U.P.B. Sci. Bull., Series A, Vol.73, Iss. 4, 2011 ISSN 1223-7027

ACCURATE ELEMENT METHOD NUMERICAL INTEGRATION OF A NONLINEAR FIRST ORDER ORDINARY DIFFERENTIAL EQUATION BY SOLVING AN EQUIVALENT NONLINEAR ALGEBRAIC EQUATION

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O metodologie pentru integrarea numerică a ecuatiilor diferențiale ordinare (ODE) neliniare a fost prezentată în [4], dar ea este greoaie şi se bazează pe un calcul iterativ. Aceiaşi problemă se rezolva aici pe altă cale în care nu mai se recurge la iterații, ci se transformă integrarea unei ODE neliniare într-o banală rezolvare a unei ecuații algebrice neliniare. Se consideră ODE neliniară cu coeficienți constanți A, B

$$\phi' = A\,\phi + B\,\phi^n + W(x)$$

Pentru determinarea valorii functiei ϕ_T *la capatul T al domeniului de integrare trebuie rezolvată ecuația neliniară*

$$K_1(\phi_T)^{2n-1} + K_2(\phi_T)^n + K_3(\phi_T)^{n-1} + K_4\,\phi_T + K_5 = 0$$

în care coeficienții $K_i(i=1,2,3,4,5)$ *se obțin direct pe parcursul procesului de calcul. Precizia calculului poate fi evaluată direct prin determinarea reziduului, fără a se efectua alte operații ajutatoare. Se prezintă deasemenea metodologia de rezolvare a ODE cu coeficienți variabili.*

A methodology for the numerical integration of first order non-linear Ordinary Differential Equations (ODE) has been already developed by the Accurate Element Method (AEM) in [4], but it is cumbersome and based on an iterative procedure. Here the same problem is solved without any iteration, by a more natural approach that transforms the problem of a nonlinear ODE in a trivial nonlinear algebraic equation. For example considering the ODE with constant coefficients A, B

$$\phi' = A\,\phi + B\,\phi^n + W(x)$$

the value ϕ_T *of the function at the target abscissa* ϕ_T *can be obtained by solving the polynomial non-linear equation*

$$K_1(\phi_T)^{2n-1} + K_2(\phi_T)^n + K_3(\phi_T)^{n-1} + K_4\,\phi_T + K_5 = 0$$

where the coefficients $K_i(i=1,2,3,4,5)$ *can be obtained directly during the computation procedure. The accuracy of the computation can be evaluated without any modification of the integration mesh or any further work using a powerful tool developed by AEM, namely the residual. The methodology for solving ODEs with variable coefficients is also described.*

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1. ODE to be integrated

Let be the following first order Ordinary Differential Equation (ODE)

$$\phi' = A\,\phi + B\,F(\phi) + W(x) \tag{1.1}$$

In order to simplify the exposure, the coefficients A and B will be initially considered as constants[2]. Besides $\frac{d\phi}{dx}$, ϕ and the free term W(x), the ODE (1.1) includes also a *non-linear term* $F[(\phi(x)]$ that can be **any function of ϕ** such as

$$F(\phi) = \phi^n \tag{1.2}$$

or $F(\phi) = \cos(\phi)$, $F(\phi) = \sqrt[4]{\phi}\,\cos(\sqrt[3]{\phi})$, $F(\phi) = \phi\sin(\phi) + \phi^2\ln(\phi)\ \ldots$

The ODE will be integrated numerically between a starting point x_S and a target point x_T by dividing the domain in sub-domains (elements). The initial (start) condition is

$$\phi(x = x_S) = \phi_S \tag{1.3}$$

The ends of each element will be noted *x=L* (Left) and *x=R* (Right), so that the left end condition becomes

$$\phi(x = L) = \phi_L \tag{1.4}$$

2. The integral equation

The “classical” numerical integration methods accepted today, such as Euler, Heun, Runge-Kutta, avoid carefully any integration procedure. On the contrary the Accurate Element Method (AEM) starts by the integration of the ODE (1.1) that leads to an integral equation

$$\int_L^R \frac{d\phi}{dx}\,dx = A\int_L^R \phi(x)\,dx + B\int_L^R F(\phi)\,dx + \int_L^R W(x)\,dx \tag{2.1}$$

The first and the last integrals can be performed straightforward

$$\int_L^R \frac{d\phi}{dx}\,dx = \phi\Big|_L^R = \phi_R - \phi_L \quad (2.2a) \quad ; \quad (\mathrm{Int}\,W) = \int_L^R W(x)\,dx \quad (2.2b)$$

(Int W) being usually a trivial integral. On the contrary the two remaining integrals cannot be performed directly because both $\phi(x)$ and $F\phi)$ are unknown

$$(\mathrm{Int}\,\phi) = \int_L^R \phi(x)\,dx \quad (2.3) \quad ; \quad (\mathrm{Int}\,F) = \int_L^R F(\phi)\,dx \quad (2.4)$$

The Accurate Element Method performs numerically these integrals replacing $\phi(x)$ and $F(\phi)$ by Concordant Functions (CF) [2,3,4,5,6,7]. A

[2] The case of ODEs with variable coefficients is analyzed in §10

methodology for the integration of nonlinear ODEs based on CFs was already established [4], but the approach developed there lead to cumbersome procedures. A small formal modification of the methodology transforms the quite complicated problem of the integration of a nonlinear ODE into a trivial problem of solving a one dimensional nonlinear algebraic or transcendental equation.

