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TSENG SPLITTING ALGORITHM FOR MONOTONE INCLUSION AND VARIATIONAL INEQUALITY PROBLEMS

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This paper aims to investigate a new forward-backward algorithm for solving a pseudomonotone variational inequality problem and a monotone inclusion problem in real Hilbert spaces. Under very mild conditions, we prove a weak convergence theorem for the proposed algorithm by using projection technique and self-adaptive step sizes. The results improve and extend the corresponding ones announced by some others in the earlier and recent literature.

Keywords: forward-backward algorithm, monotone inclusion, variational inequality, Hilbert spaces

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1. Introduction

Throughout this paper, let H be a real Hilbert spaces with inner product $\langle\cdot,\cdot\rangle$, and C be a nonempty closed convex subset of H. Fix(T) is denoted as the set of fixed points of a nonlinear mapping T. We use $x_n \to x$ and $x_n \rightharpoonup x$ to indicate the strong convergence and the weak convergence of the sequence $\{x_n\}$ to x, respectively.

First, we recall some notations which are needed in sequel. A mapping $F: H \to H$ is called

(a) monotone if

$$\langle Fx - Fy, x - y\rangle \geq 0, \quad \forall x, y \in H,$$

(b) pseudomonotone if

$$\langle Fx, y - x\rangle \geq 0 \Rightarrow \langle Fy, y - x\rangle \geq 0, \quad \forall x, y \in H,$$

(c) η-strongly monotone if there exists $\eta > 0$ such that

$$\langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in H,$$

(d) L-Lipschitz continuous if there is $L > 0$ such that

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$$\|Fx - Fy\| \leq L\|x - y\|, \quad \forall x, y \in H.$$

It is easy to see that (c) ⇒ (a) ⇒ (b), but the converse is not true.

A multi-valued mapping $B : D(B) \subseteq H \to 2^H$ is called monotone if, for all $x, y \in D(B)$, $u \in Bx$ and $v \in By$ such that

$$\langle x-y, u-v\rangle \geq 0.$$

A monotone mapping B is maximal if the Graph(B) is not properly contained in the graph of any other monotone mapping. It is well known that a monotone mapping B is maximal if and only if for $(x, u) \in D(B) \times H, \langle x - y, u - v\rangle \geq 0$ for every$(y, v) \in$Graph(B) implies that $u \in Bx$.

For every point $x \in H$, there exists a unique nearest point in C denoted by $P_C x$ such that

$$\|x-P_C x\| \leq \|x-y\|, \quad \forall y \in C.$$

P_C is called the metric projection of H onto C. It is known that P_C is nonexpansive mapping and satisfies the following inequalities:

$$\langle x-P_C x, y-P_C x\rangle \leq 0, \quad \forall x \in H, y \in C.$$

Given a nonlinear mapping $F : H \to H$, the variational inequality problem (VIP) is to find $u \in C$ such that

$$\langle Fu, v-u\rangle \geq 0, \quad \forall v \in C, \tag{1}$$

the set of solutions of the VIP (1) is denoted by $VI(C, F)$.There are several different approaches towards solving this problem infinite dimensional and infinite dimensional spaces; see e.g., [1, 2, 3].

The monotone inclusion problem (MIP) is to find $x \in H$ such that

$$0 \in (A + B)x, \tag{2}$$

where $A : H \to H$ is a single-valued mapping and $B: H \to 2^H$ is a set-valued mapping. The solution set of MIP (2) is denoted by $\Omega := (A+B)^{-1}(0)$. The monotone inclusion problem has already been used in convex minimization problems, variational inequalities and equilibrium problems, and is also at the core of the modeling of machine learning, signal processing and image restoration, see [4, 5, 6].

In 1979, Lions et al. [7] introduced the forward-backward algorithm for MIP (2) by the following way

$$x_{n+1} = (I + \lambda_n B)^{-1}(I - \lambda_n A)x_n, \tag{3}$$

where the mapping A, B are $1/L$-co-coercive and maximally monotone, respectively, $(I-\lambda_n A)$ is called a forward operator and $(I+\lambda_n B)^{-1}$ is a backward operator.

Based on the forward-backward algorithm (3), Tseng [8] proposes a modified algorithm which is known as Tseng splitting algorithm:

$$y_n = (I + \lambda_n B)^{-1}(I - \lambda_n A)x_n,$$
$$x_{n+1} = y_n - \lambda_n(Ay_n - Ax_n), \tag{4}$$

where A is L-lipschitz continuous and $\lambda_n \in (0, 1/L)$. Under some mild restrictions on the parameters, they obtained a strong convergence theorem. The Tseng splitting algorithm has been widely studied in recent years, since many real-world problems such as signal processing and image reconstruction can be cast as the modeling, and many iterative algorithms and existence results based on Tseng splitting algorithm for the MIP have been studied; see e.g., [9, 10, 11].

Inertial method was fistly introduced by Alvarez et al. [12] which is designed as the following scheme:

$$x_{n+1} = x_n + \delta_n(x_n - x_{n-1}),$$

this procedure is a good tool to speeding up the convergence rate of algorithms. As a result, many researchers have studied all kind of algorithms by utilizing inertial methods for solving VIP (1) and MIP (2), see e.g., [13, 14, 15].

In 2018, Yang et al. [16] proposed the following inertial algorithm for the VIP (1):

$$y_n = x_n + \delta_n(x_n - x_{n-1}),$$
$$x_{n+1} = P_C(x_n - \lambda_n F y_n),$$

where F : C → H is monotone. Under some mild restrictions on the parame- ters, they obtained a weak convergence theorem.

