A MODIFIED HYBRID ALGORITHM FOR SOLVING PSEUDOMONOTONE EQUILIBRIUM PROBLEMS

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This work modifies a hybrid algorithm that combines the subgradient extragradient algorithm with the inertial technique to solve a pseudomonotone equilibrium problem with a Lipschitz-like condition in a real Hilbert space. A strong convergence theorem is established under certain mild conditions for the bifunction and the control parameters.

Keywords: pseudomonotone equilibrium problem, hybrid algorithm, inertial technique, process innovation.
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1. Introduction

Let C be a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}$. Suppose that $f : C \times C \to \mathbb{R}$ is a bifunction with $f(z, z) = 0$ for all $z \in C$. The equilibrium problem is stated as follows: find an element x^* in C such that

$$f(x^*, y) \geq 0, \tag{1}$$

for all $y \in C$. We denote by $EP(f, C)$ the solution set of the problem (1).

The equilibrium problem (1) finds versatile applications in solving a myriad of real-world problems, encompassing variational inequalities, split feasibility problems, minimization problems, linear programming problems, saddle point problems, and Nash equilibrium problems, among others, as extensively documented in references [1, 2, 16, 18].

In 2008, Tran et al. [12] introduced the two-step extragradient method (TSEM) as a solution approach for equilibrium problem (1). This method drew inspiration from the extragradient method [6], designed for solving variational inequalities. However, it is worth noting that the TSEM demonstrates weak convergence when applied in Hilbert spaces.

Recently, Cholamjiak and Suparatulatorn [4] proposed the modified inertial viscosity subgradient extragradient to obtain strong convergence for addressing the equilibrium problem (1) under the bifunction f is pseudomonotone and satisfies the Lipschit-type condition. Furthermore, the algorithm finds applications in solving problems associated with diabetes mellitus classification. Significant research has been undertaken in the domain of algorithm development aimed at solving equilibrium problems, with notable exemplars documented in references [10, 14, 17, 19].

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In 2003, Nakajo and Takahashi [8] demonstrated the strong convergence of the hybrid projection method for nonexpansive mappings in Hilbert spaces. Various other methods to address fixed point problems have been proposed by several authors, see also [3, 5, 11, 13, 15].

Inspired by the aforementioned studies, this paper introduces a novel extragradient-type algorithm designed to solve the equilibrium problem (1) in a real Hilbert space. Our proposed iterative approach takes into account the pseudo-monotonicity of the bifunction associated with the problem (1). Furthermore, we establish the strong convergence of the generated sequence under mild conditions and within a framework of appropriate iterative control parameters.

2. **Preliminaries**

In what follows, recall that $\mathcal{H}$ is a real Hilbert space. Let C be a nonempty, closed and convex subset of $\mathcal{H}$. We denote $\rightharpoonup$ and $\to$ as weak and strong convergence, respectively, and the notation $\omega_w(x_n)$ is the weak ω-limit set of the sequence $\{x_n\}$. We next collect some necessary definitions and lemmas for proving our main results. For $u \in \mathcal{H}$, define the metric projection P_C from $\mathcal{H}$ onto C by

$$P_C u := \arg\min_{v \in C} \|u - v\|.$$

A normal cone of C at $x \in C$ is defined by

$$N_C(x) = \{z \in \mathcal{H} : \langle z, y - x\rangle \leq 0, \text{ for all } y \in C\}.$$

Let $g : C \to \mathbb{R}$ be a convex function and subdifferential of g at $x \in C$ is defined by

$$\partial g(x) = \{z \in \mathcal{H} : g(y) - g(x) \geq \langle z, y - x\rangle, \text{ for all } y \in C\}.$$

A bifunction $f : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is said to be

(i) pseudomonotone on C if for all $u, v \in C$,

$$f(u, v) \geq 0 \Longrightarrow f(v, u) \leq 0;$$

(ii) to satisfy a Lipschitz-like condition on C if there exist two positive constants c_1, c_2 such that for all $u, v, w \in C$,

$$f(u, w) \leq f(u, v) + f(v, w) + c_1\|u - v\|^2 + c_2\|v - w\|^2.$$

Lemma 2.1. [9] *Let $g : C \to \mathbb{R}$ be a subdifferentiable, convex and lower semi-continuous function on C. Suppose C has nonempty interior, or g is continuous at a point $x \in C$. Then, x is a minimizer of g if and only if*

