TWO SOLUTIONS FOR A PROBLEM OF MATHEMATICAL PHYSICS EQUATIONS

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În această lucrare sunt prezentate două metode de rezolvare şi caracterizare a soluţiilor, subsoluţiilor si supersoluţiilor pentru o problemă a ecuaţiilor fizicii matematice. Prima metodă foloseşte principiul variaţional Ekeland şi o condiţie de tip Palais-Smale pentru a obţine un rezultat pentru p*-laplacian. A doua metodă se bazează pe câteva teoreme de surjectivitate aparţinând autoarei care sunt demonstrate şi aplicate in găsirea unor rezultate noi pentru* p*-laplacian.*

In this paper two approach methods to obtain and characterize weak solutions or subsolutions and supersolutions for a problem of mathematical phisics equations are presented. In the first, Ekeland variational principle and a condition of Palais Smale type are both involved in order to obtain some results for the p*-Laplacian. In the second approach method, some original surjectivity theorems are established to state two original results which describe new properties of the* p*-Laplacian.*

Keywords: Ekeland varitional principle, Palais-Smale condition, critical point, weak solution, Nemytskii operator, weak subsolution, weak supersolution, Carathéodory function, Sobolev space, p - Laplacian

1. Introduction

This paper is mainly based on the organization of the concepts from [1] corroborated with the results obtained by the author in [2] and [3]. The general theory is developed towards two distinct trends as in [3] and [4]. In the first, beginning with the Ekeland principle ([5], [6], [1]), and following a work of Ghoussoub [7], a variational method to discuss some problems of partial differential equations has been presented in the manner of [1]. In the second direction, a series of propositions from [8], [9], [10] and others are generalized or used in order to obtain a sequence of original results.

The aim of this work is to compare these two approaches, and particularly to highlight new results in weak solutions for some types of partial differential equations.

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2. Critical points and weak solutions for elliptic type equation

2.1. Preliminaries

Ekeland principle ([5], [6], [1]). *Let* (X, d) *be a complete metric space and* $\varphi : X \to (-\infty, +\infty]$ *bounded from below, lower semicontinuous and proper. For any* $\varepsilon > 0$ *and* u *of* X *with*

$$\varphi(u) \le \inf \varphi(X) + \varepsilon$$

and for any $\lambda > 0$, *there exists* v_ε *in* X *such that*

$$\varphi(v_\varepsilon) < \varphi(w) + \frac{\varepsilon}{\lambda} d(v_\varepsilon, w) \ \forall\, w \in X \setminus \{v_\varepsilon\}$$

and

$$\varphi(v_\varepsilon) \le \varphi(u), \qquad d(v_\varepsilon, u) \le \lambda.$$

Let *X* be a real normed space, β a bornology[2] on *X* and $\varphi: X \to \mathbf{R}$.

Definition. Let *c* be in **R** and *F* a nonempty subset of *X*. φ verifies the *Palais - Smale condition on the level* c *around* F (or *relative to* F), $(PS)_{c,F}$, with respect to β, when $\forall\, (u_n)_{n\ge1}$ a sequence of points in *X* for which

$$\lim_{n\to\infty} \varphi(u_n) = c, \ \lim_{n\to\infty} \left\|\nabla_\beta \varphi(u_n)\right\| = 0 \text{ and } \lim_{n\to\infty} \operatorname{dist}(u_n, F) = 0, \tag{1}$$

this sequence has a convergent subsequence.

We will see that through the minimization on *F* of a functional (*minimization with constraints*) it can obtain global critical points of this.

As a preliminary –

Proposition 1. *Let* H *be a Hilbert space, let* $\varphi: H \to \mathbf{R}$ *be of* C^1- *Fréchet class and let* F *be a closed subset of* H *such that*:

for every u *from* F *with* $\varphi'(u) \ne 0$, *there is*, *for sufficiently small* $r > 0$, $f_u : S_r(0) \to \mathbf{R}$ *Fréchet differentiable such that, denoting* $g_u(\delta) = f_u\left(\delta \frac{\varphi'(u)}{\|\varphi'(u)\|}\right)$, $\delta \in [0, r]$, *we have*

$$g_u(0) = 1 \textit{ and } g_u(\delta)\left(u - \delta \frac{\varphi'(u)}{\|\varphi'(u)\|}\right) \in F. \tag{2}$$

Then, *if* φ *is lower bounded on* F, *for every* $(v_n)_{n\ge1}$ *a minimizing sequence for* φ *on* F *there exists a sequence* $(u_n)_{n\ge1}$ *in* F *such that*

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[2] Let *X* be a real normed space. A nonempty set β of bounded parts of *X*, with the properties: 1° $\bigcup_{A\in\beta} A = X$, 2° $A \in \beta \Rightarrow -A \in \beta$ and $\lambda A \in \beta$ $(\lambda > 0)$, 3° for every *A*, *B* in β there exists *C* in β such that $A \subset C$ and $B \subset C$, is named *bornology on* X.

$$\left\|\varphi'(u_n)\right\| \le \sqrt{\varepsilon_n}\left[1+\|u_n\|\left|g'_{u_n}(0)\right|\right]+\left|g'_{u_n}(0)\right| \ |\varphi'(u_n)(u_n)|, \quad (3)$$

$$\varphi(u_n) \le \varphi(v_n) \ \forall n, \quad (4)$$

$$\lim_{n\to\infty}\|u_n - v_n\| = 0, \quad (5)$$

where $\varepsilon_n > 0$ *and* $\varepsilon_n \to 0$ ([7]).