3. Concordant Function f(x)

A Concordant Function (CF) is a polynomial of high or very high degree whose coefficients are obtained rigorously using the information furnished by the governing ODE (1.1) itself. Though the procedure for finding the coefficients was extensively described in [4], it will be shortly presented here in order to introduce a small modification of the approach.

The CF analyzed below is a third degree polynomial noted as f(x), which will be written in the following matrix form

$$f(x) = C_1 + C_2 x + C_3 x^2 + C_4 x^3 = \left[X_4\right]\left[\overline{C}\right]^T \tag{3.1}$$

where $\left[X_4\right] = \left[1 \quad x \quad x^2 \quad x^3\right]$ (3.2) ; $\left[\overline{C}\right] = \left[C_1 \quad C_2 \quad C_3 \quad C_4\right]^T$ (3.3)

The first derivative of (3.1) is

$$f'(x) = C_2 + 2C_3 x + 3C_4 x^2 \tag{3.4}$$

Let be **Q(x)** the function that has to be replaced by the Concordant Function **f(x)**. Because (3.1) includes *four constants* that have to be obtained, it is referred as CF4. These constants are obtained by the Accurate Element Method using the following four conditions obtained by equating the end values of f(x) and Q(x) and their derivatives at the two ends L and R

Condition 1: $$f(L) = C_1 + C_2 L + C_3 L^2 + C_4 L^3 = Q(x = L) = Q_L \tag{3.5a}$$

Condition 2: $$f(R) = C_1 + C_2 R + C_3 R^2 + C_4 R^3 = Q(x = R) = Q_R \tag{3.5b}$$

Condition 3: $$f'(L) = C_2 + 2C_3 L + 3C_4 L^2 = Q'(x = L) = Q'_L \tag{3.5c}$$

Condition 4: $$f'(R) = C_2 + 2C_3 R + 3C_4 R^2 = Q'(x = R) = Q'_R \tag{3.5d}$$

Based on the four conditions (3.5,a,b,c,d) it results the system of equations

$$\begin{bmatrix} 1 & L & L^2 & L^3 \\ 1 & R & R^2 & R^3 \\ 0 & 1 & 2L & 3L^2 \\ 0 & 1 & 2R & 3R^2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} Q_L \\ Q_R \\ Q'_L \\ Q'_R \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} Q_L + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} Q_R + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} Q'_L + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} Q'_R \tag{3.6}$$

The four end values Q_L, Q_R, Q'_L, Q'_R have been separated by multiplying them with an appropriate vector. Suppose – in order to simplify the exposure– that L=0 and R=1, in which case the inverse of the square matrix [4×4] is

$$\begin{bmatrix} 1 & L & L^2 & L^3 \\ 1 & R & R^2 & R^3 \\ 0 & 1 & 2L & 3L^2 \\ 0 & 1 & 2R & 3R^2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{bmatrix} \tag{3.7}$$

If (3.6) is multiplied by (3.7) it results the vector of the unknown coefficients

$$\left[\overline{C}\right] = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix} Q_L + \begin{bmatrix} 0 \\ 0 \\ 3 \\ -2 \end{bmatrix} Q_R + \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} Q'_L + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} Q'_R \tag{3.8}$$

From (3.1) it results now CF4 as a sum of four 3rd degree polynomials

$$\begin{aligned} f(x) &= [X_4]\left[\overline{C}\right]^T = \\ &= (1-3x^2+2x^3)Q_L + (3x^2-2x^3)Q_R + (x-2x^2+x^3)Q'_L + (-x^2+x^3)Q'_R \end{aligned} \tag{3.9}$$

If L≠0 or R≠1 the same procedure will lead to the general relation

$$f(x) = \bar{f}_1(x)Q_L + \bar{f}_2(x)Q_R + \bar{f}_3(x)Q'_L + \bar{f}_4(x)Q'_R \tag{3.10}$$

where $\bar{f}_1(x), \bar{f}_2(x), \bar{f}_3(x), \bar{f}_4(x)$ are four known polynomials.

4. Integration of *Int* ϕ (2.3) and *Int F (2.4)*

4.1 Integration of the term A(Int ϕ)

In order to perform the integral (2.3), the function that has to be replaced here is **ϕ(x)** instead of **Q(x)**, while the Concordant Function will be noted as **y(x)** instead of **f(x)**. Consequently, (3.10) becomes

$$y(x) = \bar{y}_1(x)\phi_L + \bar{y}_2(x)\phi_R + \bar{y}_3(x)\phi'_L + \bar{y}_4(x)\phi'_R \tag{4.1}$$

If (4.1) is replaced in (2.3) it results

$$\begin{aligned} \mathrm{Int}\,\phi &= \int_L^R \phi(x)\,dx \approx \int_L^R y(x)\,dx = \int_L^R \left(\bar{y}_1(x)\phi_L + \bar{y}_2(x)\phi_R + \bar{y}_3(x)\phi'_L + \bar{y}_4(x)\phi'_R\right)dx \\ &= I_{y1}\phi_L + I_{y2}\phi_R + I_{y3}\phi'_L + I_{y4}\phi'_R \end{aligned} \tag{4.2}$$

where $I_{y1} = \int_L^R \bar{y}_1(x)\,dx\ ;\ I_{y2} = \int_L^R \bar{y}_2(x)\,dx\ ;\ I_{y3} = \int_L^R \bar{y}_3(x)\,dx\ ;\ I_{y4} = \int_L^R \bar{y}_4(x)\,dx$ (4.3)