$$0 \in \partial g(x) + N_C(x).$$

Lemma 2.2. [8] *Let $x \in \mathcal{H}$ and $y \in C$. Then the following inequality holds:*

$$\|y - P_C x\|^2 + \|x - P_C x\|^2 \leq \|x - y\|^2.$$

Lemma 2.3. [7] *Let $\mathbb{V}$ be a closed and convex subset of $\mathcal{H}$, $\{x_n\} \subset \mathcal{H}$ and $v \in \mathcal{H}$. If $\omega_w(x_n) \subset \mathbb{V}$ and $\|x_n - v\| \leq \|v - P_{\mathbb{V}} v\|$ for all $n \in \mathbb{N}$. Then $x_n \to P_{\mathbb{V}} v$ as $n \to \infty$.*

3. **Main result**

To study the convergence analysis, consider the following conditions.
($\mathcal{C}$1) C has nonempty interior or $f(z,\cdot)$ is continuous at some point in C for every $z \in C$, f is pseudomonotone on C, and the solution set $EP(f,C)$ is nonempty;
($\mathcal{C}$2) f meet the Lipschitz-like condition on $\mathcal{H}$ through $c_1 > 0$ and $c_2 > 0$;
($\mathcal{C}$3) $f(z,\cdot)$ is convex, subdifferentiable and lower semicontinuous function on $\mathcal{H}$ for each fixed $z \in \mathcal{H}$;
($\mathcal{C}$4) $\limsup_{n\to\infty} f(z_n, y) \le f(z^*, y)$ for each $y \in C$ and $\{z_n\} \subset C$ satisfies $z_n \rightharpoonup z^*$.

Algorithm 3.1

Initialization: Let $C_1 = \mathcal{H}$. Select arbitrary elements $x_0, x_1 \in \mathcal{H}$ and set $n := 1$.
Iterative Steps: Construct $\{x_n\}$ by using the following steps:
Step 1. Set $\rho_n = x_n + \delta_n(x_n - x_{n-1})$, where $\{\delta_n\}$ is a bounded sequence and compute

$$y_n = \arg\min_{y\in C}\left\{\lambda_n f(\rho_n, y) + \frac{1}{2}\|\rho_n - y\|^2\right\},$$

where $0 < \lambda_n \le \lambda < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$. If $\rho_n = y_n$, then stop. Otherwise
Step 2. Compute

$$u_n = \arg\min_{y\in \mathcal{H}_n}\left\{\lambda_n f(y_n, y) + \frac{1}{2}\|\rho_n - y\|^2\right\},$$

where $w_n \in \partial_2 f(\rho_n, y_n)$ satisfying $\rho_n - \lambda_n w_n - y_n \in N_C(y_n)$ and construct a half-space

$$\mathcal{H}_n = \{z \in \mathcal{H} : \langle \rho_n - \lambda_n w_n - y_n, z - y_n\rangle \le 0\}.$$

Step 3. Compute

$$x_{n+1} = P_{C_{n+1}} x_1,$$

where

$$C_{n+1} = \{c \in C_n : \|u_n - c\|^2 + (1 - 2c_1\lambda_n)\|\rho_n - y_n\|^2 + (1 - 2c_2\lambda_n)\|y_n - u_n\|^2 \le \|\rho_n - c\|^2 + \eta_n\}$$

and $\eta_n \in [0,\infty)$.
Replace n by $n+1$ and then repeat **Step 1.**

Lemma 3.1. *Let $\rho_n = y_n$ in Algorithm 3.1, then $\rho_n \in EP(f,C)$.*

Proof. By the definition of y_n with Lemma 2.1, we have

$$0 \in \partial_2\left(\lambda_n f(\rho_n,\cdot) + \frac{1}{2}\|\rho_n - \cdot\|^2\right)(y_n) + N_C(y_n)$$

Thus, we can write $\lambda_n \tilde{w}_n + y_n - \rho_n + \bar{w}_n = 0$, where $\tilde{w}_n \in \partial_2 f(\rho_n, y_n)$ and $\bar{w}_n \in N_C(y_n)$. Due to $\rho_n = y_n$ implies that $\lambda_n \tilde{w}_n + \bar{w}_n = 0$. Thus, we have

$$\lambda_n\langle \tilde{w}_n, y - y_n\rangle + \langle \bar{w}_n, y - y_n\rangle = 0$$

for all $y \in C$. By $\bar{w}_n \in N_C(y_n)$ implies $\langle \bar{w}_n, y - y_n \rangle \le 0$ for all $y \in C$ and through above expression, we obtain