Very recently, Inkrong et al. [17] studied a double inertial forward-backward algorithm for MIP (2), and they design the following scheme:

$$w_n = u_n + \alpha_n(u_n - u_{n-1}) + \beta_n(u_{n-1} - u_{n-2}),$$
$$y_n = (I + \delta_n G)^{-1}(I - \delta_n F)w_n,$$
$$u_{n+1} = y_n - \delta_n(Fy_n - Fw_n),$$

where

$$\delta_{n+1} = \begin{cases} min\{\dfrac{\mu\|w_n - y_n\|}{\|Fw_n - Fy_n\|}, \delta_n + \zeta_n\}, & \|Fw_n - Fy_n\| \neq 0, \\ \delta_n + \zeta_n & , \text{ otherwise;} \end{cases}$$

As a result, they proved a weak convergence of the algorithm above under appropriate assumptions on the parameters.

Motivated and inspired by the results above, we introduce a new Tseng splitting algorithm for solving a pseudomonotone variational inequality problem and a monotone inclusion problem. Under some suitable assumptions, a weak convergence of the proposed algorithm is proved by using projection technique and self-adaptive step sizes.

2. Preliminaries

In this section, we first recall some lemmas which are needed in sequel.

Lemma 2.1. ([18]) *Let H be a real Hilbert space. Then the following inequality holds:*

(i) $\|tx+(1-t)y\|^2=t\|x\|^2+(1-t)\|y\|^2-t(1-t)\|x-y\|^2, \quad \forall t\in[0,1], x, y\in H$;
(*ii*) $\|x\pm y\|^2=\|x\|^2\pm 2\langle x , y\rangle+\|y\|^2, \quad \forall x, y\in H$.
Lemma 2.2. ([19]) *Assuming $A : H \to H$ is L-Lipschitz and monotone, and $B : H \to 2^H$ is a maximally monotone operator, it follows that the operator $A+B$ is also maximally monotone.*
Lemma 2.3. ([20]) *Let $A : H \to H$ be a mapping and $B : H \to 2^H$ be a maximally monotone mapping. Define $T_\lambda := (I + \lambda B)^{-1}(I-\lambda A)$* for $\lambda>0$. *Then* $\mathrm{Fix}(T_\lambda)=\{x : T_\lambda x =x\}=(A+B)^{-1}(0)$.
Lemma 2.4. ([21]) *Assume that C is a closed and convex subset of a real Hilbert space H. Let operator $F : H \to H$ be continuous and pseudomonotone. Then, x^* is a solution of* VIP (1) *if and only if $\langle Fx, x - x^*\rangle \geq 0, \forall x \in C$.*
Lemma 2.5. ([22]) *Let $\{\phi_n\}$, $\{\delta_n\}$ and $\{\alpha_n\}$ be sequences in* $[0, +\infty)$ *such that*

$$\langle Fx, x - x^*\rangle \geq 0, \forall x \in C.$$

and there exists a real number α with $0 \leq \alpha_n \leq \alpha < 1$ for all $n \in \mathbb{N}$. Then the following hold:
(i) $\sum_{n=1}^{\infty} [\varphi_n-\varphi_{n-1}]_+<\infty$ *where* $[t]_+:=\max\{t,0\}$;
(ii) there exists $\varphi^*\in[0,+\infty)$ *such that* $\lim_{n\to\infty}\varphi_n=\varphi^*$.
Lemma 2.6. ([23]) *Let $\{\psi_n\},\{\varphi_n\}$ and $\{b_n\}$ be nonnegative sequences that satisfy*

$$\psi_{n+1}\leq (1+b_n)\psi_n+\phi_n, \quad n\geq 1.$$

If $\sum_{n=1}^{\infty} b_n<+\infty$ *and* $\sum_{n=1}^{\infty}\phi_n<+\infty$ *, then* $\lim_{n\to\infty}\psi_n$ *exists.*
Lemma 2.7. ([24]) *Let Ω be a subset of H and $\{u_n\}$ be a sequence in H that satisfy the following:*
(i) for every $u\in\Omega$, $\lim_{n\to\infty}\|u_n-u\|$ exists;
(ii) each weak-cluster point of the sequence $\{u_n\}$ is in Ω. Then $\{u_n\}$ converges weakly to an element in Ω.
Lemma 2.8. ([17]) *Let ϕ_{-1}, $\phi_0 \geq 0$ and $\{\phi_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of nonnegative real numbers that satisfy the following conditions:*

$$\varphi_{n+1}\leq (1+\alpha_n)\varphi_n+(\alpha_n+\beta_n)\varphi_{n-1}+\beta_n\varphi_{n-2}, \quad n\in N.$$

Then $\varphi_{n+1}\leq K\prod_{j=1}^{n}(1+2\alpha_j+2\beta_j)$, *where* $K=\max\{\phi_{-1}, \phi_0, \phi_1\}$. *Furthermore, if* $\sum_{n=1}^{\infty}\alpha_n<+\infty$ *and* $\sum_{n=1}^{\infty}\beta_n<+\infty$ *, then $\{\phi_n\}$ is bounded.*

3. Main results

In this section, we introduce a new forward-backward algorithm with double inertial step for finding a common solution of a pseudomonotone variational inequality problem and a monotone inclusion problem in real Hilbert spaces. Subsequently, we give the main results about our algorithm. We firstly give the following assumptions:

(C1) The mappings $A : H \to H$ is L_1-Lipschitz continuous and monotone and $B : H \to 2^H$ is maximally monotone.

(C2) P_C is the metric projection of H, $F : H \to H$ is pseudo-monotone and L_2-Lipschitz continuous with $L_2 > 0$.

(C3) The common solution set of the VIP (1) and MIP (2) is nonempty, that is, $\Omega \cap VI(C, F) \neq \emptyset$.

