KOHLENBACH HYPERBOLIC SPACES

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This paper is a continuation to the study of generalized quasi contractive operators, essentially due to Akhtar et al. [A multi-step implicit iterative process for common fixed points of generalized C^q*-operators in convex metric spaces, Sci. Int., 25(4) (2013), 887-891], in spaces of nonpositive sectional curvature. We aim to establish results concerning convergence characteristics of the classical iterative algorithms such as Picard, Mann, Ishikawa and Xu-Noor iterative algorithms associated with the proposed class of generalized quasi contractive operators. Moreover, we adopt the concept introduced by Berinde [Comparing Krasnosel'skii and Mann iterative methods for Lipschitzian generalized pseudo-contractions, Int. Conference on Fixed Point Theory Appl., 15-26, Yokohama Publ., Yokohama, 2004.] for a comparison of the corresponding rates of convergence of these iterative algorithms in such setting of spaces. The results presented in this paper improve and extend some recent corresponding results in the literature.*

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1. Introduction

Fixed point theory (FPT) contributes significantly to the theory of nonlinear functional analysis. Iterative algorithms, with respect to various nonlinear mappings, are ubiquitous in FPT and have been successfully applied in the study of a variety of nonlinear phenomena. The theory of iterative construction of fixed points of a nonlinear mapping under suitable set of control conditions is coined as metric fixed point theory (MFPT). MFPT is a fascinating field of research and has emerged as a powerful tool to solve various nonlinear real world problems, such as Fredholm and Volterra integral equations, ordinary differential equations, partial differential equations and image processing. MFPT has its roots in the celebrated Banach Contraction Principle (BCP) which not only guarantees the existence of a unique fixed point of a contraction but also describes an approximant for the construction of such a unique fixed point. It is worth

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mentioning that the BCP also gives a geometric rate of convergence for the classical Picard iterative algorithm to the unique fixed point. The BCP is a frequently cited result in the whole theory of analysis and dominates FPT for the class of contractions.

It is worth mentioning that the simplicity and applicability of the BCP paved the way for developing a new class of mappings satisfying generalized contractive condition. Most of the generalizations of the BCP possess the same characteristics regarding the existence of a unique fixed point which can be constructed by the Picard iterative algorithm. However, there are certain contractive or nonexpansive type mappings for which the construction of fixed points is also possible via Krasnosel'skii [21], Mann [13, 23], Ishikawa [14], Sintunavarat and Pitea [28], Thakur et al. [29, 30, 31] and Xu-Noor [32] iterative algorithms. In MFPT, different iterative algorithms can be evaluated with respect to various characteristics, inter alia, convergence characteristics and rates of convergence. The later concept has its own importance in MFPT and therefore we adopt the concept introduced by Berinde [3] for a comparison of the rates of convergence of different iterative algorithms involving a nonlinear mapping.

Since a variety of problems corresponding to the real world nonlinear phenomena can be transformed into fixed point problems (FPP). Therefore, it is natural to study FPP associated with a class of mappings in a suitable nonlinear framework. The term nonlinear framework for FPT is referred as a metric space embedded with a "convex structure". It is remarked that the non-positively curved hyperbolic space, introduced by Kohlenbach [20], provides rich geometrical structures suitable for MFPT of various classes of mappings. For the results concerning MFPT in Kohlenbach hyperbolic spaces, see, for example, [8, 10, 15, 16, 17, 18, 19] and the references cited therein. We are, therefore, interested into iterative construction of fixed points of the class of quasi contractive mappings in Kohlenbach hyperbolic spaces. As a consequence, we establish results concerning rates of convergence associated with the modified Mann, Ishikawa and Xu-Noor iterative algorithms, involving the class of quasi contractive mappings, in comparison to the classical Picard iterative algorithm in Kohlenbach hyperbolic spaces.

2 Preliminaries

Throughout this paper, we work in the setting of hyperbolic spaces introduced by Kohlenbach [20] and hence the term Kohlenbach hyperbolic spaces as one can find different notions of hyperbolic spaces in the current literature, see [11, 12, 25, 26].