Proof. Denote $c := \inf \varphi(F)$ and let n be from **N**. For $\varepsilon_n := \varphi(v_n) - c + \frac{1}{n}$, hence $\varepsilon_n > 0$, we have $\varphi(v_n) < c + \varepsilon_n$. Apply *Ekeland principle* with $\lambda = \sqrt{\varepsilon_n}$, $\exists\, u_n$ in F with known properties. Thus we get the sequence $(u_n)_{n\ge1}$ satisfying (4), (5) ($\|u_n - v_n\| \le \sqrt{\varepsilon_n}$, $\varepsilon_n \to 0$) and

$$\varphi(v) \ge \varphi(u_n) - \sqrt{\varepsilon_n}\|v - u_n\| \quad \forall\, v \in F. \quad (6)$$

Verify (3). It is sufficient to work under the assumption $\|\varphi'(u_n)\| > 0 \ \forall\ n$. Thus apply the hypothesis made in the statement with respect to F with $u = u_n$ and denoting, for $\delta \in (0,r]$, $v_\delta := g_{u_n}(\delta)\left(u_n - \delta\frac{\varphi'(u_n)}{\|\varphi'(u_n)\|}\right)$ ($\in F$), replace v_δ in (6) and find, taking into account the Fréchet derivative definition,

$$\sqrt{\varepsilon_n}\|v_\delta - u_n\| \ge \varphi(u_n) - \varphi(v_\delta) = \varphi'(v_\delta)(u_n - v_\delta) + o(\delta),$$

where $\lim_{\delta\to0}\frac{o(\delta)}{\delta} = 0$, hence

$$\sqrt{\varepsilon_n}\|v_\delta - u_n\| \ge \left[1 - g_{u_n}(\delta)\right]\varphi'(v_\delta)(u_n) + \delta g_{u_n}(\delta)\varphi'(v_\delta)\left(\frac{\varphi'(u_n)}{\|\varphi'(u_n)\|}\right) + o(\delta). \quad (7)$$

But $\lim_{\delta\to0}\frac{1-g_{u_n}(\delta)}{\delta} = \lim_{\delta\to0}\frac{g_{u_n}(0)-g_{u_n}(\delta)}{\delta} = -g'_{u_n}(0)$, $\lim_{\delta\to0} v_\delta = u_n$, $\lim_{\delta\to0}\frac{1}{\delta}\|v_\delta - u_n\|$ $= \left\|g'_{u_n}(0)u_n - \frac{\varphi'(u_n)}{\|\varphi'(u_n)\|}\right\| \le 1 + \left|g'_{u_n}(0)\right|\|u_n\|$, consequently if in (7) it divides by δ and takes the limit for $\delta \to 0$ one obtains

$$\sqrt{\varepsilon_n}\left(1+\left|g'_{u_n}(0)\right|\|u_n\|\right) \ge -g'_{u_n}(0)\varphi'(u_n)(u_n) + \varphi'(u_n)\left(\frac{\varphi'(u_n)}{\|\varphi'(u_n)\|}\right)$$

and it remains only to remark that $\varphi'(u_n)\left(\frac{\varphi'(u_n)}{\|\varphi'(u_n)\|}\right) = \|\varphi'(u_n)\|$ in order to get *a fortiori* (3).■

Notation. $\varphi: X \to \mathbf{R}$ β-differentiable, $c \in \mathbf{R} \Rightarrow$

$$K_c(\varphi) := \{x \in X : \varphi(x) = c, \nabla_\beta \varphi(x) = 0\}.$$

Proposition 2. *Let* H *be a Hilbert space*, φ: H → **R** *be of* C^1-*Fréchet class and* F *be a nonempty convex closed subset such that* (I − φ′)(F) ⊂ F, I *the identity map. If* φ *is lower bounded on* F, *then for every* $(v_n)_{n\geq 1}$ *a minimizing sequence for* φ *on* F, *there is a sequence* $(u_n)_{n\geq 1}$ *in* F *such that*

$$\varphi(u_n) \leq \varphi(v_n)\ \forall\ n,\ \lim_{n\to\infty}\|u_n - v_n\| = 0,\ \lim_{n\to\infty}\|\varphi'(u_n)\| = 0.$$

Moreover, *if* φ *satisfies* $(PS)_{c,F}$ *where* c = inf φ(F), *then*

$$F \cap K_c(\varphi) \neq \Phi.$$

Proof. Apply Proposition 1 with $g_u = 1$. (2) is indeed satisfied: if $u \in F$ and $\varphi'(u) \neq 0$, then, F being convex,

$$u - \delta\frac{\varphi'(u)}{\|\varphi'(u)\|} = \left(1 - \frac{\delta}{\|\varphi'(u)\|}\right)u + \frac{\delta}{\|\varphi'(u)\|}\left(I - \varphi'\right)(u) \in F.$$

Let $(u_n)_{n\geq 1}$ be the sequence given by the statement., $c \leq \varphi(u_n) \leq \varphi(v_n)\ \forall\ n$, hence $\varphi(u_n) \to c$. Since $g'_{u_n}(0) = 0$, $\|\varphi'(u_n)\| \overset{(3)}{\leq} \sqrt{\varepsilon_n}$, hence $\|g'(u_n\| \to 0$, clearly dist $(u_n, F) = 0$, consequently $(u_n)_{n\geq 1}$ has a convergent subsequence $(u_{k_n})_{n\geq 1}$, $u_{k_n} \to u_0 \in F$. This implies $\|\varphi'(u_{k_n})\| \to \|\varphi'(u_0)\| = 0$, u_0 is a global critical point of φ contained in F. ■

2.2. Weak solutions

Open set of C^1 class in $\mathbf{R}^N$. We use the notations (the norm is the Euclidean norm from $\mathbf{R}^{N-1}$):

$$\mathbf{R}^N_+ = \{x = (x', x_N) : x_N > 0\},$$

$$Q = \{x = (x', x_N) : \|x'\| < 1, |x_N| < 1\},$$

$$Q_+ = Q \cap \mathbf{R}^N_+$$

$$Q_0 = \{x = (x', x_N) : \|x'\| < 1, x_N = 0\}.$$

Let Ω be an open nonempty set of $\mathbf{R}^N$, $\Omega \neq \mathbf{R}^N$ and ∂Ω its boundary. By definition, Ω is of C^1 *class* if ∀ x from ∂Ω ∃ U a neighborhood of x in $\mathbf{R}^N$ and f: $Q \to U$ one-to-one such that $f \in C^1(\overline{Q})$, $f^{-1} \in C^1(\overline{U})$, $f(Q_+) = U \cap \Omega$, $f(Q_0) = U \cap \partial\Omega$.