These integrals are performed using the known polynomials $\bar{y}_1(x)$, $\bar{y}_2(x)$, $\bar{y}_3(x)$, $\bar{y}_4(x)$. For instance if *L=0* and *R=1* – as it was considered in (3.7) – it results

$I_{y1}=1/2$; $I_{y2}=1/2$; $I_{y3}=1/12$; $I_{y4}=-1/12$ (4.4)

The product **A(Int ϕ)** results finally as

$$A(\mathrm{Int}\,\phi) = J_1\phi_L + J_2\phi_R + J_3\phi'_L + J_4\phi'_R \tag{4.5}$$

where

$$J_1 = AI_{y1} \;;\; J_2 = AI_{y2} \;;\; J_3 = AI_{y3} \;;\; J_4 = AI_{y4} \tag{4.6}$$

Besides J_i (i=1,2,3,4) the relation (4.5) includes also

1. The Left end value of the function ϕ_L that is **known** as initial condition (1.3).
2. The Right end value ϕ_R, which represents the **unknown** that has to be found.
3. The Left end value of the derivative ϕ'_L can be evaluated using the ODE (1.1) itself

$$\phi'_L = A\,\phi_L + B\,F_L + W_L \tag{4.7}$$

where $W_L = W(x{=}L)$ and – if F is given by (1.2) – $F_L = (\phi_L)^n$.

4. The Right end value of the derivative ϕ'_R is **unknown**, but using also ODE (1.1) it can be expressed as function of ϕ_R. Because $W_R = W(x{=}R)$ and $F_R = (\phi_R)^n$

$$\phi'_R = A\,\phi_R + B\,F_R + W_R = A\,\phi_R + B(\phi_R)^n + W_R \tag{4.8}$$

4.2 Integration of *Int F* (2.4)

The integral (2.4) can be performed replacing in (3.10) **Q** by **F** while the Concordant Function will be noted as **z(x)** instead of **f(x)**. It is necessary to specify that the Concordant Function used for the integration of (2.4) can still be CF4 (third degree polynomial) used above or a higher degree polynomial as those that will be established in §7. At this stage the exposure is simplified if as z(x) will be used the same CF4, so that the replacing function (3.10) becomes

$$z(x) = \bar{z}_1(x)F_L + \bar{z}_2(x)F_R + \bar{z}_3(x)F'_L + \bar{z}_4(x)F'_R \tag{4.9}$$

The integral (2.4) is given by

$$\mathrm{Int}\,F = \int_L^R F[\phi(x)]dx \approx \int_L^R z(x)\,dx = \int_L^R \left(\bar{z}_1(x)F_L + \bar{z}_2(x)F_R + \bar{z}_3(x)F'_L + \bar{z}_4(x)F'_R\right)dx =$$
$$= I_{z1}F_L + I_{z2}F_R + I_{z3}F'_L + I_{z4}F'_R \tag{4.10}$$

where

$$I_{z1} = \int_L^R \bar{z}_1(x)\,dx \;;\; I_{z2} = \int_L^R \bar{z}_2(x)\,dx \;;\; I_{z3} = \int_L^R \bar{z}_3(x)\,dx \;;\; I_{z4} = \int_L^R \bar{z}_4(x)\,dx \tag{4.11}$$

Remark. If the Concordant Function is still CF4 these integrals are in fact (4.4)

$$I_{z1}=1/2 \;;\; I_{z2}=1/2 \;;\; I_{z3}=1/12 \;;\; I_{z4}=-1/12 \tag{4.12}$$

The term **B(Int F)** results finally as

$$B(\mathrm{Int}\,F) = G_1F_L + G_2F_R + G_3F'_L + G_4F'_R \tag{4.13}$$

where

$$G_1 = BI_{z1} \;;\; G_2 = BI_{z2} \;;\; G_3 = BI_{z3} \;;\; G_4 = BI_{z4} \tag{4.14}$$

All the terms from (4.13) depending on $F(\phi)$ have to be thoroughly analyzed. If $F(\phi)=\phi^n$ (1.2) it results

1. The value of the function F at the Left end $F_L=(\phi_L)^n$ is **known**.

2. The value of the function F at the Right end

$$F_R=(\phi_R)^n \tag{4.15}$$

is **unknown** because it depends on the **unknown** value ϕ_R .

3. The Left end derivative F'_L requires a special analysis. In fact F is a function of ϕ, while the integral *Int F* (2.4) depends on *x*, so that

$$\frac{dF(\phi)}{dx}=\frac{dF(\phi)}{d\phi}\frac{d\phi}{dx}=n\phi^{n-1}\frac{d\phi}{dx} \tag{4.16}$$

The Left end it results as

$$F'_L=\left(\frac{dF}{dx}\right)_L=\left(\frac{dF}{d\phi}\right)_L\left(\frac{d\phi}{dx}\right)_L=n\,(\phi_L)^{n-1}\,\phi'_L \tag{4.17}$$

F'_L is **known** because ϕ_L is (1.4) and ϕ'_L is given by (4.7).