A Kohlenbach hyperbolic space X is a metric space (X, d) together with a convexity mapping $W: X^2 \times [0,1] \rightarrow X$ satisfying

$$(\mathrm{W1})\, d(u, W(x,y,\alpha)) \leq \alpha d(u,x) + (1-\alpha) d(u,y)$$
$$(\mathrm{W2})\, d(W(x,y,\alpha), W(x,y,\beta)) = |\alpha - \beta| d(x,y)$$
$$(\mathrm{W3})\, W(x,y,\alpha) = W(y,x,(1-\alpha))$$
$$(\mathrm{W4})\, d(W(x,z,\alpha), W(y,w,\alpha)) \leq \alpha d(x,y) + (1-\alpha) d(z,w)$$

for all $x,y,z,w \in X$ and $\alpha, \beta \in [0,1]$. A subset K of a hyperbolic space X is convex if $W(x,y,\alpha) \in K$ for all $x,y \in K$ and $\alpha \in [0,1]$. A hyperbolic space X is uniformly convex [22] if for all $u,x,y \in X$, $r > 0$ and $\varepsilon \in (0,2]$, there exists $\delta \in (0,1]$ such that $d\left(W(x,y,\frac{1}{2}),u\right) \leq (1-\delta) r$ whenever $d(x,u) \leq r, d(y,u) \leq r$ and $d(x,y) \geq r\varepsilon$.

A mapping $\eta: (0,\infty) \times (0,2] \to (0,1]$ providing such $\delta = \eta(r,\varepsilon)$ for given $r > 0$ and $\varepsilon \in (0,2]$ is called modulus of uniform convexity. For more on hyperbolic spaces, we refer the reader to [20, p.384].

We now recall some mappings satisfying generalized contractive condition. A mapping $T: X \to X$ is called:

(i) Zamfirescu mapping [33], if there exist real numbers a, b and c satisfying $a \in (0,1)$ and $b, c \in \left(0, \frac{1}{2}\right)$ such that for each pair of points x, y in X, we have

$$\begin{aligned} &(\mathrm{Z1})\, d(Tx,Ty) \leq a d(x,y) \\ &(\mathrm{Z2})\, d(Tx,Ty) \leq b[d(x,Tx) + d(y,Ty)] \\ &(\mathrm{Z3})\, d(Tx,Ty) \leq c[d(x,Ty) + d(y,Tx)]; \end{aligned} \tag{2.1}$$

(ii) C^q-mapping [7], if for some $h \in [0,1)$ and for all $x, y \in X$, we have

$$d(Tx,Ty) \leq h\max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}; \tag{2.2}$$

(iii) generalized contractive mapping [24], if for some $h \in [0,1)$ and for all $x, y \in X$, we have

$$d(Tx,Ty) \leq h\max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty) + d(y,Tx)\}. \tag{2.3}$$

(iv) generalized C^q-mapping [1], if for some $h \in [0,1)$ and for all $x, y \in X$, we have

$$d(Tx,Ty) \leq h\max\{d(x,y), d(x,Tx) + d(y,Ty), d(x,Ty) + d(y,Tx)\}. \tag{2.4}$$

Remark 2.1. It is evident from the above definitions that the class of mappings defined in (2.4) contains properly the corresponding classes of mappings defined in (2.1)-(2.3). However, the class of Zamfirescu mapping is one of the most studied class of contractive mappings. For more on contractive type mapping, we refer the reader to [6].

We now introduce different iterative algorithm, required in the sequel, in Kohlenbach hyperbolic spaces. Let $T: X \to X$ be a given mapping and $x_0 \in X$ be chosen arbitrarily, then the Picard, Mann, Ishikawa and Xu-Noor iterative algorithms be defined, respectively, as follows:

$$x_{n+1} = Tx_n, \tag{2.5}$$

$$x_{n+1} = W(Tx_n, x_n, \alpha_n), \tag{2.6}$$

where $\{\alpha_n\} \subset [0,1]$,

$$\begin{aligned} x_{n+1} &= W(Ty_n, x_n, \alpha_n) \\ y_n &= W(Tx_n, x_n, \beta_n), \end{aligned} \tag{2.7}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$,

$$\begin{aligned} x_{n+1} &= W(Ty_n, x_n, \alpha_n) \\ y_n &= W(Tz_n, x_n, \beta_n) \\ z_n &= W(Tx_n, x_n, \gamma_n), \end{aligned} \tag{2.8}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [0,1]$.

We now recall the concept introduced by Berinde [3] for a comparison of the rates of convergence of different iterative algorithms involving a nonlinear mapping.