Weak solution. Let Ω be an open bounded nonempty set in $\mathbf{R}^N$, $N > 1$, $f : \Omega \times \mathbf{R}^N \to \mathbf{R}$ and $u_0 \in H_0^1(\Omega)\left(= W_0^{1,2}(\Omega)\right)$. Consider the problem

$$\begin{cases} -\Delta u = f(x,u) & \text{in} \quad \Omega, \\ u = u_0 & \text{on} \ \ \partial\Omega. \end{cases} \tag{*}$$

Actually $u = u_0$ means $u \mid \partial\Omega = u_0$, where γ: $u \to u \mid \partial\Omega$ is the *trace operator*, a continuous linear mapping from $W^{1,p}(\Omega)$ in $\mathbf{L}^p(\partial\Omega)$, $p \in [1, +\infty)$. We have $\gamma^{-1}(0)$ $= W_0^{1,p}(\Omega) \left(= \overline{C_c^1(\Omega}^{W^{1,p}(\Omega)} \right)$ and $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \Rightarrow \gamma(u) = u\big|\partial\Omega$.

$\overline{u}$ from $H_0^1(\Omega)$ is by definition a *weak solution* for (*) if $\overline{u} = u_0$ on $\partial\Omega$ and

$$\int_\Omega \nabla\overline{u} \cdot \nabla v dx - \int_\Omega f(x,\overline{u}(x))v dx = 0 \ \forall\, v \in C_c^\infty(\Omega) \tag{8}$$

(∇w, the weak gradient, is equal to $\left(\frac{\partial w}{\partial x_1},...,\frac{\partial w}{\partial x_N}\right)$, $\frac{\partial w}{\partial x_i}$ the weak derivatives).

Nemytskii operator. Let be $\mathbf{R}^N$, $N \geq 1$, μ the Lebesgue measure in $\mathbf{R}$, Ω open non-empty Lebesgue measurable (L.m.) and $\mathcal{M}(\Omega) := \{u\colon \Omega \to \mathbf{R} \mid u$ Lebesgue measurable (L.m.)$\}$.

By definition $f : \Omega \times \mathbf{R} \to \mathbf{R}$ is a *Carathéodory function* if

1° $f(\cdot, s)$ is L.m. $\forall s \in \mathbf{R}$,

2° $f(x, \cdot)$ is continuous $\forall\, x \in \Omega \setminus A$, $\mu(A) = 0$.

In this case, for every u from $\mathcal{M}(\Omega)$ it can consider the function

$$N_f : \mathcal{M}(\Omega) \to \mathcal{M}(\Omega),$$
$$N_f u : N_f u(x) = f(x, u(x)),$$

Nemytskii operator.

Suppose $\mu(\Omega) < +\infty$. Then $u_n(x) \underset{x\in\Omega}{\overset{\mu}{\to}} u_0(x) \Rightarrow N_f u_n(x) \underset{x\in\Omega}{\overset{\mu}{\to}} N_f u_0(x)$.

Suppose that f satisfies the *growth condition* :

$|f(x,s)| \leq c|s|^{p-1} + \beta(x)$, $\forall\, x \in \Omega \setminus A$ with $\mu(A) = 0$, $\forall\, s \in \mathbf{R}$, where $c \geq 0$, $p > 1$ and $\beta \in \mathbf{L}^q(\Omega)$, $q \in [1, +\infty]$. (9)

Then

$$N_f(\mathbf{L}^{(p-1)q}(\Omega)) \subset \mathbf{L}^q(\Omega); \tag{10}$$

N_f is continuous ($q < +\infty$) and bounded on $\mathbf{L}^{(p-1)q}(\Omega)$; (11)

(12) If Ω is bounded and $\frac{1}{p} + \frac{1}{q} = 1$, then $N_f(\mathbf{L}^p(\Omega)) \subset \mathbf{L}^q(\Omega)$ with N_f continuous; moreover, $N_F(\mathbf{L}^p(\Omega)) \subset \mathbf{L}^1(\Omega)$ with N_F continuous, where

$F(x,s)=\int_0^s f(x,t)dt$, and Φ: $\mathbf{L}^p(\Omega)\to\mathbf{R}$, $\Phi(u)=\int_\Omega F(x,u(x))dx$ is of C^1-Frechét class and $\Phi'=N_f$.

Theorem 1. *Let* Ω *be an open bounded nonempty set in* $\mathbf{R}^N$ *and* $f:\Omega\times\mathbf{R}\to\mathbf{R}$ *a Carathéodory function with the growth condition*

$$|f(x,s)|\le c|s|^{p-1}+b(x), \tag{13}$$

where $c>0$, $2\le p\le\frac{2N}{N-2}$ *when* $N\ge 3$ *and* $2\le p<+\infty$ *when* $N=1,2$, *and where* $b\in\mathbf{L}^q(\Omega)$, $\frac{1}{p}+\frac{1}{q}=1$.

Then the energy functional $\varphi:H_0^1(\Omega)\to\mathbf{R}$,

$$\varphi(u)=\frac{1}{2}\int_\Omega|\nabla u(x)|^2dx-\int_\Omega F(x,u(x))dx, \tag{14}$$

where $F(x,s)=\int_0^s f(x,t)dt$, *is of* C^1 *- Fréchet class and*

$$\varphi'(u)(v)=\int_\Omega\nabla u\cdot\nabla v\,dx-\int_\Omega f(x,u(x))v dx\ \forall u,v\in H_0^1(\Omega).$$

Explanation. $|\nabla u|^2=\sum_{i=1}^N\left|\frac{\partial u}{\partial x_i}\right|^2$, $\frac{\partial u}{\partial x_i}$ the weak derivatives.