Algorithm 1 Given $\mu, \rho \in (0, 1)$, $\alpha_1, \beta_1, \lambda_1, \xi_1, \tau_1, \theta_1 > 0$ and choose three arbitrary initial guesses $u_{-1}, u_0, u_1 \in H$. For $n \in \mathbb{N}$, let $\{u_n\}$ be a sequence of H generated by the following steps:

Step 1 Choose α_n, β_n and compute

$$w_n = u_n + \alpha_n(u_n - u_{n-1}) + \beta_n(u_{n-1} - u_{n-2}),$$
$$y_n = (I + \lambda_n B)^{-1}(I - \lambda_n A)w_n,$$

where

$$\lambda_{n+1} = \begin{cases} min\{\dfrac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|}, \lambda_n + \tau_n\}, \|Aw_n - Ay_n\| \neq 0, \\ \lambda_n + \tau_n \qquad , \text{ otherwise,} \end{cases}$$

Step 2 Calculate

$$z_n = P_C(w_n - \xi_n F w_n),$$

where

$$\xi_{n+1} = \begin{cases} min\{\dfrac{\rho\|u_n - z_n\|}{\|Fu_n - Fz_n\|}, \xi_n\}, \|Fu_n - Fz_n\| \neq 0, \\ \xi_n \qquad , \text{ otherwise,} \end{cases}$$

Step 3 Compute

$$u_{n+1} = \theta_n[y_n - \lambda_n(Ay_n - Aw_n)] + (1 - \theta_n)[z_n - \xi_n(Fz_n - Fw_n)].$$

Set $n := n+1$ and return to Step 1.

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\lambda_n\}$, $\{\xi_n\}$, $\{\tau_n\}$ and $\{\theta_n\}$ are parameters and the following conditions hold:

(i) $\alpha_n, \beta_n > 0, \sum_{n=1}^{\infty} \alpha_n < \infty, \sum_{n=1}^{\infty} \beta_n < \infty, \sum_{n=1}^{\infty} \tau_n < \infty$;

(ii) $0 < \theta < \theta_n < \frac{1}{1+\sigma}$, $\sigma \in (0, +\infty)$.

(iii)

Lemma 3.1. *The sequences* $\{\lambda_n\}$ *and* $\{\xi_n\}$ *from Algorithm 1 satisfy the following properties:*

(1) $\{\lambda_n\}$ *is bounded with* $\{\lambda_n\} \subset [\min\{\mu/L_1, \lambda_1\}, \lambda_1 + \Gamma]$ *and there exists* $\lambda > 0$ *such that* $\lim_{n\to\infty} \lambda_n = \lambda \in [\min\{\mu/L_1, \lambda_1\}, \lambda_1 + \Gamma]$, *where* $\Gamma = \sum_{n=1}^{\infty} \tau_n$.

(2) There exists $\xi > 0$ *such that* $\lim_{n\to\infty} \xi_n = \xi$ *and* $\xi_n \geq \min\{\rho/L_2, \xi_1\}$.

Proof. (1) First, by the definition of $\{\lambda_n\}$, if $Aw_n \neq Ay_n$, we have

$$\frac{\mu\|w_n-y_n\|}{\|Aw_n-Ay_n\|}\geq\frac{\mu\|w_n-y_n\|}{L_1\|w_n-y_n\|}=\frac{\mu}{L_1},$$

which implies

$$\lambda_{n+1}=\min\{\frac{\mu\|w_n-y_n\|}{\|Aw_n-Ay_n\|},\lambda_n+\tau_n\}\geq\min\{\frac{\mu}{L_1},\lambda_n\},$$

By induction, we obtain

$$\lambda_n\geq\min\{\frac{\mu}{L_1},\lambda_{n-1}\}\geq\min\{\frac{\mu}{L_1},\min\{\frac{\mu}{L_1},\lambda_{n-2}\}\}\geq\cdots\geq\min\{\frac{\mu}{L_1},\lambda_1\}. \tag{5}$$

Since $\Gamma=\sum_{n=1}^{\infty}\tau_n$, we have

$$\lambda_{n+1}\leq\lambda_n+\tau_n\leq\lambda_1+\sum_{n=1}^{\infty}\tau_n=\lambda_1+\Gamma. \tag{6}$$

In view of (5) and (6), we obtain

$$\min\{\frac{\mu}{L_1},\lambda_1\}\leq\lambda_n\leq\lambda_1+\Gamma,$$

which implies $\{\lambda_n\}$ is bounded. Setting $[\lambda_{n+1}-\lambda_n]_- := \max\{0, \lambda_n-\lambda_{n+1}\}$and $[\lambda_{n+1}-\lambda_n]_+ := \max\{0, \lambda_{n+1}-\lambda_n\}$, we deduce that

$$\lambda_{n+1}-\lambda_n=[\lambda_{n+1}-\lambda_n]_+-[\lambda_{n+1}-\lambda_n]_-,$$

by induction, we derive

$$\lambda_{n+1}-\lambda_1=\sum_{i=1}^{n}[\lambda_{i+1}-\lambda_i]_+-\sum_{i=1}^{n}[\lambda_{i+1}-\lambda_i]_-.$$

Since $\{\lambda_n\}$ is bounded and $\sum_{n=1}^{\infty}[\lambda_{n+1}-\lambda_n]_+<\sum_{n=1}^{\infty}\tau_n=\Gamma$, we have $\sum_{n=1}^{\infty}[\lambda_{n+1}-\lambda_n]_-$ is convergent. Thus, $\lambda_n<\lambda_1+\Gamma$. In addition, we note that

$$\lambda_{n+1}\leq\lambda_n+\tau_n=\lambda_n+0\,(\lambda_{n+1}-\lambda_n)+\tau_n.$$

By using Lemma 2.5 in the inequality above, there exists $\lambda\in[\min\{\mu/L_1, \lambda_1\}, \lambda_1+\Gamma]$ such that $\lim_{n\to\infty}\lambda_n=\lambda$.

(2) It follows from the definition of $\{\xi_{n+1}\}$ that $0\leq\xi_{n+1}\leq\xi_n$, which implies that there exists $\xi>0$ such that $\lim_{n\to\infty}\xi_n=\xi$. Since F is L_2-Lipschitz continuous, we get

$$\|Fw_n-Fy_n\|\leq L_2\|w_n-y_n\|,$$

which implies

$$\frac{\rho\|u_n-z_n\|}{\|Fw_n-Fy_n\|}\geq\frac{\rho}{L_2}\ \text{if } Fw_n\neq Fy_n,$$

it follows that $\xi_n\geq\min\{\xi_1,\frac{\rho}{L_2}\}$ and $\xi\geq\min\{\xi_1,\frac{\rho}{L_2}\}$. This completes the proof.