$$\lambda_n \langle \tilde{w}_n, y - y_n \rangle \ge 0 \tag{2}$$

for all $y \in C$. Due to $\tilde{w}_n \in \partial_2 f(\rho_n, y_n)$ and using the subdifferential definition, we obtain

$$\langle \tilde{w}_n, y - y_n \rangle \le f(\rho_n, y) - f(\rho_n, y_n) \tag{3}$$

for all $y \in C$. From the inequalities (2) and (3) with $0 < \lambda_n \le \lambda$ implies that $f(\rho_n, y) \ge 0$ for all $y \in C$, that is, $\rho_n \in EP(f, C)$. □

Lemma 3.2. *Suppose that $f : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ meet the items $(\mathcal{C}1) - (\mathcal{C}3)$, we have*

$$\|u_n - \bar{\xi}\|^2 + (1 - 2c_1\lambda_n)\|\rho_n - y_n\|^2 + (1 - 2c_2\lambda_n)\|y_n - u_n\|^2 \le \|\rho_n - \bar{\xi}\|^2 + \eta_n \tag{4}$$

for all $\bar{\xi} \in EP(f, C)$.

Proof. Let $\bar{\xi} \in EP(f, C)$, then by using Lemma 2.1, we have

$$0 \in \partial_2 \left(\lambda_n f(y_n, \cdot) + \frac{1}{2}\|\rho_n - \cdot\|^2 \right)(u_n) + N_{\mathcal{H}_n}(u_n)$$

Thus, we can write $\lambda_n \tilde{w}_n + u_n - \rho_n + \bar{w}_n = 0$, where $\tilde{w}_n \in \partial_2 f(y_n, u_n)$ and $\bar{w}_n \in N_{\mathcal{H}_n}(u_n)$. This implies that

$$\langle \rho_n - u_n, y - u_n \rangle = \lambda_n \langle \tilde{w}_n, y - u_n \rangle + \langle \bar{w}_n, y - u_n \rangle$$

for all $y \in \mathcal{H}_n$. Given that $\bar{w}_n \in N_{\mathcal{H}_n}(u_n)$ then $\langle \bar{w}_n, y - u_n \rangle \le 0$ for all $y \in \mathcal{H}_n$. Therefore, we have

$$\langle \rho_n - u_n, y - u_n \rangle \le \lambda_n \langle \tilde{w}_n, y - u_n \rangle \tag{5}$$

for all $y \in \mathcal{H}_n$. Since $\tilde{w}_n \in \partial_2 f(y_n, u_n)$, we have

$$\langle \tilde{w}_n, y - u_n \rangle \le f(y_n, y) - f(y_n, u_n) \tag{6}$$

for all $y \in \mathcal{H}$. From (5) and (6), we get

$$\langle \rho_n - u_n, y - u_n \rangle \le \lambda_n f(y_n, y) - \lambda_n f(y_n, u_n) \tag{7}$$

for all $y \in \mathcal{H}_n$. Substituting $y = \bar{\xi}$ in (7), we obtain

$$\langle \rho_n - u_n, \bar{\xi} - u_n \rangle \le \lambda_n f(y_n, \bar{\xi}) - \lambda_n f(y_n, u_n). \tag{8}$$

Given $\bar{\xi} \in EP(f, C)$ imply that $f(\bar{\xi}, y_n) \ge 0$ and owing to the item $(\mathcal{C}1)$ gives that $f(y_n, \bar{\xi}) \le 0$. Thus, we obtain

$$\langle \rho_n - u_n, u_n - \bar{\xi} \rangle \ge \lambda_n f(y_n, u_n). \tag{9}$$

Following the condition $(\mathcal{C}2)$, we have

$$f(y_n, u_n) \ge f(\rho_n, u_n) - f(\rho_n, y_n) - c_1\|\rho_n - y_n\|^2 - c_2\|y_n - u_n\|^2. \tag{10}$$

Combining (9) and (10), we get

$$\begin{aligned} \langle \rho_n - u_n, u_n - \bar{\xi} \rangle \ge{}& \lambda_n f(\rho_n, u_n) - \lambda_n f(\rho_n, y_n) - c_1\lambda_n\|\rho_n - y_n\|^2 \\ &- c_2\lambda_n\|y_n - u_n\|^2. \end{aligned} \tag{11}$$