Corollary 1. *Let* Ω *and* f *be as in Theorem* 1. *Then the weak solutions of* $(*)$ *are precisely the critical points of the functional* $\varphi:H_0^1(\Omega)\to\mathbf{R}$,

$$\varphi(u)=\frac{1}{2}\int_\Omega|\nabla u(x)|^2dx-\int_\Omega F(x,u(x))dx,\ F(x,s):=\int_0^s f(x,t)dt.$$

Proof. Indeed, if $\overline{u}$ is a weak solution of $(*)$, then $\varphi'(\overline{u})(v)=0$ ((1), Theorem 1) $\forall v\in C_c^\infty(\Omega)$, hence $\varphi'(\overline{u})=0$, because φ' is continuous and $\overline{C_c^\infty(\Omega)}=H_0^1(\Omega)$. The converse is obviously. ■

Weak subsolutions and weak supersolutions of $(*)$. Let Ω be an open bounded set of C^1 class in $\mathbf{R}^N$, $N\ge 3$, $f:\Omega\times\mathbf{R}\to\mathbf{R}$ a Carathéodory function and $\overline{u}\in H_0^1(\Omega)$.

Definition. $\overline{u}$ is a *weak subsolution* respectively *weak supersolution* of $(*)$ if

$$\bar{u} \le u_0 \text{ on } \partial\Omega \text{ respectively } \bar{u} \ge u_0 \text{ on } \partial\Omega \tag{15}$$

and

$$\begin{cases} \int_\Omega \nabla\bar{u}\cdot\nabla v dx \le \int_\Omega f(x,\bar{u}(x))v dx \ \forall\, v \in C_c^\infty(\Omega),\ v \ge 0 \\ \text{resp.} \\ \int_\Omega \nabla\bar{u}\cdot\nabla v dx \ge \int_\Omega f(x,\bar{u}(x))v dx \ \forall\, v \in C_c^\infty(\Omega),\ v \le 0. \end{cases} \tag{16}$$

Proposition 3. *Let* Ω *be an open bounded of* C^1 *class set in* $\mathbf{R}^N$, $N \ge 3$, $f : \Omega \times \mathbf{R} \to \mathbf{R}$ *a Carathéodory function and* u_1 , u_2 *from* $H_0^1(\Omega)$ *bounded weak subsolution* resp. *weak supersolution of* $(*)$ *with* $u_1(x) \le u_2(x)$ *a. e. on* Ω.

Suppose that f *verifies* (16) *and there is* $\rho > 0$ *such that the function* g: $g(x, s) = f(x, s) + \rho s$ *is strictly increasing in* s *on* $[\inf u_1(\Omega), \sup u_2(\Omega)]$. *Then a weak solution* $\bar{u}$ *of* $(*)$ *exists in* $H_0^1(\Omega)$ *with the property*

$u_1(x) \le \bar{u}(x) \le u_2(x)$ *a. e. on* Ω

Proof. One can suppose $u_0 = 0$. Take the equivalent norm on $H_0^1(\Omega)$

$$\|u\|^2 = \rho\|u\|_2^2 + \|\nabla u\|_2^2$$

and let $<\cdot,\cdot>$ the corresponding scalar product. Consider the functional $\varphi : H_0^1(\Omega) \to \mathbf{R}$,

$$\varphi(u) = \frac{1}{2} < u,u > - \int_\Omega G(x,u(x))dx, \tag{17}$$

$$G(x,s) := \int_0^s g(x,t)dt.$$

φ is of C^1 class and its critical points are the weak solutions of $(*)$. Use Proposition 2. Let be

$$\mathscr{F} := \{u \in H_0^1(\Omega) : u_1(x) \le u(x) \le u_2(x) \text{ a. e. on } \Omega\}.$$

$\mathscr{F}$ is closed convex. We also get

$$(I - \varphi')\mathscr{F} \subset \mathscr{F}. \tag{18}$$

Indeed, let u be in $\mathscr{F}$ and $v := (I - \varphi')(u)$. For every w in $C_c^\infty(\Omega)$, $w \ge 0$ we have

$$< v,w > \overset{(17)}{=} < u,w > - < u,w > + \int_\Omega g(x,u(x))\, w dx,$$

$$< v - u_1, w > \overset{(16)}{\geq} \int_\Omega |g(x,u(x)) - g(x,u_1(x))| w dx \geq 0,$$

apply the maximum principle, we get $u_1(x) \leq v(x)$ a.e. on Ω. In the same way, we get $v(x) \leq u_2(x)$ a. e. on Ω and hence (18). φ is lower bounded on $\mathscr{F}$: $\forall$ u in $\mathscr{F}$ we have $\varphi(u) \overset{(17)}{\geq} \frac{1}{2} < u,u > - c_1$, $c_1 \in \mathbf{R}$ and moreover $(PS)_{c,F}$ is verified, $c := \inf \varphi(F)$ ([7], Theorem 1.16, proof). Finish the proof applying Proposition 2. ■

Example. Consider the problem (Ω open bounded of C^1 class in $\mathbf{R}^N$, $N \geq 3$)

$$\begin{cases} -\Delta u = \alpha(x) u |u|^{p-2} \\ u = u_0 \quad \text{on } \partial\Omega, \end{cases} \tag{19}$$

where $p = \frac{2N}{N-2}$, α is continuous with $1 \leq \alpha(x) \leq a < +\infty$ on Ω and $u_0 \in C^1(\overline{\Omega})$, $u_0(x) = 1$ on $\partial\Omega$. Then $u_1 := 1$ is a weak subsolution, $u_2 := M$, $M > 1$ sufficiently big, is a weak supersolution, $|f(x, s)| \leq a|s|^{p-1}$ (condition (13)) and $s \to \alpha(x)s|s|^{p-2} + s$ is increasing in s on $[1, M]$, consequently, according to Proposition 3, (19) has a weak solution $\overline{u}$ with $1 \leq \overline{u}(x) \leq M$ a. e. on Ω.

3. Surjectivity of the operators $\lambda J_\varphi - S$. Applications to partial differential equations

3.1. Surjectivity of the operators $\lambda T - S$

An extension of the Theorem 1.1 from [8] is proved with weakened assumption: normed space instead of Banach space, bijection with continuous inverse instead of homeomorphism. Two corollaries of the author are also presented.