Let $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ be two sequences of positive numbers that converge to a, b, respectively. Assume that the limit $l = \lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|}$, exists. If $l = 0$, then the sequence $\{a_n\}_{n=0}^{\infty}$ converges to a faster than $\{b_n\}_{n=0}^{\infty}$ to b. If $0 < l < \infty$, then we say that the two sequence $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ have the same rate of convergence. It is remarked that the results concerning rates of convergence associated with the classes of mappings defined in (2.1)-(2.3) have been established in [2, 4, 5, 27]. See, also, [9] and the references cited therein. We are now in a position to prove our main results.

3 Main Results

This section is devoted to establish the results concerning iterative construction of fixed points of the class of generalized C^q-mappings and consequent rates of convergence for the modified Mann, Ishikawa and Xu-Noor iterative algorithms in comparison to the classical Picard iterative algorithm in Kohlenbach hyperbolic spaces.

Theorem 3.1. Let K be a nonempty closed convex subset of a uniformly convex Kohlenbach hyperbolic space X and let $T: K \to K$ be a generalized C^q-mapping. Assume that $F(T)$, the set of fixed points of T, is nonempty and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ satisfies the following conditions:

(C1): $0 \leq \alpha_n < 1$;

(C2): $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then the iterative algorithms defined in (2.5) and (2.6) converges to a fixed point p of T provided that the iterative algorithms have same initial guess $x_0 \in K$. Moreover, iterative algorithm defined in (2.5) converges faster than (2.6) to the fixed point of T.

Proof. Since T is a generalized C^q-mapping, therefore, if $d(Tx, Ty) \leq h\{d(x, Tx) + d(y, Ty)\}$, then (2.4) becomes

$$d(Tx,Ty) \le h\{d(x,Tx) + d(y,x) + d(x,Tx) + d(Tx,Ty)\}.$$

So, we have

$$d(Tx,Ty) \le \frac{h}{1-h}\{d(x,y) + 2d(x,Tx)\}. \tag{3.1}$$

If $d(Tx,Ty) \le h\{d(x,Ty) + d(y,Tx)\}$, then (2.4) becomes

$$d(Tx,Ty) \le h\{d(x,Tx) + d(Tx,Ty) + d(y,x) + d(x,Tx)\}.$$

Again, we have $$d(Tx,Ty) \le \frac{h}{1-h}\{d(x,y) + 2d(x,Tx)\}.$$

Letting $\lambda = \max\left\{h, \frac{h}{1-h}\right\}$, the above estimate implies that

$$d(Tx,Ty) \le \lambda d(x,y) + 2\lambda d(x,Tx). \tag{3.2}$$

Similarly, we can calculate the following inequality

$$d(Tx,Ty) \le \lambda d(x,y) + 2\lambda d(y,Tx). \tag{3.3}$$

Let $p \in F(T)$, then it follows from the estimate (3.2) and the sequence (2.5) that

$$d(x_{n+1},p) = d(Tx_n,p) \le \lambda d(x_n,p).$$

Continuing in this fashion, we have

$$d(x_{n+1},p) \le \lambda^n d(x_0,p). \tag{3.4}$$

Since $\lambda \in [0,1)$, therefore, (3.4) implies that

$$\lim_{n\to\infty} d(x_{n+1},p) = 0. \tag{3.5}$$

Now utilizing the estimate (3.2) for the sequence (2.6), we get

$$\begin{aligned} d(x_{n+1},p) &\le \alpha_n d(Tx_n,p) + (1-\alpha_n)d(x_n,p) \\ &\le (1-(1-\lambda)\alpha_n)d(x_n,p). \end{aligned} \tag{3.6}$$

The estimate (3.6) inductively yields

$$d(x_{n+1},p) \le \prod_{k=1}^{n} (1-(1-\lambda)\alpha_k)d(x_0,p). \tag{3.7}$$

Making use of conditions (C1) and (C2), the estimate (3.7) implies that

$$\lim_{n\to\infty} d(x_{n+1},p) = 0. \tag{3.8}$$

Hence the convergence of iterative algorithms (2.5) and (2.6) follows from the estimates (3.5) and (3.8), respectively. In order to compare the rates of convergence of iterative algorithms (2.5) and (2.6), we let $a_n = \lambda^n$ and $b_n = \prod_{k=1}^{n} (1-(1-\lambda)\alpha_k)d(x_0,p)$.