By using the half-space definition, we have $\langle \rho_n - \lambda_n w_n - y_n, u_n - y_n \rangle \leq 0$, which implies that

$$\langle \rho_n - y_n, u_n - y_n \rangle \leq \lambda_n \langle w_n, u_n - y_n \rangle. \tag{12}$$

Since $w_n \in \partial_2 f(\rho_n, y_n)$, we obtain

$$\langle w_n, y - y_n \rangle \leq f(\rho_n, y) - f(\rho_n, y_n)$$

for all $y \in \mathcal{H}$. By replacing $y = u_n$, we obtain

$$\langle w_n, u_n - y_n \rangle \leq f(\rho_n, u_n) - f(\rho_n, y_n). \tag{13}$$

It follows from inequalities (12) and (13) that

$$\langle \rho_n - y_n, u_n - y_n \rangle \leq \lambda_n f(\rho_n, u_n) - \lambda_n f(\rho_n, y_n). \tag{14}$$

From (11) and (14), we have

$$\langle \rho_n - u_n, u_n - \bar{\xi} \rangle \geq \langle \rho_n - y_n, u_n - y_n \rangle - c_1 \lambda_n \|\rho_n - y_n\|^2 - c_2 \lambda_n \|y_n - u_n\|^2. \tag{15}$$

Now, we obtain the following equalities:

$$\|\rho_n - \bar{\xi}\|^2 - \|u_n - \rho_n\|^2 - \|u_n - \bar{\xi}\|^2 = 2\langle \rho_n - u_n, u_n - \bar{\xi} \rangle$$

and

$$\|\rho_n - y_n\|^2 + \|u_n - y_n\|^2 - \|\rho_n - u_n\|^2 = 2\langle \rho_n - y_n, u_n - y_n \rangle.$$

Combining the above equalities with expression (15), it can be implied that

$$\begin{aligned} \|u_n - \bar{\xi}\|^2 + (1 - 2c_1\lambda_n)\|\rho_n - y_n\|^2 + (1 - 2c_2\lambda_n)\|y_n - u_n\|^2 &\leq \|\rho_n - \bar{\xi}\|^2 \\ &\leq \|\rho_n - \bar{\xi}\|^2 + \eta_n. \end{aligned}$$

□

Lemma 3.3. *Assume that the items* (C1) – (C4) *hold. If there is a subsequence* $\{\rho_{n_k}\}$ *of* $\{\rho_n\}$ *such that* $\rho_{n_k} \rightharpoonup x^* \in \mathcal{H}$ *and*

$$\lim_{k\to\infty} \|\rho_{n_k} - y_{n_k}\| = \lim_{k\to\infty} \|\rho_{n_k} - u_{n_k}\| = \lim_{k\to\infty} \|u_{n_k} - y_{n_k}\| = 0. \tag{16}$$

Then $x^* \in EP(f, C)$.

Proof. From $y_n \in C$, $\rho_{n_k} \rightharpoonup x^*$ and $\lim_{k\to\infty} \|\rho_{n_k} - y_{n_k}\| = 0$, we get $y_{n_k} \rightharpoonup x^* \in C$. This follows from $\lim_{k\to\infty} \|u_{n_k} - y_{n_k}\| = 0$ that the subsequence $\{u_{n_k}\}$ is bounded. For any $y \in \mathcal{H}_n$, using (7), (10) and (14), we have

$$\begin{aligned} \lambda_{n_k} f(y_{n_k}, y) &\geq \lambda_{n_k} f(y_{n_k}, u_{n_k}) + \langle \rho_{n_k} - u_{n_k}, y - u_{n_k} \rangle \\ &\geq \lambda_{n_k} f(\rho_{n_k}, u_{n_k}) - \lambda_{n_k} f(\rho_{n_k}, y_{n_k}) - c_1 \lambda_{n_k} \|\rho_{n_k} - y_{n_k}\|^2 \\ &\quad - c_2 \lambda_{n_k} \|y_{n_k} - u_{n_k}\|^2 + \langle \rho_{n_k} - u_{n_k}, y - u_{n_k} \rangle \\ &\geq \langle \rho_{n_k} - y_{n_k}, u_{n_k} - y_{n_k} \rangle + \langle \rho_{n_k} - u_{n_k}, y - u_{n_k} \rangle \\ &\quad - c_1 \lambda_{n_k} \|\rho_{n_k} - y_{n_k}\|^2 - c_2 \lambda_{n_k} \|y_{n_k} - u_{n_k}\|^2. \end{aligned}$$

This implies by (16) and the boundedness of $\{u_{n_k}\}$ that the right hand side tends to zero. Due to $0 < \lambda_{n_k} \leq \lambda < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$, the condition (C4), and $y_{n_k} \rightharpoonup x^*$, we obtain

$0 \leq \limsup_{k\to\infty} f(y_{n_k}, y) \leq f(x^*, y)$ for all $y \in \mathcal{H}_n$. Since $C \subset \mathcal{H}_n$, we get $f(x^*, y) \geq 0$ for all $y \in C$, that is, $x^* \in EP(f, C)$. □

With the above results we are now ready for the main convergence theorem.