Firstly, to have a short expression,

Definition. $T: X \to Y$, X and Y normed spaces, is (K, L, a), where $K > 0$, $L > 0$, $a > 0$, if

$$K\,\|x\|^a \leq \|Tx\| \leq L\,\|x\|^a, \ \forall\, x \text{ from } X.$$

Proposition 4. *Let* X, Y *be a real normed spaces*, *let* T: X → Y (K, L, a) *be an odd bijection with continuous inverse and* S: X → Y *an odd compact operator*. *For any* $\lambda \neq 0$, *if*

$$\lim_{\|x\| \to +\infty} \|\lambda Tx - Sx\| = +\infty,$$

Then $\lambda T - S$ *is surjective.*

Proof. Let z_0 be from Y, we state that

$$\exists\, x_0 \text{ in } X \text{ e.g. } \lambda T x_0 - S x_0 = z_0 . \tag{20}$$

Take $R > 0$ with the property (see the hypothesis)

$$\|x\| \geq R \Rightarrow \|\lambda Tx - Sx\| > \|z_0\| \tag{21}$$

and the open ball from Y $\Omega := S(0, r)$, $r := |\lambda| L R^{a}$. If $y \in \partial\,\Omega$ and $y = \lambda Tx$, then $\|x\| \geq R$ and hence (22) $\|\lambda Tx - Sx\| \overset{(21)}{>} \|z_0\|$.

Let be the operator A: $Y \to Y$,

$$Ay = ST^{-1}\left(\frac{y}{\lambda}\right).$$

A is compact, odd and $Ay \neq y$ when $y \in \partial\sigma$ (*par absurdum*, put y in the form λTx and takes into account (22), i.e. $0 \notin (I - A)(\partial\sigma)$. Applying the Borsuk theorem, it follows that the Leray-Schauder degree, $d(I - A, \sigma, 0)$, is odd. But

$$H : [0, 1] \times \overline{\sigma} \to Y,\ H(t, y) = Ay + tz_0$$

being a homotopy of compact transforms on $\overline{\sigma}$, we have

$$d(I - H(0, \cdot\,), \sigma, 0) = d(I - H(1, \cdot\,), \sigma, 0),$$

i.e.

$$d(I - A, \sigma, 0) = d(I - A - z_0 , 0),$$

consequently, $d(I - A, \sigma, 0)$ is an odd number, particularly different from zero, therefore $\exists\, y_0$ in σ so that $(I - A - z_0)(y_0) = 0$ and we have only to take x_0 in X with $y_0 = \lambda T x_0$, to obtain (20). ■

Corollary 2. *Let* X, Y *be real normed spaces*, T : X → Y *odd* (K, L, a) *bijection with continuous inverse*, S: X → Y *odd compact operator and* $\alpha := \overline{\lim_{\|x\| \to +\infty}} \dfrac{\|Sx\|}{\|x\|^{a}} < +\infty$. If

$$|\lambda| > \frac{\alpha}{K},\ \lambda \in \mathbf{R},$$

then $\lambda T - S$ *is surjective*.

Explanations. ([11], vol. II, V, § 5, 11.9_1). $f : X \to Y$, X and Y normed spaces,

$$\overline{\lim_{\|x\|\to+\infty}} \|f(x)\| \overset{def}{=} \inf_{\rho>0} \sup_{\substack{x\in X \\ \|x\|\geq\rho}} \|f(x)\| = \lim_{\rho\to+\infty} \sup_{\substack{x\in X \\ \|x\|\geq\rho}} \|f(x)\|.$$

If $\alpha = \overline{\lim\limits_{\|x\|\to+\infty}} \|f(x)\|$, then $x_n \in X$ $\forall$ n from **N** and $\| x_n \| \to +\infty$ implies $\lim\limits_{n\to+\infty} \|f(x_n)\| \leq \alpha$.

If $\alpha = \overline{\lim\limits_{n\to+\infty}} \|f(x_n)\|$ for any (x_n) with x_n from X and $\|x_n\| \to +\infty$, then $\alpha = \overline{\lim\limits_{\|x\|\to+\infty}} \|f(x)\|$.

It remains to state

$$\overline{\lim_{\|x\|\to+\infty}} \|\lambda Tx - Sx\| = +\infty. \tag{23}$$

Assuming, *par absurdum*, the contrary, one obtains $\rho > 0$ and a sequence $(x_n)_{n\geq0}$, $x_n \in X$, $\|x_n\| \to +\infty$ e.g.

$$\|\lambda Tx_n - Sx_n\| \leq \rho \quad \forall n \geq 1. \tag{24}$$

From (24),

$$\lim_{n\to+\infty} \left\| \frac{\lambda Tx_n}{\|x_n\|^a} - \frac{Sx_n}{\|x_n\|^a} \right\| = 0,$$

hence $\lim\limits_{n\to+\infty} \left[\dfrac{|\lambda| \|Tx_n\|}{\|x_n\|^a} - \dfrac{\|S(x_n)\|}{\|x_n\|^a} \right] = 0$ and as $\overline{\lim\limits_{n\to+\infty}} \dfrac{\|S(x_n)\|}{\|x_n\|^a} \leq \alpha$, it results

$$\overline{\lim_{n\to+\infty}} \frac{|\lambda| \|Tx_n\|}{\|x_n\|^a} \leq \alpha. \tag{25}$$

But the condition (K, L, a) imposes

$$K \leq \overline{\lim_{n\to\infty}} \frac{\|Tx_n\|}{\|x_n\|^a}. \tag{26}$$

From (25) and (26) results $K \le \frac{\alpha}{|\lambda|}$. If $\alpha \ne 0$, then $|\lambda| \le \frac{\alpha}{K}$, which contradicts the hypothesis, and if $\alpha = 0$, then $K = 0$, also in contradiction with the hypothesis, and consequently (23). ■

Corollary 3. *In the condition of the Corollary* 2, *if* $\alpha = 0$, *then* $\lambda T - S$ *is surjective for any* λ *from* $\mathbf{R} \setminus \{0\}$.