Lemma 3.2. *Assume that the sequence $\{u_n\}$ is generated by Algorithm 1. Then for all $p\in\Omega\cap VI(C,F)$, the following assertions hold:*

(1) $\|y_n-\lambda_n(Ay_n-Aw_n)-p\|^2\leq\|w_n-p\|^2-(1-\frac{\mu^2\lambda_n^2}{\lambda_{n+1}^2})\|w_n-y_n\|^2.$

(2) $\|z_n-\xi_n(Fz_n-Fw_n)-p\|^2 \leq \|w_n-p\|^2-(1-\frac{\rho^2\xi_n^2}{\xi_{n+1}^2})\|z_n-w_n\|^2$.

Proof. (1) First, we note that

$$\begin{aligned}
&\|y_n-\lambda_n(Ay_n-Aw_n)-p\|^2\\
&= \|y_n-p\|^2+\lambda_n^2\|Ay_n-Aw_n\|^2-2\lambda_n\langle y_n-p,Ay_n-Aw_n\rangle\\
&= \|y_n-w_n\|^2+\|w_n-p\|^2+2\langle y_n-w_n,w_n-p\rangle\\
&\quad+\lambda_n^2\|Ay_n-Aw_n\|^2-2\lambda_n\langle y_n-p,Ay_n-Aw_n\rangle\\
&= \|y_n-w_n\|^2+\|w_n-p\|^2-2\langle y_n-w_n, y_n-w_n\rangle+2\langle y_n-w_n, y_n-p\rangle\\
&\quad+\lambda_n^2\|Ay_n-Aw_n\|^2-2\lambda_n\langle y_n-p, Ay_n-Aw_n\rangle\\
&= \|w_n-p\|^2-\|y_n-w_n\|^2+2\langle y_n-w_n, y_n-p\rangle\\
&\quad+\lambda_n^2\|Ay_n-Aw_n\|^2-2\lambda_n\langle y_n-p, Ay_n-Aw_n\rangle\\
&= \|w_n-p\|^2-\|y_n-w_n\|^2-2\langle y_n-p, w_n-y_n-\lambda_n(Aw_n-Ay_n)\rangle\\
&\quad+\lambda_n^2\|Ay_n-Aw_n\|^2,
\end{aligned} \tag{7}$$

from the definition of $\{\lambda_n\}$, one has

$$\|Ay_n-Aw_n\|^2 \leq \frac{\mu^2}{\lambda_{n+1}^2}\|y_n-w_n\|^2. \tag{8}$$

In addition, since A is monotone, we have

$$\langle Ay_n-Ap, y_n-p\rangle\geq 0. \tag{9}$$

Moreover, since B is maximally monotone and $y_n=(I+\lambda_nB)^{-1}(I-\lambda_nA)w_n$, we deduce that

$$\frac{w_n-y_n-\lambda_nAw_n}{\lambda_n}\in By_n,$$

thus

$$\frac{w_n-y_n-\lambda_nAw_n}{\lambda_n}+Ay_n\in(A+B)y_n.$$

It follows from the monotonicity of A, B and Lemma 2.2 that $A+B$ is maximally monotone. Since $p\in \Omega \cap VI(C, F)\subset\Omega$, we deduce $0\in(A+B)p$ and $-Ap\in Bp$. We get from the maximal monotonicity of B that

$$\langle\frac{w_n-y_n-\lambda_nAw_n}{\lambda_n}+Ap, y_n-p\rangle\geq 0,$$

which together with (9), implies that

$$\langle\frac{w_n-y_n-\lambda_nAw_n}{\lambda_n}+Ay_n, y_n-p\rangle\geq 0,$$

that is,

$$\langle w_n-y_n-\lambda_n(Aw_n-Ay_n), y_n-p\rangle\geq 0. \quad (10)$$

It follows from (7), (8) and (10) that

$$\|y_n-\lambda_n(Ay_n-Aw_n)-p\|^2\leq\|w_n-p\|^2-(1-\frac{\mu^2\lambda_n^2}{\lambda_{n+1}^2})\|w_n-y_n\|^2.$$

(2) First, we claim

$$\|Fz_n-Fw_n\| \leq\frac{\rho}{\xi_{n+1}}\|z_n-w_n\|. \quad (11)$$

Indeed, if $Fz_n = Fw_n$, then (11) holds clearly. Otherwise, from the definition of $\{\xi_{n+1}\}$, we have

$$\xi_{n+1}= \min\{\frac{\rho\|z_n-w_n\|}{\|Fz_n-Fw_n\|}, \xi_n\} \leq \frac{\rho\|z_n-w_n\|}{\|Fz_n-Fw_n\|},$$

which implies

$$\|Fz_n-Fw_n\| \leq \frac{\rho}{\xi_{n+1}}\|z_n-w_n\|.$$

Next, we observe that

$$\begin{aligned}
&\|z_n-\xi_n(Fz_n-Fw_n)-p\|^2\\
&=\|z_n-p\|^2+\xi_n^2\|Fz_n-Fw_n\|^2-2\xi_n\langle z_n-p,Fz_n-Fw_n\rangle\\
&=\|w_n-p\|^2+\|z_n-w_n\|^2+2\langle z_n-w_n,w_n-p\rangle\\
&\quad+\xi_n^2\|Fz_n-Fw_n\|^2-2\xi_n\langle z_n-p,Fz_n-Fw_n\rangle\\
&=\|w_n-p\|^2+\|z_n-w_n\|^2-2\langle z_n-w_n,z_n-w_n\rangle+2\langle z_n-w_n,z_n-p\rangle\\
&\quad+\xi_n^2\|Fz_n-Fw_n\|^2-2\xi_n\langle z_n-p,Fz_n-Fw_n\rangle\\
&=\|w_n-p\|^2-\|z_n-w_n\|^2+2\langle z_n-w_n,z_n-p\rangle\\
&\quad+\xi_n^2\|Fz_n-Fw_n\|^2-2\xi_n\langle z_n-p,Fz_n-Fw_n\rangle. \quad (12)
\end{aligned}$$