4. The **unknown** right end derivative F'_R can be expressed as function of ϕ_R, if ϕ'_R is replaced by (4.8)

$$F'_R=n(\phi_R)^{n-1}\phi'_R=nA(\phi_R)^n+nB(\phi_R)^{2n-1}+n(\phi_R)^{n-1}W_R \tag{4.18}$$

5. The integral equation

Because all the integrals included in (2.1) have received a specific form, it results replacing (2.2a), (2.2b), (4.5) and (4.13)

$$\boxed{\phi_L-\phi_R+(J_1\phi_L+J_2\phi_R+J_3\phi'_L+J_4\phi'_R)+(G_1F_L+G_2F_R+G_3F'_L+G_4F'_R)+\mathrm{Int}\,W=0} \tag{5.1}$$

If (4.8), (4.15), (4.18) are used it results a nonlinear equation whose unknown is the target value ϕ_R. This equation transforms in fact **the integration of the nonlinear ODE (1.1), in another more simple problem, namely to find the appropriate root of a nonlinear one dimension equation.**

If F is given by (1.2) it results the non-linear equation

$$K_1(\phi_R)^{2n-1}+K_2(\phi_R)^n+K_3(\phi_R)^{n-1}+K_4\phi_R+K_5=0 \tag{5.2}$$

whose coefficients K_i(i=1,2,3,4,5) can be evaluated straightforward

$$K_1=nBG_4\ ;\ K_2=BJ_4+G_2+nAG_4\ ;\ K_3=nG_4W_R\ ;\ K_4=J_2+A\,J_4-1$$

$$K_5=\phi_L+J_1\phi_L+J_3\phi'_L+G_1F_L+G_3F'_L+J_4W_R+\mathrm{Int}\,W$$

If n=2 the equation (5.2) becomes $K_1(\phi_R)^3+K_2(\phi_R)^2+(K_3+K_4)\phi_R+K_5=0$ (5.3)

6. Verification of the solution

6.1 Residual evaluation

The root of (5.1) or – as a particular case – (5.2) is represented by the target value ϕ_R. Thus $F_R = (\phi_R)^2$ is also known, so that the right end derivative ϕ'_R can be obtained form (4.8). The replacing function (4.1) that result can be considered as a *quasi-analytic solution* [6] of the ODE (1.1). The best way to verify its accuracy to replace (4.1) in the ODE (1.1). Because (4.1) is not an exact solution it will result a *residual function* different from zero given by

$$\mathrm{Res}(x) = Ay - \frac{dy}{dx} + B\,F(y) + W(x) \tag{6.1}$$

A simple way to appreciate numerically the accuracy of the solution is to divide the integration interval in NP points having the abscissas x_i (i=1,2..NP) where the residual is computed. This allows getting a synthetic result as the mean square root value

$$R_{MS} = \frac{1}{NP}\sqrt{\sum_{i=1}^{i=NP}\left[\mathrm{Res}(x_i)\right]^2} \tag{6.2}$$

The problem of the accuracy of the numerical results obtained by ODEs numerical integration is analyzed thoroughly in [Fox], from which we quote: “A good method for solving any differential equation will not only produce an accurate result but also give at least a reasonable indication of the accuracy achieved. With one unrefined application of the any step-by-step method there is no possibility of assessing the accuracy, and at least some further work is needed”.

The residual reflected by (6.2) presents many particularities as compared to other methods used for the appreciation of the accuracy achieved:

1. *The residual is evaluated using the information already found*. It makes possible assessing the accuracy without any “further work”.

2. *The value of R_{MS} (5.2) obtained for a given element does not depend on the previous history of the computation.* Consequently, evaluating the truncation and/or round-up errors is useless for AEM.

3. The residual reflects the closeness between y(x) and the exact solution[3]. The value of R_{MS} (5.2) represents a tool that allows following the evolution of the computation at each step. For instance a sudden increase of R_{MS} can indicate that the solution is moving off from the exact solution. In such case the results are no more reliable and some decision[4] has to be taken to improve them.

4. *The residual reflects the evolution of a complex relation, not of a particular parameter*. The R_{MS} evaluation is based on the ODE, which includes

[3] If such a solution exists

[4] Modifying the dimensions of the elements or/and using a higher degree polynomial as CF

besides ϕ_R at least the derivative ϕ'_R. Therefore the value of R_{MS} indicates *the evolution of the whole integration procedure* and can not be used a direct appreciation of the accuracy of the Target value ϕ_R.

Example 1. The ODE

$$\frac{d\phi}{dx} = \phi' = 4\phi + 3\phi^2 - 16 - 70x + 6x^2 + 40x^3 - 75x^4 + 36x^5 - 12x^6 \qquad (6.3)$$

will be integrated imposing as initial condition $\phi_{Start} = 2$ (6.4)

Here the nonlinear term is (1.2), where **n=2**. Because as replacing functions y(x) and z(x) will be used CF4 (which is a poor replacing function that can give only lowery results, similar to those obtained by the fourth order Runge-Kutta method) the integration domain will be quite small, being limited at x_{Final}=0.1. With the purpose to follow the behavior of CF4 when the integration interval is modified, several integrations will be performed, each time *the entire domain being covered by a single element*. There were considered 10 different intervals, starting form x_{Target}=0.01 and increasing step 0.01 until the x_{Final}=0.1 is reached.