3.2. Surjectivity of the operators of the form $\lambda J_\varphi - S$, J_φ the duality map

Proposition 6. *Let* X *be a real Banach space, reflexive and with the property* (H), J_φ *the duality map on* X *with* φ *a* (K, L, a) *function*, $S: X \to X^*$ *an odd compact operator and*

$$\alpha := \overline{\lim_{\|x\| \to +\infty}} \frac{\|Sx\|}{\|x\|^a} < +\infty.$$

Then

1^0 $\alpha > 0 \Rightarrow \lambda J_\varphi - S$ *surjective* $\forall \lambda$ with $|\lambda| > \frac{\alpha}{K}$;

2^0 $\alpha = 0 \Rightarrow \lambda J_\varphi - S$ *surjective* $\forall \lambda \ne 0$.

Explanation. A Banach space has the (H) property if it is strictly convex and satisfies: $x_n \xrightarrow{w} x$ and $\|x_n\| \to \|x\| \Rightarrow x_n \to x$.

Proof. J_φ is odd and bijective with continuous inverse (X being reflexive, smooth and with (H) property, any duality map J_φ on X is bijective with its inverse continuous by rapport to the strong topologies on X and X^* [2]). Moreover, since

$$Kt^a \le \varphi(t) \le Lt^a \quad \forall\, t \ge 0,$$

we have

$$K\|x\|^a \le \varphi(\|x\|) = \|J_\varphi x\| \le L\|x\|^a \quad \forall\, x \text{ from } X,$$

i.e. J_φ is (K, L, a). Apply Corollaries 2 and 3. ■

Proposition 7. *Let* X *be a real reflexive Banach space*, *smooth with the property* (H), *and* J_φ *the duality map on* X *with* $\varphi(t) = t^{p-1}$, $p \in (1, +\infty)$. *Suppose that* X *is compact embedded by the linear injection* i *in a Banach space* Z,

$$\|i(u)\| \le c_0 \|u\| \quad \forall\, u \text{ from } X. \tag{27}$$

and $N: Z \to Z^*$ *is odd hemi-continuous operator with the property*

$$\|Nx\| \le c_1\|x\|^{q-1} + c_2 \;\forall\, x \text{ from } Z,\; c_1, c_2 \ge 0,\; q \in (1, p). \tag{28}$$

Then $\lambda J_\varphi - N$ *is surjective for any* $\lambda \neq 0$.

Explanation. N is the short notation for the operator, which acts from X to X^*, i' o N o i, i' the adjoint of i.

Proof. It follows to state that

$$\forall\ h \text{ from } X^* \ \exists\ u \text{ in } X \text{ e.g. } \lambda J_\varphi u - (i' \circ N \circ i)u = h. \tag{29}$$

Apply Proposition 6 with $T = J_\varphi$ (correctly as φ is (K, L, a) with $K = L = 1$, $a = p - 1$), $S = i'$ o N o i. Obviously, S is odd, and also compact: let $(x_n)_{n\in \mathbf{N}}$, $x_n \in X$ be bounded sequence, $(i(x_n))_{n\geq 1}$ has a convergent subset, let be $i(x_{k_n}) \to \gamma$, $\gamma \in Z$, then $N(i(x_{k_n})) \xrightarrow{w} N(\gamma)$ and consequently $i'(N(i(x_{k_n}))) \to i'(N(\gamma))$ because i' is also compact (Schauder theorem). So, to obtain the conclusion it remains to proof

$$\overline{\lim_{\|u\|\to+\infty}} \frac{\|(i' \circ N \circ i)u\|}{\|u\|^{p-1}} = 0. \tag{30}$$

$$\|i'(N(i(u)))\| \leq \|i'\|\ \|N(i(u))\| \overset{(27),(28)}{\leq} c_0(c_1 \|i(u)\|^{q-1} + c_2) \leq c_0(c_0^{q-1} c_1 \|u\|^{q-1} + c_2),$$

from which results (30) (was used $\|i'\| \leq \|i\| \leq c_0$). ■

In the following one searches the surjectivity of the operator $\lambda J_\varphi - N$, when N verifies the growth condition (28) where $q = p$, i.e.

$$\|Nx\| \leq c_1 \|x\|^{p-1} + c_2 \ \ \forall\ x \text{ from } Z,\ c_1, c_2 \geq 0. \tag{31}$$

For this reason, one presents the statement

Proposition 8. *Let* X *be a real reflexive Banach space compactly embedded by the linear injection* i *in the Banach space* Z,

$$\|i(u)\| \leq c_0 \|u\| \ \ \forall\ u \text{ from } X. \tag{32}$$

If

$$\lambda_1 := \inf\{\frac{\|u\|^p}{\|i(u)\|^p} : u \in X \setminus \{0\}\},\ p \in (1, +\infty),$$

then

1^0 λ_1 *is attained and nonzero*;

2^0 $\lambda_1^{-\frac{1}{p}}$ *is optimal for* (32) (i.e. $\lambda_1^{-\frac{1}{p}} \leq c_0$, for any c_0);

3^0 *If* X *and* Z *are smooth and* J_{XX^*} : $X \to X^*$, $J_{ZZ^*} : Z \to Z^*$ duality maps relative to the same weight φ: $\varphi(t) = t^{p-1}$, *then* λ_1 *is the smallest eigenvalue of the couple* (J_{XX^*} , J_{YY^*}) ([9]).

Proof. The set from the statement is correctly defined: $u \neq 0 \Rightarrow i(u) \neq 0$.

1^0 We have

$$\lambda_1 = \inf \{\|v\|^p : v \in X, \|i(v)\| = 1\}$$

(the two sets coincide, as $\left\| i\left(\frac{u}{i(u)}\right)\right\| = 1$), let $(v_n)_{n\geq 1}$, $v_n \in X$ be with $\|i(v_n)\| = 1$ and $\|v_n\| \to \lambda_1^{\frac{1}{p}}$.

Since X is reflexive, then $(v_n)_{n\geq 1}$ admits a subsequence, similarly denoted, which is weekly convergent in X, $v_n \xrightarrow{w} v$ (Kakutani theorem). Then

$$\|v\| \leq \lim_{n\to\infty} \|v_n\|,$$

$$\|v\|^p \leq \lambda_1 , \tag{33}$$

On the other hand, because i is compact, we have $i(v_n) \to i(v)$, this implies $\|i(v_n)\| \to \|i(v)\|$, $\|i(v)\| = 1$ and hence $\|v\|^p \geq \lambda_1$, $\|v\|^p \overset{(33)}{=} \lambda_1$, λ_1 is attained and *a fortiori* nonzero.