Since $z_n =P_C(w_n - \xi_nFw_n)$, by the property of projection, we deduce

$$\langle z_n-w_n+\xi_nFw_n, z_n-p\rangle=\langle P_C(w_n-\xi_nFw_n)-(w_n-\xi_nFw_n), P_C(w_n-\xi_nFw_n)-p\rangle\leq 0,$$

that is,

$$\langle z_n-w_n, z_n-p\rangle\leq-\xi_n\langle Fw_n, z_n-p\rangle. \quad (13)$$

By $p\in VI(C, F)$, we get $\langle Fp, y_n-p\rangle\geq 0$. It follows from the pseudomono- tonicity of F that $\langle Fz_n, z_n-p\rangle\geq 0$, this together with (11), (12) and (13), yields that

$$\begin{aligned}
&\|z_n-\xi_n(Fz_n-Fw_n)-p\|^2\\
&=\|w_n-p\|^2-\|z_n-w_n\|^2+2\langle z_n-w_n, z_n-p\rangle\\
&\quad+\xi_n^2\|Fz_n-Fw_n\|^2-2\xi_n\langle z_n-p, Fz_n-Fw_n\rangle\\
&\leq\|w_n-p\|^2-\|z_n-w_n\|^2-2\xi_n\langle Fw_n, z_n-p\rangle\\
&\quad+\frac{\rho^2\xi_n^2}{\xi_{n+1}^2}\|w_n-z_n\|^2-2\xi_n\langle z_n-p, Fz_n-Fw_n\rangle
\end{aligned}$$

$$=\|w_n-p\|^2-(1-\frac{\rho^2\xi_n^2}{\xi_{n+1}^2})\|w_n-z_n\|^2-2\xi_n\langle z_n-p, Fz_n\rangle$$
$$\leq\|w_n-p\|^2-(1-\frac{\rho^2\xi_n^2}{\xi_{n+1}^2})\|w_n-z_n\|^2.$$

This completes the proof.

Theorem 3.1. *Assume that the sequence {un} is generated by Algorithm 1. Assume that assumptions* (C1)-(C3) *and conditions (i), (ii) hold. Then* $\{u_n\}$ *converges weakly to a point in* $\Omega\cap VI(C, F)$.

Proof. Take $p\in\Omega\cap VI(C, F)$. It follows from the definition of y_n that $(I-\lambda_nA)w_n\in(I+\lambda_nB)y_n$. Since B is maximally monotone, there exists $v_n\in By_n$ with $(I-\lambda_nA)w_n=y_n+\lambda_nv_n$, that is,

$$v_n=\frac{1}{\lambda_n}(w_n-y_n-\lambda_nAw_n). \tag{14}$$

Moreover, we have $0\in(A+B)p$ and $Ay_n+v_n\in(A+B)y_n$. By the monotonicity of A, B and Lemma 2.2, we have $A+B$ is maximally monotone, which implies that

$$\langle Ay_n+v_n, y_n-p\rangle\geq 0. \tag{15}$$

In view of (14) and (15), we have

$$\frac{1}{\lambda_n}\langle w_n-y_n-\lambda_nAw_n+\lambda_nAy_n, y_n-p\rangle\geq0.$$

In addition, by using Lemma 3.2, we obtain

$$\|u_{n+1}-p\|^2=\|\theta_n[y_n-\lambda_n(Ay_n-Aw_n)]+(1-\theta_n)[z_n-\xi_n(Fz_n-Fw_n)]-p\|^2$$
$$=\|\theta_n[y_n-\lambda_n(Ay_n-Aw_n)-p]+(1-\theta_n)[z_n-\xi_n(Fz_n-Fw_n)-p]\|^2$$
$$\leq\theta_n\|y_n-\lambda_n(Ay_n-Aw_n)-p\|^2+(1-\theta_n)\|z_n-\xi_n(Fz_n-Fw_n)-p\|^2$$
$$\leq\theta_n(\|w_n-p\|^2-(1-\frac{\mu^2\lambda_n^2}{\lambda_{n+1}^2})\|w_n-y_n\|^2)$$
$$+(1-\theta_n)(\|w_n-p\|^2-(1-\frac{\rho^2\xi_n^2}{\xi_{n+1}^2})\|z_n-w_n\|^2)$$
$$=\|w_n-p\|^2-\theta_n(1-\frac{\mu^2\lambda_n^2}{\lambda_{n+1}^2})\|w_n-y_n\|^2-(1-\theta_n)(1-\frac{\rho^2\xi_n^2}{\xi_{n+1}^2})\|z_n-w_n\|^2 \tag{16}$$

On the other hand, we observe that

$$\|w_n-p\|=\|u_n+\alpha_n(u_n-u_{n-1})+\beta_n(u_{n-1}-u_{n-2})-p\|$$
$$\leq\|u_n-p\|+\alpha_n\|u_n-u_{n-1}\|+\beta_n\|u_{n-1}-u_{n-2}\|. \tag{17}$$

From Lemma 3.1, there exist λ, ξ such that $\lim_{n\to\infty}\lambda_n=\lambda$, $\lim_{n\to\infty}\xi_n=\xi$, it follows that $\lim_{n\to\infty}(1-\frac{\mu^2\lambda_n^2}{\lambda_{n+1}^2})=1-\mu^2>0$ and $\lim_{n\to\infty}(1-\frac{\rho^2\xi_n^2}{\xi_{n+1}^2})=1-\rho^2>0$. Thus, there is $n_0\in\mathbb{N}$ such that