Remark. The exact solution of (6.3) is $\phi_{exact} = 2 + 4x - 3x^2 + 2x^3$ (6.5)

The results of the computations are summarized in Table 1. Because the exact target value can be obtained (in this particular case) from (6.5), the true errors have been given in the fourth column. The variations of the absolute value of the true errors and of the R_{MS} [calculated with (6.2)] are represented in Fig.1.

Table 1

Computations using a single element				
x_{Target}	Target value ϕ_R		True error	Residual (R_{MS})
	Computed	Exact		
(1)	(2)	(3)	(4)	(5)
0.01	2.039698012094537	2.039702	-1.95×10^{-6}	2.39×10^{-8}
0.02	2.078784243263860	2.078816	-1.52×10^{-5}	3.28×10^{-7}
0.03	2.117247609102766	2.117354	-5.02×10^{-5}	1.59×10^{-6}
0.04	2.155078795607963	2.155328	-1.15×10^{-4}	4.91×10^{-6}
0.05	2.192271629931732	2.192750	-2.18×10^{-4}	1.17×10^{-5}
0.06	2.228820795332329	2.229632	-3.63×10^{-4}	2.44×10^{-5}
0.07	2.264712455934159	2.265986	-5.62×10^{-4}	4.63×10^{-5}
0.08	2.299924713646576	2.301824	-8.25×10^{-4}	7.96×10^{-5}
0.09	2.334441517690581	2.337158	-1.16×10^{-3}	1.24×10^{-4}
0.10	2.368254112961083	2.372000	-1.58×10^{-3}	1.84×10^{-4}

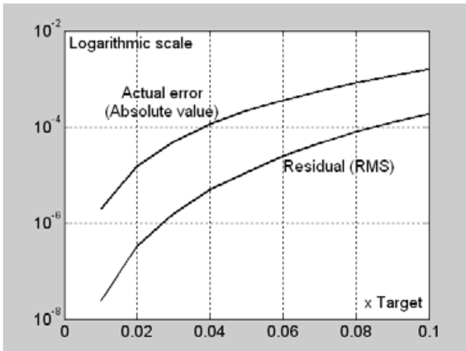

Fig.1

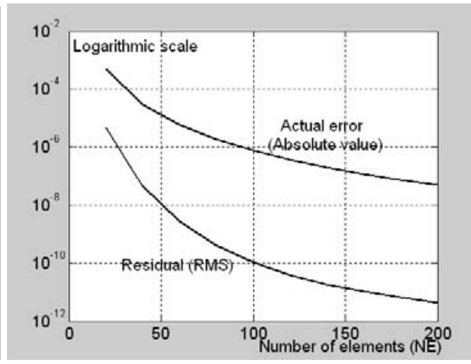

Fig.2

It results from Table 1 and from Fig.1 that the R_{MS} values are more optimistic than the true errors. It is useful to observe that both curves have a similar trend. This fact is important because the values of the R_{MS} are available to the user at any phase of the computation, so that R_{MS} can be a tool to follow the evolution of the computation. It seems that residuals like $R_{MS} \approx (10^{-4} \dots 10^{-5})$ show that the corresponding target value is far from accurate. Obviously these values – resulted from a small number of tests – are disputable.

Example 2. The same ODE (6.3) will be integrated on a larger domain between x_{Start}=0 and x_{Target}=0.5, using each time a different number of elements between NE=20 and NE=200. The results (together with the true errors) are given in Table 2 and Fig.2.

Table 2

Exact Target Value = 3.5			
NE	Target value ϕ_R	True error	Residual (R_{MS})
(1)	(2)	(3)	(4)
20	3.498298373701107	-4.86×10^{-4}	4.90×10^{-6}
40	3.499893369734073	-3.04×10^{-5}	4.50×10^{-8}
60	3.499978932029412	-6.02×10^{-6}	2.82×10^{-9}
80	3.499993333507704	-1.90×10^{-6}	4.13×10^{-10}
100	3.499997269341086	-7.80×10^{-7}	1.05×10^{-10}
120	3.499998683076844	-3.76×10^{-7}	3.97×10^{-11}
140	3.499999289156937	-2.03×10^{-7}	1.92×10^{-11}
160	3.499999583312941	-1.19×10^{-7}	1.08×10^{-11}
180	3.499999739848328	-7.43×10^{-8}	6.59×10^{-12}
200	3.499999829339440	-4.87×10^{-8}	4.25×10^{-12}

The descending trends of both R_{MS} and true errors are to be expected, because as NE increases the length of each element decreases leading therefore to a better result. From this example it results that a $R_{MS} \approx (10^{-11} \dots 10^{-12})$ may indicate a very good value of the target value, which can be considered as accurate.

The results reflected in Fig.2 especially for higher NEs seem to be reassuring, but they can be delusive. In fact the exact solution (6.5) has the same degree as CF4. If the exact solution is a higher degree polynomial or non-polynomial, the results could be inaccurate. Obviously, if the residuals indicate that the solution is inaccurate, one can resort to a trivial solution, namely to increase the number of elements. But the Accurate Element Method furnishes a better alternative: the use of higher order CFs.

7. Higher degree polynomials used as Concordant Functions

Higher degree polynomials – up to CF16 (15^{th} degree polynomial) – have been used in [4] in order to improve the accuracy. Here the spectrum will be limited to CF6 (5^{th} degree polynomial), CF8 (7^{th} degree polynomial) and CF10 (9^{th} degree polynomial).