2^0 Take into account the definition of λ_1 and 1^0.

3^0 Firstly show that λ_1 is eigenvalue for the couple (J_{ZZ^*} , J_{XX^*}), i.e. $\exists\, u_0 \neq 0$ in X e.g.

$$\lambda_1 (i' \circ J_{ZZ^*} \circ i)u_0 = J_{XX^*} u_0 . \tag{34}$$

Take the functional $\Phi: X \to \mathbf{R}$,

$$\Phi(u) = \frac{1}{p} \|u\|^p - \frac{\lambda_1}{p} \|i(u)\|^p.$$

$\Phi(u) \geq 0 \ \ \forall\, u$ from X (see the definition of λ_1) and, for $u_0 \neq 0$ e.g. $\lambda_1 \overset{1^0}{=} \left(\frac{\|u_0\|}{\|i(u_0)\|}\right)^p$, $\Phi(u_0) = 0$, which imposes (one takes into account that X is a smooth space iff its norm is Gâteaux differentiable on $X \setminus \{0\}$)

$$\Phi'(u_0) = 0 \text{ (Gâteaux derivative).} \tag{35}$$

Then we use the formulae: "X smooth $\Rightarrow J_\varphi x = \varphi(\|x\|)\|x\|'$, $x \neq 0$" and "if X, Y real normed spaces and $f : X \to Y$ Gâteaux differentiable, $F : Y \to \mathbf{R}$ of Gâteaux C^1 class, then $g := F \circ f$ is Gâteaux differentiable ([9]) and $g'(x) = F'(f(x)) \circ f'(x)$" to obtain: $\forall$ u from X, $0 \overset{(35)}{=} \Phi'(u_0)(u) = \left\langle \|u_0\|^{p-1} \| \|'(u_0), u \right\rangle - \lambda_1 \left\langle \|i(u_0)\|^{p-1} \| \|'(i(u_0)), i(u) \right\rangle = (J_{XX^*}\ u_0)(u) - \lambda_1(J_{ZZ^*}(i(u_0))(i(u))) = \langle J_{XX^*} u_0 - \lambda_1(i' \circ J_{ZZ^*} \circ i)(u_0), u \rangle$, i.e. (34).

Let now be λ eigenvalue for the couple (J_{ZZ^*}, J_{XX^*}) and u a corresponding eigenvector. Then

$$\|u\|^p = (J_{XX^*} u)(u) = \lambda \langle J_{ZZ^*}(i(u)), i(u) \rangle = \lambda \|i(u)\|^p,$$

hence

$$\lambda = \frac{\|u\|^p}{\|i(u)\|^p} \geq \lambda_1 . \blacksquare$$

We can now state

Proposition 9. *Let* X *be a real reflexive Banach space and smooth with the* (H) *property*, J_φ *the duality map on* X *with* $\varphi(t) = t^{p-1}$, $p \in (1, +\infty)$. *Suppose that* X *is compactly embedded with the linear injection* i *in the Banach space* Z *and let* $N: Z \to Z^*$ *be an odd hemi-continuous operator with*:

$$\|Nx\| \leq c_1 \|x\|^{p-1} + c_2 \ \ \forall \ x \textit{ from } Z, \ c_1, c_2 \geq 0.$$

Then, for any λ, *if*

$$|\lambda| > \overline{\lim_{\|u\| \to +\infty}} \frac{\|(i' \circ N \circ i)u\|}{\|u\|^{p-1}}, \textit{ a fortiori if } |\lambda| > c_1 \lambda_1^{-1}, \textit{ where}$$

$$\lambda_1 := \inf\{\frac{\|u\|^p}{\|i(u)\|^p} : u \in X \setminus \{0\}\},$$

$\lambda J_\varphi - N$ *is surjective* (N *means* $i' \circ N \circ i$).

Proof. Due to Proposition 8, 1° it follows that $\lambda_1 \neq 0$, hence λ_1^{-1} exists. We apply, as for Proposition 7, Proposition 6 with $T = J_\varphi$, $S = i' \circ N \circ i$. We prove

$$\overline{\lim_{\|u\|\to+\infty}} \frac{\|(i' \circ N \circ i)u\|}{\|u\|^{p-1}} \le c_1\lambda_1^{-1}, \tag{36}$$

By using Proposition 8, which is sufficient to establish the conclusion. $\|(i' \circ N \circ i)u\| \le \|i'\| \ \|N(i(u))\| \le \lambda_1^{-\frac{1}{p}} (c_1 \lambda_1^{\frac{1-p}{p}} \|u\|^{p-1} + c_2)$ $\left(\|i'\| \le \|i\| \le \lambda_1^{-\frac{1}{p}}\right)$ and (36) becomes obviously. ■

3.3. Existence of the solution for the problem

$$\begin{cases} -\lambda \operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(\cdot, u(\cdot)) + h, \ x \in \Omega \\ u \mid \partial\Omega \end{cases}, \ p \in (1, +\infty)$$

The operator - Δ_p, p ∈ (1, + ∞) (the *p*-Laplacian)

Ω is a open set, with finite measure Lebesgue, from $\mathbf{R}^N$, $N \ge 2$. The norm on $W_0^{1,p}(\Omega)$ will be $u \to \|u\|_{1,p} = \| |\nabla u| \|_{L^p(\Omega)}$. The dual of $(W_0^{1,p}(\Omega). \| \cdot \|_{1,p})$ is designates by $W^{-1,p'}(\Omega)$, p' the conjugate with the exponent p.