Now, consider

$$\begin{aligned} (1-\lambda)\alpha_n &\le (1-\lambda) \\ -(1-\lambda) &\le -(1-\lambda)\alpha_n \\ 1-(1-\lambda) &\le 1-(1-\lambda)\alpha_n \\ \lambda &\le 1-(1-\lambda)\alpha_n \\ \frac{\lambda}{1-(1-\lambda)\alpha_n} &\le 1. \end{aligned}$$

Moreover $$\frac{\min\lambda}{\max[1-(1-\lambda)\alpha_n]} < 1.$$

Since $\prod_{k=1}^{n} \frac{\lambda^k}{[1-(1-\lambda)\alpha_k]} < \left(\frac{\min\lambda^k}{\max[1-(1-\lambda)\alpha_k]}\right)^n$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.

Hence (2.5) converges faster than (2.6) to the fixed point of T. ■

Theorem 3.2. Let K be a nonempty closed convex subset of a uniformly convex Kohlenbach hyperbolic space X and let $T: K \to K$ be a generalized C^q-mapping. Assume that $F(T)$, the set of fixed points of T, is nonempty and the sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ satisfy the following conditions:
(C1): $0 \leq \alpha_n, \beta_n < 1$;
(C2): $\sum_{n=0}^{\infty} \alpha_n = \infty$.
Then the iterative algorithms defined in (2.6) and (2.7) converges to a fixed point p of T provided that the iterative algorithms have same initial guess $x_0 \in K$. Moreover, iterative algorithm defined in (2.6) converges faster than (2.7) to the fixed point of T.

Proof. Note that the convergence of (2.6) has already been established in Theorem 3.1. It remains to establish the convergence of (2.7) involving the class of generalized C^q-mapping. For this, we proceed with the following estimate:

$$d(x_{n+1}, p) \leq \alpha_n d(Ty_n, p) + (1-\alpha_n) d(x_n, p).$$

On using (3.2), we get

$$d(x_{n+1}, p) \leq \alpha_n \lambda d(y_n, p) + (1-\alpha_n) d(x_n, p). \tag{3.9}$$

Consider

$$d(y_n, p) \leq \beta_n d(Tx_n, p) + (1-\beta_n) d(x_n, p).$$

Again, using (3.2), we get

$$\begin{aligned} d(y_n, p) &\leq \beta_n \lambda d(x_n, p) + (1-\beta_n) d(x_n, p) \\ &= [\beta_n \lambda + (1-\beta_n)] d(x_n, p). \end{aligned}$$

Substituting the above estimate in (3.9), we have

$$\begin{aligned} d(x_{n+1}, p) &\leq \alpha_n \lambda \beta_n \lambda + (1-\beta_n)] d(x_n, p) + (1-\alpha_n) d(x_n, p) \\ &= [\alpha_n \beta_n \lambda^2 + \alpha_n \lambda (1-\beta_n) + (1-\alpha_n)] d(x_n, p) \\ &= [1 - \alpha_n (1 - \lambda + \beta_n \lambda - \beta_n \lambda^2)] d(x_n, p) \\ &= [1 - \alpha_n ((1-\lambda) + \beta_n \lambda (1-\lambda))] d(x_n, p) \\ &= [1 - \alpha_n (1-\lambda)(1+\beta_n \lambda)] d(x_n, p). \end{aligned} \tag{3.10}$$

Consider

$$\begin{aligned} 1 - \lambda &\leq 1 + \beta_n \lambda \\ \alpha_n (1-\lambda)(1-\lambda) &\leq \alpha_n (1-\lambda)(1+\beta_n \lambda) \\ -\alpha_n (1-\lambda)(1+\beta_n \lambda) &\leq -\alpha_n (1-\lambda)(1-\lambda) \\ 1 - \alpha_n (1-\lambda)(1+\beta_n \lambda) &\leq 1 - \alpha_n (1-\lambda)^2. \end{aligned}$$

Utilizing the above assertion, the estimate (3.10) implies that

$$d(x_{n+1}, p) \leq 1 - \alpha_n (1-\lambda)^2] d(x_n, p). \tag{3.11}$$

Continuing in this fashion, we have

$$d(x_{n+1}, p) \leq \prod_{k=1}^{n} [1 - \alpha_k (1-\lambda)^2] d(x_0, p).$$

Using the fact that $\lambda \in [0,1)$ and conditions (C1)-(C2), we get

$$\lim_{n \to \infty} d(x_{n+1}, p) = 0. \tag{3.12}$$

The estimate (3.12) implies that the iterative algorithm (2.7) converges to the fixed point of T. In order to compare the rates of convergence of (2.6) and (2.7), we must compare $a_n = \prod_{k=1}^{n}[1-\alpha_k(1-\lambda)]$ and $b_n = \prod_{k=1}^{n}[1-\alpha_k(1-\lambda)^2]$. For this, we reason as follow:

$$\alpha_k(1-\lambda)(1-\lambda) \leq \alpha_k(1-\lambda)$$
$$-\alpha_k(1-\lambda) \leq -\alpha_k(1-\lambda)^2$$
$$1-\alpha_k(1-\lambda) \leq 1-\alpha_k(1-\lambda)^2$$
$$\frac{1-\alpha_k(1-\lambda)}{1-\alpha_k(1-\lambda)^2} \leq 1.$$

Also
$$\frac{\min\{1-\alpha_k(1-\lambda)\}}{\max\{1-\alpha_k(1-\lambda)^2\}} < 1.$$

Since $\prod_{k=1}^{n}\frac{[1-\alpha_k(1-\lambda)]}{[1-\alpha_k(1-\lambda)^2]} < \left(\frac{\min\{1-\alpha_k(1-\lambda)\}}{\max\{1-\alpha_k(1-\lambda)^2\}}\right)^n$, then $\lim_{n\to\infty}\frac{a_n}{b_n} = 0.$

Hence (2.6) converges faster than (2.7) to the fixed point of T. ∎

Theorem 3.3. Let K be a nonempty closed convex subset of a uniformly convex Kohlenbach hyperbolic space X and let $T: K \to K$ be a generalized C^q-mapping. Assume that $F(T)$, the set of fixed points of T, is nonempty and the sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ satisfy the following conditions:
(C1): $0 \leq \alpha_n, \beta_n, \gamma_n < 1$;
(C2): $\sum_{n=0}^{\infty}\alpha_n = \infty$.
Then the iterative algorithms defined in (2.7) and (2.8) converges to a fixed point p of T provided that the iterative algorithms have same initial guess $x_0 \in K$. Moreover, iterative algorithm defined in (2.7) converges faster than (2.8) to the fixed point of T.

Proof. Note that the convergence of (2.7) has already been established in Theorem 3.2. It remains to establish the convergence of (2.8) involving the class of generalized C^q-mapping. For this, we proceed with the following estimates:

$$\begin{aligned} d(z_n,p) &= d(W(Tx_n,x_n,\gamma_n),p) \\ &\leq \gamma_n d(p,Tx_n) + (1-\gamma_n)d(x_n,p) \\ &\leq (1-\gamma_n(1-\lambda))d(x_n,p) \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} d(y_n,p) &= d(W(Tz_n,x_n,\beta_n),p) \\ &\leq \beta_n d(p,Tz_n) + (1-\beta_n)d(x_n,p) \\ &\leq \beta_n\lambda d(z_n,p) + (1-\beta_n)d(x_n,p). \end{aligned} \tag{3.14}$$

Substituting (3.13) in (3.14), we have

$$d(y_n,p) \leq \beta_n\lambda[(1-\gamma_n(1-\lambda))d(x_n,p)] + (1-\beta_n)d(x_n,p). \tag{3.15}$$

Moreover

$$\begin{aligned} d(x_{n+1},p) &= d(W(Ty_n,x_n,\alpha_n),p) \\ &\leq \alpha_n d(p,Ty_n) + (1-\alpha_n)d(x_n,p) \\ &\leq \alpha_n\lambda d(y_n,p) + (1-\alpha_n)d(x_n,p). \end{aligned} \tag{3.16}$$

Substituting (3.15) in (3.16), we get

$$d(x_{n+1},p) \leq [\alpha_n\lambda\{\beta_n\lambda(1-\gamma_n(1-\lambda))+1-\beta_n\}+(1-\alpha_n)]d(x_n,p)$$

$$=\{1-\alpha_n[1-\beta_n\lambda^2(1-\gamma_n+\gamma_n\lambda)-\lambda(1-\beta_n)]\}d(x_n,p)$$

$$=\{1-\alpha_n[1-\beta_n\lambda^2+\beta_n\gamma_n\lambda^2-\beta_n\gamma_n\lambda^3-\lambda+\beta_n\lambda]\}d(x_n,p)$$

$$=\{1-\alpha_n[1-\lambda+(1-\lambda)(\beta_n\lambda+\beta_n\gamma_n\lambda^2)]\}d(x_n,p)$$
$$=\{1-\alpha_n(1-\lambda)[1+\beta_n\gamma_n\lambda^2+\beta_n\lambda]\}d(x_n,p)$$
$$\leq \{1-\alpha_n(1-\lambda)\}d(x_n,p).$$