The Concordant Function referred as *CF6* is a fifth degree polynomial given by

$$f(x) = C_1 + C_2 x + C_3 x^2 + C_4 x^3 + C_5 x^4 + C_6 x^5 \tag{7.1}$$

From the six conditions necessary to obtain the unknown coefficients, four are those represented by (3.5a), (3.5b), (3.5c), (3.5d) using obviously (7.1) instead of (3.1). The two more necessary conditions are obtained using the second derivative of (7.1)

Condition 5: $f''(L) = 2C_3 + 6C_4 L + 12C_5 L^2 + 20C_6 L^3 = Q''_L$

Condition 6: $f''(R) = 2C_3 + 6C_4 R + 12C_5 R^2 + 20C_6 R^3 = Q''_R$

The procedure that follows is similar to that described in §3, §4.1 and §4.2 including some obvious modifications due to the increase of the polynomial degree. The replacing functions (4.1) and (4.9) become

$$y(x) = \overline{y}_1(x)\phi_L + \overline{y}_2(x)\phi_R + \overline{y}_3(x)\phi'_L + \overline{y}_4(x)\phi'_R + \overline{y}_5(x)\phi''_L + \overline{y}_6(x)\phi''_R \tag{7.2a}$$

$$z(x) = \overline{z}_1(x)F_L + \overline{z}_2(x)F_R + \overline{z}_3(x)F'_L + \overline{z}_4(x)F'_R + \overline{z}_5(x)F''_L + \overline{z}_6(x)F''_R \tag{7.2b}$$

The second derivatives are replaced using the first derivative of the ODE (1.1) [4].

The integrals (4.5) and (4.15) have will include six terms

$$A(\text{Int}\,\phi) = J_1\phi_L + J_2\phi_R + J_3\phi'_L + J_4\phi'_R + J_5\phi''_L + J_6\phi''_R \tag{7.3}$$

$$B(\text{Int}\,F) = G_1F_L + G_2F_R + G_3F'_L + G_4F'_R + G_5F''_L + G_6F''_R \tag{7.4}$$

The final form of the integral equation will include four more terms as compared to (5.1), but leads also to a nonlinear one-dimensional equation whose appropriate root gives the target value ϕ_R

$$\begin{aligned} &\phi_L - \phi_R + \left(J_1\phi_L + J_2\phi_R + J_3\phi'_L + J_4\phi'_R + J_5\phi''_L + J_6\phi''_R\right) + \\ &+ \left(G_1F_L + G_2F_R + G_3F'_L + G_4F'_R + G_5F''_L + G_6F''_R\right) + \text{Int}\,W = 0 \end{aligned} \tag{7.5}$$

The residual results by replacing (7.2a) into the ODE (1.1).

The procedure for obtaining CF8 and CF10 is described in [4]. It is useful to mention that the final equation will include four more terms for CF8 and eight more terms for CF10, as compared to (7.5).

8. ODEs integrated with four different Concordant Functions

The following examples will be solved using all the four *CFs* mentioned above. The comparison of the results leads to some useful conclusions.

Example 3. The ODE
$$\frac{d\phi}{dx} = \phi' = 4\phi + 3\cos(\phi) - 2 - 3x - 5x^2 \tag{8.1}$$
will be integrated between $x_{Start} = 0$ and $x_{Target} = 1$, the initial condition being $\phi_{Start} = 0.1$.

This time the integration procedure is extended to a larger domain and has also another goal: *finding the appropriate number of elements NE so that the R_{MS} of the residual is around 10^{-9}*. The results obtained are given in Table 3.

Because for the ODE (8.1) no exact solution is known by the author, the Table 3 includes besides the general information represented by NE, only the results of the computation. Some values included in Table 3 deserve to be analyzed:

Table 3

Concordant Function	NE	Element length	Target value (x=1)	R_{MS}
CF4	900	0.00111	–**0.759194888** 4170064	5.82×10^{-9}
CF6	70	0.01428	–**0.759194888** 2709806	7.25×10^{-9}
CF8	32	0.03125	–**0.759194888** 8472719	2.96×10^{-9}
CF10	17	0.0588	–**0.759194888** 4170064	1.57×10^{-9}

1. The number of elements necessary for obtaining $R_{MS} \approx 10^{-9}$ with the four *CFs* is totally different. The CF4 needs 53 times more elements as compared to CF10.

2. *The Target value for all the four CFs is practically the same if the condition $R_{MS} \approx 10^{-9}$ is fulfilled.*

3. The possibility to integrate the same ODE with different CFs gives to the user a powerful tool to know *how accurate the computed Target value is*. In fact from Table 3 it results that the Target values coincide – for all the four CFs – with **9 decimal digits**, so that
$$\boldsymbol{\phi_{Target} = -0.759194888} \tag{8.2}$$
can be considered as accurate.