$1-\frac{\mu^2\lambda_n^2}{\lambda_{n+1}^2}>0$ and $1-\frac{\rho^2\xi_n^2}{\xi_{n+1}^2}>0$, $\forall n\geq n_0$, which together with (16) and (17), implies that, for $n \geq n_0$,

$$\begin{aligned}\|u_{n+1}-p\|&\leq\|w_n-p\|\\ &\leq\|u_n-p\|+\alpha_n\|u_n-u_{n-1}\|+\beta_n\|u_{n-1}-u_{n-2}\|\\ &\leq\|u_n-p\|+\alpha_n(\|u_n-p\|+\|u_{n-1}-p\|)+\beta_n(\|u_{n-1}-p\|+\|u_{n-2}-p\|)\\ &=(1+\alpha_n)\|u_n-p\|+(\alpha_n+\beta_n)\|u_{n-1}-p\|+\beta_n\|u_{n-2}-p\|. \qquad (18)\end{aligned}$$

Using Lemma 2.8, we obtain

$$\|u_{n+1}-p\| \leq K\prod_{j=1}^{n}(1+2\alpha_j+2\beta_j),\ \ n\geq n_0,$$

and $\|u_n-p\|$ is bounded, where $K=\max\{\|u_{n_0-2}-p\|, \|u_{n_0-1}-p\|, \|u_{n_0}-p\|\}$.

Next, we prove that u_n converges weakly to a point in $\Omega\cap VI(C,F)$. It follows from condition (i) that $\sum_{n=1}^{\infty}\alpha_n\|u_n-u_{n-1}\|<+\infty$ and $\sum_{n=1}^{\infty}\beta_n\|u_{n-1}-u_{n-2}\|<+\infty$. Set $\psi_n:=\|u_n-p\|$, $\phi_n:=\alpha_n\|u_n-u_{n-1}\|+\beta_n\|u_{n-1}-u_{n-2}\|$. Then, we have $\sum_{n=1}^{\infty}\phi_n<\infty$. From (18), we get

$$\|u_{n+1}-p\| \leq \|u_n-p\|+\alpha_n\|u_n-u_{n-1}\|+\beta_n\|u_{n-1}-u_{n-2}\|,$$

That is,

$$\psi_{n+1} \leq \psi_n+\phi_n.$$

Applying Lemma 2.6 in the inequality above, we deduce that $\lim_{n\to\infty}\psi_n$ exists, i.e., $\lim_{n\to\infty}\|u_n-p\|$ exists.

Now, we note that

$$\begin{aligned}\|w_n-p\|^2 &= \|u_n+\alpha_n(u_n-u_{n-1})+\beta_n(u_{n-1}-u_{n-2})-p\|^2\\ &= \|(u_n-p)+\alpha_n(u_n-u_{n-1})+\beta_n(u_{n-1}-u_{n-2})\|^2\\ &= \|(u_n-p)+\alpha_n(u_n-u_{n-1})\|^2+\beta_n^2\|u_{n-1}-u_{n-2}\|^2\\ &\quad+2\langle u_n-p+\alpha_n(u_n-u_{n-1}), \beta_n(u_{n-1}-u_{n-2})\rangle\\ &= \|u_n-p\|^2+\alpha_n^2\|u_n-u_{n-1}\|^2+2\langle u_n-p, \alpha_n(u_n-u_{n-1})\rangle\\ &\quad+\beta_n^2\|u_{n-1}-u_{n-2}\|^2+2\langle u_n-p+\alpha_n(u_n-u_{n-1}), \beta_n(u_{n-1}-u_{n-2})\rangle\\ &= \|u_n-p\|^2+\alpha_n^2\|u_n-u_{n-1}\|^2+2\langle u_n-p, \alpha_n(u_n-u_{n-1})\\ &\quad+\beta_n^2\|u_{n-1}-u_{n-2}\|^2+2\langle u_n-p, \beta_n(u_{n-1}-u_{n-2})\rangle\\ &\quad+2\langle\alpha_n(u_n-u_{n-1}), \beta_n(u_{n-1}-u_{n-2})\rangle\\ &\leq \|u_n-p\|^2+\alpha_n^2\|u_n-u_{n-1}\|^2+2\alpha_n\|u_n-p\|\|u_n-u_{n-1}\|\\ &\quad+\beta_n^2\|u_{n-1}-u_{n-2}\|^2+2\beta_n\|u_n-p\|\|u_{n-1}-u_{n-2}\|\\ &\quad+2\alpha_n\beta_n\|u_n-u_{n-1}\|\|u_{n-1}-u_{n-2}\|. \qquad (19)\end{aligned}$$

Substituting (19) into (16), we deduce

$$\begin{aligned}\|u_{n+1}-p\|^2 \leq & \|u_n-p\|^2+\alpha_n^2\|u_n-u_{n-1}\|^2+2\alpha_n\|u_n-p\|\|u_n-u_{n-1}\| \\ & +\beta_n^2\|u_{n-1}-u_{n-2}\|^2+2\beta_n\|u_n-p\|\|u_{n-1}-u_{n-2}\|+2\alpha_n\beta_n\|u_n-u_{n-1}\|\|u_{n-1}-u_{n-2}\| \\ & -\theta_n(1-\frac{\mu^2\lambda_n^2}{\lambda_{n+1}^2})\|w_n-y_n\|^2-(1-\theta_n)(1-\frac{\rho^2\xi_n^2}{\xi_{n+1}^2})\|z_n-w_n\|^2,\end{aligned}$$

which means that

$$\begin{aligned}&\theta_n(1-\frac{\mu^2\lambda_n^2}{\lambda_{n+1}^2})\|w_n-y_n\|^2+(1-\theta_n)(1-\frac{\rho^2\xi_n^2}{\xi_{n+1}^2})\|z_n-w_n\|^2 \\ &\quad \leq (\|u_n-p\|^2-\|u_{n+1}-p\|^2)+\alpha_n^2\|u_n-u_{n-1}\|^2+\beta_n^2\|u_{n-1}-u_{n-2}\|^2+2\alpha_n\|u_n-p\|\|u_n-u_{n-1}\| \\ &\quad\quad +2\beta_n\|u_n-p\|\|u_{n-1}-u_{n-2}\|+2\alpha_n\beta_n\|u_n-u_{n-1}\|\|u_{n-1}-u_{n-2}\|. \qquad (20)\end{aligned}$$