Making use of conditions (C1) and (C2), the above estimate implies that

$$\lim\ d(x_{n+1},p)=0. \tag{3.17}$$

Now we use the estimate (3.3) for the iterative algorithm (2.8) to get the following estimates:

$$\begin{aligned} d(z_n,p) &= d(W(Tx_n,x_n,\gamma_n),p) \\ &\leq \gamma_n d(Tx_n,p)+(1-\gamma_n)d(x_n,p) \\ &\leq 3\lambda\gamma_n d(x_n,p)+(1-\gamma_n)d(x_n,p) \\ &= [3\lambda\gamma_n+(1-\gamma_n)]d(x_n,p), \end{aligned} \tag{3.18}$$

and

$$d(y_n,p) \leq 3\lambda\beta_n d(z_n,p)+(1-\beta_n)d(x_n,p). \tag{3.19}$$

Substituting (3.18) in (3.19), we get

$$d(y_n,p) \leq 3\lambda\beta_n[3\lambda\gamma_n+(1-\gamma_n)+(1-\beta_n)]d(x_n,p). \tag{3.20}$$

Now, consider

$$d(x_{n+1},p) \leq 3\lambda\alpha_n d(y_n,p)+(1-\alpha_n)d(x_n,p). \tag{3.21}$$

Substituting (3.20) in (3.21) and then simplifying the terms, we have

$$\begin{aligned} d(x_{n+1},p) &\leq [1-\alpha_n(1-3\lambda)\{1+9\lambda^2\beta_n\gamma_n+3\lambda\beta_n\}]d(x_n,p) \\ &\leq [1-\alpha_n(1-3\lambda)]d(x_n,p). \end{aligned} \tag{3.22}$$

Again, making use of conditions (C1) and (C2), the above estimate implies that

$$\lim\ d(x_{n+1},p)=0. \tag{3.23}$$

The estimate (3.23) implies that the iterative algorithm (2.8) converges to the fixed point of T. In order to compare the rates of convergence of (2.7) and (2.8), we must compare $a_n=\prod_{k=1}^{n}[1-\alpha_k(1-\lambda)^2]$ and $b_n=\prod_{k=1}^{n}[1-\alpha_k(1-3\lambda)]$. For this, we have the following two cases:

Case (I). Let $\lambda\in[0,\frac{1}{3}]$, then $a_n\leq 1$ and $b_n=1$, therefore, we have

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)=0.$$

Case (II). Let $\lambda\in(\frac{1}{3},1)$, then again $a_n\leq 1$ and

$$b_n=\prod_{k=1}^{n}[1-\alpha_k(1-3\lambda)\{1+9\lambda^2\beta_k\gamma_k+3\lambda\beta_k\}]\geq 1.$$

So $$\frac{a_n}{b_n} = \prod_{k=1}^{n} \left[\frac{1-\alpha_k(1-\lambda)^2}{1-\alpha_k(1-3\lambda)\{1+9\lambda^2\beta_k\gamma_k+3\lambda\beta_k\}}\right] \leq 1.$$

Consequently

$$\frac{\min\{1-\alpha_k(1-\lambda)^2\}}{\max\{1-\alpha_k(1-3\lambda)\{1+9\lambda^2\beta_k\gamma_k+3\lambda\beta_k\}\}} < 1.$$

Since $\prod_{k=1}^{n} \left[\frac{1-\alpha_k(1-\lambda)^2}{1-\alpha_k(1-3\lambda)\{1+9\lambda^2\beta_k\gamma_k+3\lambda\beta_k\}}\right] < \left(\frac{\min\{1-\alpha_k(1-\lambda)^2\}}{\max\{1-\alpha_k(1-3\lambda)\{1+9\lambda^2\beta_k\gamma_k+3\lambda\beta_k\}\}}\right)^n,$

therefore, we get $$\lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

This implies that, in both cases, (2.7) converges faster than (2.8) to the fixed point of T. ■

Remark 3.4. As an applications of Theorems (3.1)-(3.3), we can establish similar kind of results for the classes of mappings defined in (2.1)-(2.3) in Kohlenbach hyperbolic spaces. As a consequence, our results generalize the corresponding results from linear spaces to more general setup of spaces.

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