Theorem 3.1. *Suppose that* $\lim_{n\to\infty} \eta_n = 0$ *and the items* $(\mathcal{C}1) - (\mathcal{C}4)$ *are satisfied. Then, the sequence* $\{x_n\}$ *generated due to Algorithm 3.1 converges strongly to* $\upsilon = P_{EP(f,C)} x_1$.

Proof. For the beginning, we separate the proof into the claims listed below.
Claim 1. $\{x_n\}$ is well defined.
Lemma 3.2 then guarantees that $EP(f, C) \subset C_n$ and thus C_n is nonempty for all $n \in \mathbb{N}$. Indeed, one sees that $C_1 = \mathcal{H}$ is closed and convex. This follows from [7, Lemma 1.3] and mathematical induction that C_n is closed and convex for all $n \in \mathbb{N}$. Thus, Claim 1 is attained.
Claim 2. $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.
Since f satisfies the conditions $(\mathcal{C}1) - (\mathcal{C}4)$, we have that the solution set $EP(f, C)$ is closed and convex, see [12]. Then, there is a unique $\upsilon \in EP(f, C)$ such that $\upsilon = P_{EP(f,C)} x_1$. Applying $EP(f, C) \subset C_n$ to the definition of $\{x_n\}$, we obtain for every $n \in \mathbb{N}$,

$$\|x_n - x_1\| \leq \|\upsilon - x_1\|, \tag{17}$$

implying that $\{x_n\}$ is bounded. Since $x_{n+1} \in C_n$, we have that
$\|x_n - x_1\| \leq \|x_{n+1} - x_1\|$ for all $n \in \mathbb{N}$, which leads to $\lim_{n\to\infty} \|x_n - x_1\|$ exists. This implies by Lemma 2.2 that $\|x_{n+1} - x_n\|^2 \leq \|x_{n+1} - x_1\|^2 - \|x_k - x_1\|^2$ and hence Claim 2 is attained.
Claim 3. $\lim_{n\to\infty} \|\rho_n - y_n\| = \lim_{n\to\infty} \|\rho_n - u_n\| = \lim_{n\to\infty} \|u_n - y_n\| = 0$.
From the boundedness of $\{\delta_n\}$, there exists some $\delta > 0$ such that $|\delta_n| \leq \delta$ for all $n \in \mathbb{N}$. Using this to Claim 2 yields that

$$\|\rho_n - x_n\| = |\delta_n| \, \|x_n - x_{n-1}\| \leq \delta \, \|x_n - x_{n-1}\| \quad \to \quad 0 \quad \text{as} \quad n \to \infty \tag{18}$$

and so

$$\|\rho_n - x_{n+1}\| \leq \|\rho_n - x_n\| + \|x_n - x_{n+1}\| \quad \to \quad 0 \quad \text{as} \quad n \to \infty. \tag{19}$$

Meanwhile, since $x_{n+1} \in C_{n+1}$, we have

$$\|u_n - x_{n+1}\|^2 + (1 - 2c_1\lambda_n)\|\rho_n - y_n\|^2 + (1 - 2c_2\lambda_n)\|y_n - u_n\|^2 \leq \|\rho_n - x_{n+1}\|^2 + \eta_n.$$

This implies by (19), $0 < \lambda_n \leq \lambda < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$ and $\lim_{n\to\infty} \eta_n = 0$ that

$$\lim_{n\to\infty} \|u_n - x_{n+1}\| = \lim_{n\to\infty} \|\rho_n - y_n\| = \lim_{n\to\infty} \|y_n - u_n\| = 0. \tag{20}$$

Further, from the expressions (19) and (20), we have

$$\|\rho_n - u_n\| \leq \|\rho_n - x_{n+1}\| + \|x_{n+1} - u_n\| \quad \to \quad 0 \quad \text{as} \quad n \to \infty. \tag{21}$$