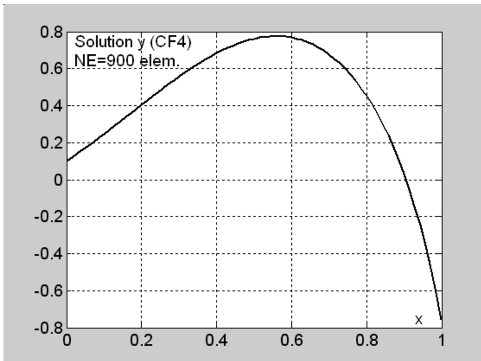

Fig.3

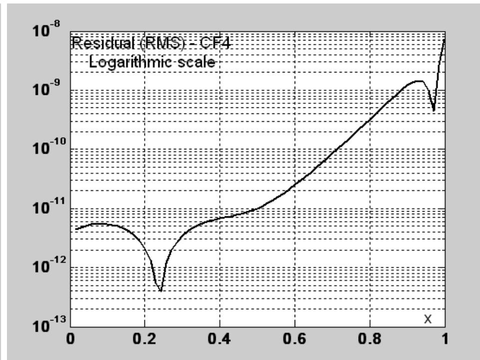

Fig.4

The variation of the function-solution along *x* is represented in Fig.3, while that of the R_{MS} is given in Fig.4. While the graphs of R_{MS} represented in Figs.1 and 2 vary monotonously, the graph given in Fig.4 shows a curious behavior, for which the author has no reliable explanation. In fact the two local variations of the R_{MS} near x≈ 0.24 and x ≈ 0.97 do not represent a tendency towards any instability, but on the contrary, is a local improvement of the result because in both cases the value of the R_{MS} is smaller. Nevertheless the variation of the function-solution (Fig.3) is smooth, the perturbations shown in Fig.4 having no relevant influence. Anyway the graph given in Fig.4 shows that one may obtain interesting information (that have to be better understood) concerning the computation progress by following the variation of the R_{MS}.

Example 4. The ODE $\phi' = 4\phi + 3\sqrt[4]{\phi}\,\cos(\sqrt[3]{\phi}) - 2 - 3x - 5x^2$ (8.3)

will be integrated between $x_{Start} = 0$ and $x_{Target} = 1$, the initial condition being $\phi_{Start} = 2$.

The same procedure as in *Example 3* will be used: *finding the appropriate number of elements NE so that the R_{MS} of the residual is around 10^{-8}*. The results of this approach are given in Table 4.

Table 4

Concordant Function	NE	Elem.length	Target value (x=1)	R_{MS}
CF4	800	0.00125	**5.84485405** 7378286	1.73×10^{-8}
CF6	30	0.03333	**5.84485405** 8499404	2.81×10^{-8}
CF8	8	0.125	**5.84485405** 7301115	4.44×10^{-8}
CF10	5	0.2	**5.84485405** 9367726	1.43×10^{-8}

Here the results are similar to those obtained in *Example 3*, namely the Target values coincide for all four *CF*s with the following 9 decimal digits

$$\boldsymbol{\phi_{Target} = 5.84485405} \quad (8.4)$$

This result can also be considered as accurate.

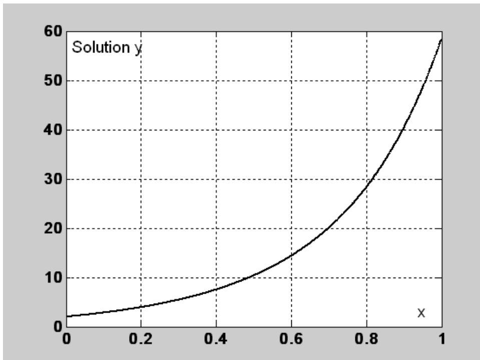

Fig.5

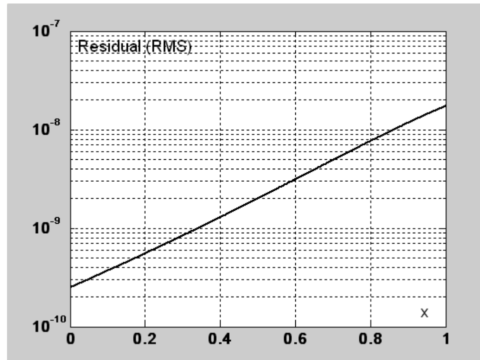

Fig.6

The variations of the function-solution *y* and of the residual are given in Figs.5 and 6, respectively. This time the R_{MS} represented on a logarithmic scale has a monotonous variation.

Remark. The Target values in *Examples 3* and *4* coincide with 9 decimal digits. These results have been obtained from **four different solutions** of the same ODE solving **four different non-linear equations**, on **four different integration sub-domains** and they **coincide with 9 digits**. It is difficult to accept that this is *a pure coincidence.* On the contrary, is more rational to consider that the results (8.2) and (8.4) are **reliable.**

9. ODE with variable coefficients

Let consider now the ODE with variable coefficients

$$D(x)\frac{d\phi}{dx} = A(x)\,\phi + B(x)\,F(\phi) + W(x) \tag{9.1}$$

where A(x), B(x), D(x) are given by three functions of *x*. From the derivative

$$\frac{d[D(x)\,\phi(x)]}{dx} = D(x)\frac{d\phi(x)}{dx} + \frac{dD(x)}{dx}\phi(x) \tag{9.2}$$

it results using (9.1)

$$\frac{d[D(x)\,\phi(x)]}{dx} - \frac{dD(x)}{dx}\phi(x) = A(x)\,\phi(x) + B(x)\,F(\phi) + W(x)\,, \quad \text{or}$$

$$\frac{d[D(x)\,\phi(x)]}{dx} = \alpha(x)\,\phi(x) + B(x)\,F(\phi) + W(x) \tag{9.3}$$

where

$$\alpha(x) = A(x) + \frac{dD(x)}{dx} \tag{9.4}$$

From (9.3) it results, similar to (2.1), the integral equation

$$\int_L^R \frac{d[D(x)\,\phi(x)]}{dx}dx = D(x)\,\phi(x)\;\Big|_{x=L}^{x=R} = D(x=R)\,\phi(x=R) - D(x=L)\,\phi(x=L)$$

$$= D_R\phi_R - D_L\phi_L = \int_L^R \alpha(x)\phi(x)\,dx + \int_L^R B(x)\,F(\phi)\,dx + \int_L^R W(x)\,dx \tag{9.5}$$