Meanwhile, since $\lim_{n\to\infty}\| u_n-p\|$ exists. It follows from conditions (i), (ii) and (20) that

$$\lim_{n\to\infty}\|w_n-y_n\|=\lim_{n\to\infty}\|z_n-w_n\|=0. \qquad (21)$$

In addition, notice that

$$\begin{aligned}\|w_n-u_n\| &= \|u_n+\alpha_n(u_n-u_{n-1})+\beta_n(u_{n-1}-u_{n-2})-u_n\| \\ &\leq \alpha_n\|u_n-u_{n-1}\|+\beta_n\|u_{n-1}-u_{n-2}\|,\end{aligned}$$

by condition (i), we deduce that

$$\lim_{n\to\infty}\|w_n-u_n\|=0.$$

Let $(s, t)\in\text{Graph}(A+B)$, which implies that $t-As\in Bs$. For $\{n_k\}\subset\{n\}$, we get $y_{n_k}= (I+\lambda_{n_k}B)^{-1}(I-\lambda_{n_k}A)w_{n_k}$, which means $(I-\lambda_{n_k}A)w_{n_k}\in(I+\lambda_{n_k}B)y_{n_k}$. It follows that $\frac{1}{\lambda_{n_k}}(w_{n_k}-y_{n_k}-\lambda_{n_k}Aw_{n_k})\in By_{n_k}$. Since B is maximally monotone, we obtain that

$$\langle s-y_{n_k}, t-As-\frac{1}{\lambda_{n_k}}(w_{n_k}-y_{n_k}-\lambda_{n_k}Aw_{n_k})\rangle\geq 0.$$

and hence

$$\begin{aligned}\langle s-y_{n_k}, t\rangle &\geq \langle s-y_{n_k}, As+\frac{1}{\lambda_{n_k}}(w_{n_k}-y_{n_k}-\lambda_{n_k}Aw_{n_k})\rangle \\ &= \langle s-y_{n_k}, As-Aw_{n_k}\rangle+\langle s-y_{n_k}, \frac{1}{\lambda_{n_k}}(w_{n_k}-y_{n_k})\rangle \\ &= \langle s-y_{n_k}, As-Ay_{n_k}\rangle+\langle s-y_{n_k}, Ay_{n_k}-Aw_{n_k}\rangle+\langle s-y_{n_k}, \frac{1}{\lambda_{n_k}}(w_{n_k}-y_{n_k})\rangle \\ &\geq \langle s-y_{n_k}, Ay_{n_k}-Aw_{n_k}\rangle+\langle s-y_{n_k}, \frac{1}{\lambda_{n_k}}(w_{n_k}-y_{n_k})\rangle. \qquad (22)\end{aligned}$$

In light of (21) and the Lipschitz continuity of A, we deduce $\lim_{k\to\infty}\| Aw_{n_k}-Ay_{n_k}\| = 0$. Let $\hat{p}$ be a weak cluster point of $\{u_n\}$. Since $\{u_n\}$ is bounded, there is a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $u_{n_i}\rightharpoonup\hat{p}$. Furthermore, $y_{n_i}\rightharpoonup\hat{p}$. It follows from (22) that $\langle s-\hat{p}, t\rangle \geq 0$. Thus, by the maximal monotonicity of $A+B$, we have $0\in(A+B)(\hat{p})$, that is $\hat{p}\in(A+B)^{-1}(0)$.

On the other hand, let

$$f(x)=\begin{cases} F(x)+N_C(x), & x\in C, \\ \emptyset, & x\in H\setminus C, \end{cases}$$

where N_C is the normal cone of C at $x\in C$. Obviously, f is maximal monotone and $f^{1}(0) = VI(C, F)$. If $(x, r)\in Graph(f)$, since $r\in f(x) = F(x)+N_C(x)$, we have $r-F(x)\in N_C(x)$, which leads to

$$\langle r-F(x), x-v\rangle \geq 0, \ \forall v\in C. \tag{23}$$

Note that $z_n = P_C(w_n - \xi_n F w_n)$, we obtain

$$\langle w_n-\xi_n Fw_n-z_n, z_n-x\rangle \geq 0, \ \forall x\in C.$$

that is,

$$\langle \frac{z_n-w_n}{\xi_n}+Fw_n, x-z_n\rangle \geq 0, \ \forall x\in C. \tag{24}$$

Since $\lim_{n\to\infty} \| z_n-w_n\| = 0$, applying (23) with $\{z_{k_j}\}_{j=0}^{\infty}$, we have

$$\langle r-F(x), x-z_{k_j}\rangle \geq 0, \ \forall x\in C. \tag{25}$$

In view of (24) and (25), we get

$$\begin{aligned}
\langle r, x-z_{k_j}\rangle &\geq \langle Fx, x-z_{k_j}\rangle \\
&\geq \langle Fx, x-z_{k_j}\rangle-\langle \frac{z_{k_j}-w_{k_j}}{\xi_{k_j}}+Fw_{k_j}, x-z_{k_j}\rangle \\
&= \langle Fx-Fw_{k_j}, x-z_{k_j}\rangle-\langle \frac{z_{k_j}-w_{k_j}}{\xi_{k_j}}, x-z_{k_j}\rangle \\
&= \langle Fx-Fz_{k_j}, x-z_{k_j}\rangle+\langle Fz_{k_j}-Fw_{k_j}, x-z_{k_j}\rangle-\langle \frac{z_{k_j}-w_{k_j}}{\xi_{k_j}}, x-z_{k_j}\rangle \\
&\geq \langle Fz_{k_j}-Fw_{k_j}, x-z_{k_j}\rangle-\langle \frac{z_{k_j}-w_{k_j}}{\xi_{k_j}}, x-z_{k_j}\rangle.
\end{aligned}$$

Thus, $\langle r, x-z_{k_j}\rangle \geq 0$. Let $j\to\infty$, we obtain $\langle r, x-\hat{p}\rangle \geq 0$. Since f is maximal monotone, we deduce that $\hat{p}\in f^{1}(0) = VI(C, F)$.Therefore, $\hat{p}\in\Omega\cap VI(C, F)$. By Lemma 2.7, $\{u_n\}$ converges weakly to a point of $\Omega\cap VI(C, F)$. This completes the proof.