Therefore, Claim 3 is established from the expressions (20) and (21).
Claim 4. $x_n \to \upsilon$ as $n \to \infty$.
Let $\bar{r} \in \omega_w(x_n)$. Then, the expression (18) gives that $\bar{r} \in \omega_w(\rho_n)$. In light of Claim 3, one can obtain by Lemma 3.3 that $\bar{r} \in EP(f, C)$ and so $\omega_w(x_n) \subset EP(f, C)$ for all

$n \in \mathbb{N}$. Finally, employing this to Lemma 2.3 with the inequality (17) delivers the desired conclusion. □

4. Conclusions

We established the strong convergence theorem of the sequence generated by the modified algorithm under suitable conditions for solving pseudomonotone equilibrium problems. In future research work, we intend to develop novel algorithms aimed at addressing the aforementioned problem and its associated problems.

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REFERENCES

[1] *E. Blum*, From optimization and variational inequalities to equilibrium problems, Math. Stud., **63**(1994), 123-145.

[2] *E. Blum and W. Oettli*, From optimization and variational inequalities to equilibrium problems, Math. Program., **63**(1994), 123-145.

[3] *T. Chalarux and K. Chaichana*, Approximation of fixed points for Garcia-Falset mappings in a uniformly convex Banach space, Results Nonlinear Anal., **4**(2021), 200-206.

[4] *W. Cholamjiak and R. Suparatulatorn*, Strong convergence of a modified extragradient algorithm to solve pseudomonotone equilibrium and application to classification of diabetes mellitus, Chaos, Solit. Fract., **168**(2023), 113108.

[5] *V. Dadashi and M. Postolache*, Forward-backward splitting algorithm for fixed point problems and zeros of the sum of monotone operators, Arab. J. Math., **9**(2020), 89-99.

[6] *G. Korpelevich*, The extragradient method for finding saddle points and other problems, Matecon, **12**(1976), 747-756.

[7] *C. Martinez-Yanes and H. K. Xu*, Strong convergence of the CQ method for fixed point iteration processes, Nonlinear Anal., **64**(2006), 2400-2411.

[8] *K. Nakajo and W. Takahashi*, Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups, J. Math. Anal. Appl., **279**(2003), 372-379.

[9] *R. T. Rockafellar*, Convex Analysis, Princeton University Press: Princeton, NJ, 1970.

[10] *R. Suparatulatorn, W. Cholamjiak and N. Jun-on*, A modified subgradient extragradient method for equilibrium problems to predict prospective mathematics teachers' digital proficiency level, Results Nonlinear Anal., **6**(2023), 1-18.

[11] *R. Suparatulatorn, S. Suantai and W. Cholamjiak*, Hybrid methods for a finite family of G-nonexpansive mappings in Hilbert spaces endowed with graphs, AKCE Int. J. Graphs Co., **14**(2017), 101-111.

[12] *D. Q. Tran, M. L. Dung and V. H. Nguyen*, Extragradient algorithms extended to equilibrium problems, Optimization, **57**(2008), 749-776.

[13] *Y. Yao, H. Li and M. Postolache*, Iterative algorithms for split equilibrium problems of monotone operators and fixed point problems of pseudo-contractions, Optimization, **71**(2022), 2451-2469.

[14] *Z. Yao and M. Postolache*, Algorithmic approach to the solution of pseudomonotone equilibrium problems and generalized variational inequalities, Filomat, **36**(2022), 3969-3980.

[15] *Y. Yao and M. Postolache*, Iterative methods for pseudomonotone variational inequalities and fixed point problems, J. Optim. Theory Appl., **155**(1)(2012), 273-287.

[16] *Y. Yao, N. Shahzad and J. C. Yao*, Convergence of tseng-type self-adaptive algorithms for variational inequalities and fixed point problems, Carpathian J. Math., **37**(2021), 541-550.

[17] *Y. Yao, N. Shahzad and J. C. Yao*, Projected subgradient algorithms for pseudomonotone equilibrium problems and fixed points of pseudocontractive operators, Mathematics, **8**(2020), Art. No. 461.

[18] *X. P. Zhao, J. C. Yao and Y. Yao*, A nonmonotone gradient method for constrained multiobjective optimization problems, J. Nonlinear Var. Anal., **6**(2022), 693-706.

[19] *L. J. Zhu, Y. Yao and M. Postolache*, Projection methods with linesearch technique for pseudomonotone equilibrium problems and fixed point problems, U.P.B. Sci. Bull. Ser. A., **83**(2021), 3-14.