Consider the operator $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$,

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u). \tag{37}$$

This acts according to ([10])

$$\langle -\Delta_p u, v \rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx \ \forall \ u, v \text{ from } W_0^{1,p}(\Omega). \tag{38}$$

The problem

$$(*)\begin{cases} -\lambda \Delta_p u) = f(\cdot, u(\cdot)) + h, \ x \in \Omega, \lambda \in \mathbf{R} \\ u \mid \partial\Omega \end{cases}$$

Proposition 10. *Let* Ω *be an open bounded of* C^1 *class set from* $\mathbf{R}^N$, $N \ge 2$, p ∈ (1, + ∞), h *from* $W^{-1,p'}(\Omega)$ *and* f : Ω × **R** → **R** *Carathéodory function with the properties*

1^0 f (x, – s) = – f (x, s) ∀s *from* **R**, ∀ x *from* Ω,

2^0 $|f(x, s)| \le c_1 |s|^{q-1} + \beta(x)$ ∀ s *from* **R**, ∀ x *from* Ω \ A, μ(A) = 0, *where* $c_1 \geq 0$, $q \in (1, p)$, $\beta \in L^{q'}(\Omega)$, $\frac{1}{q} + \frac{1}{q'} = 1$.

Then, for any $\lambda \neq 0$, *the problem* (*) *has solution in* $W_0^{1,p}(\Omega)$ *in the sense*[3] *of* $W^{-1,p'}(\Omega)$.

Explanations. The relationship $u \mid \partial\Omega$ from (*) is in the sense of the trace[4]. $f(\cdot, u) = N_f u$, N_f Nemyskii operator, and so the equation from (*) can be written as

$$-\lambda\, \Delta_p u = N_f u + h. \tag{39}$$

Let be i': $L^{q'} \to W^{-1,p'}$ the transposed of i (as $(L^q)^* = L^{q'}$).

u_0 from $W_0^{1,p}$ is *solution* for (*) in the sense of $W^{-1,p'}$, if

$$-\lambda\, \Delta_p u_0 = (i' \circ N_f \circ i) u_0 + h. \tag{40}$$

Proof. $-\Delta_p = J_\varphi$ ([9]), J_φ duality map with $\varphi(t) = t^{p-1}$, the Banach space $(W_0^{1,p}, \|\cdot\|_{1,p})$ being uniform convex ([9]), and consequently with (H) property and is reflexive (uniform convex $\Rightarrow$ reflexive). It is also smooth (its norm being Gâteaux differentiable on $W_0^{1,p} \setminus \{0\}$ ([2])). So, apply Proposition 7 (with $X = W_0^{1,p}$, $Z = L^q$, $N = N_f$ – odd continuous operator, ([2]), $Z^* = L^{q'}$, take into account $\|N_f u\|_{0,q'} \leq c_1 \|u\|_{0,q}^{q-1} + c_2$, $c_2 := \|\beta\|_{0,q'}$, $\forall\, u$ from L^q), the operator $\lambda(-\Delta_p) - S$: $W_0^{1,p} \to W^{-1,p'}$, where $S = i' \circ N_f \circ i$ is surjective, *a fortiori* the operator $-\lambda\Delta_p - S - h$ is surjective (commuting group) and hence $\exists\, u_0$ in $W_0^{1,p}$ which satisfies (40). ■

Replacing q with p in 2^0 from Proposition 10 and applying Proposition 9, one obtains

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[3] Al the terms from the first relationship from (*) are considered as elements of $W^{-1,p'}(\Omega)$.

[4] The trace is the unique linear continuous operator γ: $W^{1,p}(\Omega) \to W^{1-\frac{1}{p},p'}(\partial\Omega)$ such that it is surjective and $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \Rightarrow \gamma(u) = u \mid \partial\Omega$. So we use here the notation $u \mid \partial\Omega$ for $\gamma(u)$.

Proposition 11. *Let* Ω *bean open bounded of* C^1 *class set from* $\mathbf{R}^N$, N ≥ 2, p ∈ (1, + ∞), h *from* $W^{-1,p'}$ *and* f : Ω × **R** → **R** *a Carathéodory function with the properties*

1^0 f (x, −s) = − f (x, s) ∀ x *from* Ω, ∀ s *from* **R**,

2^0 $|f(x, s)| \le c_1 |s|^{p-1} + \beta(x)$ ∀ s *from* **R**, ∀ x *from* Ω \ A, μ(A) = 0,

where $c_1 \ge 0$, $\beta \in L^{p'}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$.

Finally, let be i: $W_0^{1,p} \to L^p(\Omega)$ *linear compact embedding. Then, for any* λ, *if*

$$|\lambda| > c_1 \lambda_1^{-1},\ \lambda_1 := \inf \left\{ \frac{\|u\|_{1,p}^p}{\|i(u)\|_{0,p}^p} : u \in W_0^{1,p} \setminus \{0\} \right\},$$

the problem (*) *has solution in* $W_0^{1,p}(\Omega)$ *in the sense of* $W^{-1,p'}(\Omega)$.

Proof. The statement is correct because ($W_0^{1,p}$, $\|\cdot\|_{1,p}$) is compact embedded in L^p (Rellich-Kondrashev theorem). Apply Proposition 9. ■

Remark. The condition from Proposition 11 can be replaced (see Proposition 9) by:

$$|\lambda| > \overline{\lim_{\|u\| \to +\infty}} \frac{\|(i' \circ N_f \circ i)u\|}{\|u\|^{p-1}}.$$

Attention to λ_1 : $\lambda_1^{-\frac{1}{p}}$ is optimal for the inequality from the enunciation, it is attained, nonzero, and the smallest eigenvalue of the couple ($J_{L^q L^{q'}}, J_{W_0^{1,p} W^{-1,p'}}$) (see Proposition 8).

6. Conclusions

An original version of some minimization results (minimization with regular constraints giving global critical points) due to the author, which involved

Ekeland principle and a condition of Palais-Smale type is presented in the first part of this work. These are involved in characterization of weak subsolutions and weak supersolutions for a partial differential equation involving also Nemytski operator.

The novel results of this paper are developed in the second section. An extension of a theorem from [8] is proved with weakened and two corollaries have been proved for it. Propositions 6, 7 and 9 are due to the author.

The Propositions 10 and 11, concerning the *p*-Laplacian, are also obtained by the author.

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