Remark 3.1. *Compared with Theorem 3.1 of Inkrong et al. [17], our Theorem 3.1 extends, improves and develops it in the following aspects:*
(i) Our iterative scheme is more general than it in [17]. Especially, an extragradient algorithm is added to construct our iteration process, which is not applied in [17].

(ii) The result in [17] can only be applied to solve a monotone inclusion problem, while our result can be applied to solve a pseudomonotone variational inequality problem and a monotone inclusion problem, which makes our result more applicable and valid.

4. Conclusions

In this paper, we introduce a new double inertial forward-backward algorithm with adaptive step size that does not depend on the knowledge of the Lipschitz constant and norms of the nonlinear operators to approximating a common solution of a monotone inclusion problem and a pseudomonotone variational inequality problem. Under some suitable assumptions on the parameters, we prove a weak convergence of our algorithm by using inertial technique, self-adaptive step sizes, and the properties of pseudomonotone mapping and monotone mapping.

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REFERENCES

[1] *A. Bnouhachem, M. Aslam Noor, Z. Hao*, Some new extragradient iterative methods for variational inequalities, Nonlinear Anal., **70** (2009), 1321-1329.
[2] *Y. Yao, Y.C. Liou, and R. Chen*, Convergence theorems for fixed point problems and variational inequality problems in Hilbert spaces, Math. Nachr., **282** (2009), 1827-1835.
[3] *G. Cai, S.Q. Bu*, Strong and weak convergence theorems for general mixed equilibrium problems and variational inequality problems and fixed point problems in Hilbert spaces, J. Comput. Appl. Math., **247** (2013), 34-52.
[4] *Y. Censor, T. Bortfeld, B. Martin, et al.*, A unified approach for inversion problems in intensity modulated radiation therapy, Phys. Med. Biol., **51** (2006), 2353-2365.
[5] *V.B. Cong*, A splitting algorithm for dual monotone inclusions involving cocoercive operators, Adv. Comput. Math., **2011** (2011), 1-15.
[6] *L. C. Ceng, L. J. Zhu, and T. C. Yin*, On generalized extragradient implicit method for systems of variational inequalities with constraints of variational inclusion and ffxed point problems, Open Mathematics, **20** (2022), 1770-1784.
[7] *R.I. Bot, R.C. Erno*, An inertial forward-backward-forward primal-dual splitting algorithm for solving monotone inclusion problems, Numer. Algor., **71** (2014), 1-22.
[8] *P. Tseng*, A modified forward-backward splitting method for maximal monotone mapping, SIAM J. Control. Optim., **38** (2000), 431-446.

[9] *D.V. Thong, P. Cholamjiak*, Strong convergence of a forward-backward splitting method with a new step size for solving monotone inclusions, Comput. Appl. Math., Series A, **38** (2019), 1-16.
[10] *H. Attouch, A. Cabot*, Convergence of a Relaxed Inertial Forward–Backward Algorithm for Structured Monotone Inclusions, Appl. Math. Optimization, **4** (2019), 1-52.
[11] *S.S. Chang, C.F. Wen, J.C. Yao*, A generalized forward–backward splitting method for solving a system of quasi variational inclusions in Banach spaces, RACSAM, **2018** (2018), 1-19.
[12] *F. Alvarez, H. Attouch*, An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping, Set-Valued Anal., **9** (2001), 3-11.
[13] *P. Cholamjiak, D.V. Hieu, Y.J. Cho*, Relaxed forward-backward splitting methods for solving variational inclusions and applications, J. Sci. Comput., **88** 85 (2021).
[14] *P. Cholamjiak, D.V. Hieu, L.D. Muu*, Inertial splitting methods without prior constants for solving variational inclusions of two operators, Bull. Iran. Math. Soc., **48** (2022), 3019-3045.
[15] *Q. Dong, D. Jiang, P. Cholamjiak*, A strong convergence result involving an inertial forward-backward algorithm for monotone inclusions, J. Fixed Point Theory Appl., **19** (2017), 3097-3118.
[16] *J. Yang, H. Liu*, A Modified Projected Gradient Method for Monotone Variational Inequalities, J. Optimization Theory Appl., **179** (2018), 1-15.
[17] *P. Inkrong, P. Cholamjiak*, Modified proximal gradient methods involving double inertial extrapolations for monotone inclusion, Math. Meth. Appl. Sci., **47** (2024), 12132-12148.
[18] *W. Takahashi*, Introduction to nonlinear and convex analysis, Yokohama Publishers, (2009).
[19] *H. Brezis*, Operateurs maximaux monotones. Chapitre II, North-Holland Math. Stud., **5** (1973), 19-51.
[20] *A. Gibali, D.V. Thong*, Tseng type methods for solving inclusion problems and its applications, Calcolo, **55** 49 (2018).
[21] *R.W. Cottle, J.C. Yao*, Pseudo-monotone complementarity problems in Hilbert space, J. Optim. Theory Appl., **75** (1992), 281-295.
[22] *P.E. Mainge*, Convergence theorems for inertial KM-type algorithms, J. Comput. Appl. Math., **219** (2008), 223-236.
[23] *M.O. Osilike, S.C. Aniagbosor*, Weak and strong convergence theorems for fixed points of asymptotically nonexpensive mappings, Math. Comput. Model., **32** (2000), 1181-1191.
[24] *Z. Opial*, Weak convergence of the sequence of successive approximations for nonexpansive mappings, Bull. Am. Math. Soc., 73 (1967), **73** (1967), 591-597.