The integrals $(\text{Int}\alpha\phi) = \int_L^R \alpha(x)\phi(x)\,dx$ and $(\text{IntBF}) = \int_L^R B(x)\,F(\phi)\,dx$ can be performed using the Concordant Functions y(x) (4.1) and z(x) (4.9), following the procedure developed in §4. Similarly to (4.2) it results

$$(\text{Int}\,\alpha\phi) \approx \int_L^R \alpha(x)y(x)\,dx = \int_L^R \alpha(x)\left(\bar{y}_1(x)\,\phi_L + \bar{y}_2(x)\,\phi_R + \bar{y}_3(x)\,\phi'_L + \bar{y}_4(x)\,\phi'_R\right)dx\ , \text{ or}$$

$$\text{Int}\alpha\,\phi = J_1\phi_L + J_2\phi_R + J_3\phi'_L + J_4\phi'_R \tag{9.6}$$

where the integrals J_i(i=1,2,3,4) result from the products

$$J_1 = \int_L^R \alpha(x)\bar{y}_1(x)\,dx\,;J_2 = \int_L^R \alpha(x)\bar{y}_2(x)\,dx\,;J_3 = \int_L^R \alpha(x)\bar{y}_3(x)\,dx\,;J_4 = \int_L^R \alpha(x)\bar{y}_4(x)\,dx \tag{9.7}$$

Following a similar procedure for **IntBF** it results, using z(x) (4.9)

$$\text{Int}\,BF \approx \int_L^R B(x)z(x)\,dx = \int_L^R B(x)\left(\bar{z}_1(x)\,F_L + \bar{z}_2(x)\,F_R + \bar{z}_3(x)\,F'_L + \bar{z}_4(x)\,F'_R\right)dx\ , \text{ or}$$

$$\text{Int}\,BF = G_1F_L + G_2F_R + G_3F'_L + G_4F'_R \tag{9.8}$$

where the integrals G_i(i=1,2,3,4) result from the products

$$G_1 = \int_L^R B(x)\bar{z}_1(x)\,dx\ ;\ G_2 = \int_L^R B(x)\bar{z}_2(x)\,dx\ ;\ G_3 = \int_L^R B(x)\bar{z}_3(x)\,dx\ ;\ G_4 = \int_L^R B(x)\bar{z}_4(x)\,dx$$

Thus the basic relation remains similar to (5.1), except the first and second terms, which have to be modified according to (9.5)

$$\boxed{D_L\phi_L - D_R\phi_R + \left(J_1\phi_L + J_2\phi_R + J_3\phi'_L + J_4\phi'_R\right) + \left(J_1F_L + J_2F_R + J_3F'_L + J_4F'_R\right) + \text{Int}\,W = 0} \tag{9.9}$$

Obviously, (5.1) results from (9.9) for $D_L= D_R=1$.

10. Some conclusions and further developments

The purpose of this paper is to present a short but comprehensive approach concerning the numerical integration of nonlinear first order ODEs. The special strategy developed here transfer the integration problem to the simpler approach of solving a trivial non-linear algebraic equation. In order to facilitate the exposure some simplifications – that can be easily discarded – have been adopted:

1. All the relations have been established in Cartesian coordinates. In this case it is necessary to compute *for each element* the inverse of the square matrix that multiply the vector $\left[\overline{C}\right]$(3.3), which can be time consuming for higher degree CFs. More than that, when the degree of the polynomial increases beyond 10 or 12, some numerical difficulties may occur for the small length elements. All these problems are implicitly discarded if one uses a *natural* (dimensionless) axes system whose abscissas vary between η_L=0 and η_R=1 [4].

2. The computation "strategy" used here is rudimentary and requires an improvement. The computation was developed by dividing the entire domain in *equal steps*, or the use of steps with different lengths becomes necessary when the solution begins to change rapidly [10].

3. No details have been given concerning the establishing of the particular form of the non-linear algebraic such as (5.2), starting from the general equation (5.1). This is in fact a routine problem that can be solved using a very simple program implemented in MAPLE, which performs symbolic operations. An appropriate connection between MATLAB and MAPLE can eliminate any intervention of the user.

The methodology developed here creates a frame for solving the more difficult problem of the numerical integration of explicit ODEs. When the derivative ϕ' can not be isolated and moved on the left side of the equal sign as in (1.1), the ODE is considered as **implicit**. Such an ODE may be difficult to be integrated by other numerical methods, such as Euler or Runge-Kutta[5], because **the starting value** $\phi'_S = \phi'(x=0)$ **can not be obtained directly**. The implicit ODEs can be integrated by AEM following a special and more sophisticated methodology that will be developed elsewhere.

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