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EXACT SOLUTIONS FOR AN ELLIPTICAL INCLUSION IN PLANE ELASTICITY

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In această lucrare obţinem soluţia exactă a problemei unei incluziuni elastice de profil eliptic într-o matrice elastică infinită. Este presupusă stare de deformaţie plană. La infinit sunt impuse tensiuni liniare, ceea ce înglobează cazurile tracţiunii, compresiunii, forfecării simple sau încovoierii. Metoda constă în determinarea potenţialilor complecşi φ and ψ în interiorul şi exteriorul incluziunii, care satisfac condiţiile de aderenţă la interfaţă. Pentru tensiuni constante impuse la infinit regăsim rezultatele lui Hardiman. In cazul stării de încovoiere, punem în evidenţă diferenţe faţă de soluţia obţinută de Sendeckyj.

In this paper, we shall give exact solutions for the problem of one elliptical elastic inclusion in an infinite body. Plane deformation state is assumed both in matrix and inclusion. Linear stresses are imposed at infinity, wich means that simple tension, all-round tension, simple shear and bending at infinity may be recovered as particular cases. The method consists in finding the complex potentials φ and ψ inside and outside the inclusion wich give the continuity of resultant stress vector and displacement across the interface. This yields much simpler equations, that not involve second order derivatives of the potentials. For constant stresses at infinity, on obtain the same results with the method used by Hardiman [1] for the plate in generalized plane stress. In the case of a bending field at infinity, the solution show that bending is induced inside the inclusion, but also a rigid translation is present wich was not signaled by Sendeckyj [4].

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1. **Problem statement**

For an isotropic elastic solid, having shear modulus μ and Poisson ratio ν, in a state of plane deformation parallel to the x, y plane, the displacements and stresses can be written in terms of Muskhelishvili's complex potentials

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$\varphi(z)$, $\psi(z)$ [3]

$$\sigma_{xx}+\sigma_{yy} = 2\left(\varphi'(z)+\overline{\varphi'(z)}\right), \tag{1}$$

$$\sigma_{yy}-\sigma_{xx}-2i\sigma_{xy} = 2\left(z\overline{\varphi''(z)}+\overline{\psi'(z)}\right), \tag{2}$$

$$2\mu(u+iv) = \kappa\varphi(z)-z\overline{\varphi'(z)}-\overline{\psi(z)}, \tag{3}$$

where $\kappa = 3-4\nu$.

Let $X_n ds$ and $Y_n ds$ be the components of the stress vector applied to an element of arc ds on the positive side of the normal. Then, for an arbitrary arc $\widehat{AB}$ the resultant stress vector is given by [3]

$$\int_{\widehat{AB}}(X_n+iY_n)\,ds = -i\left(\varphi(z)+z\overline{\varphi'(z)}+\overline{\psi(z)}\right)\Big|_A^B.$$

Let D_2 be an elastic infinite body that contains an elastic inclusion D_1. We assume that the two bodies are perfectly bonded on the interface $\Gamma = \overline{D_2}\cap\overline{D_1}$ and that both regions are is state of plane deformation.

The elastic matrix D_2 is subjected to linear stresses at infinity, described by the potentials

$$\varphi_\infty(z) = C_{11}z+C_2z^2, \psi_\infty(z) = G_1z+G_2z^2, \tag{4}$$

where C_{11} is real and $C_2 = C_{21}+iC_{22}$, $G_k = G_{k1}+iG_{k2}$ are complex constants.

Letting subscripts 1 and 2 on functions and elastic constants refer to the inclusion and matrix respectively, the boundary conditions corresponding to a perfect bond at the interface are the continuity of the displacements and tractions along the interface Γ, namely

$$\frac{1}{2\mu_2}\left(\kappa_2\varphi_2-z\overline{\varphi_2'(z)}-\overline{\psi_2(z)}\right) = \frac{1}{2\mu_1}\left(\kappa_1\varphi_1-z\overline{\varphi_1'(z)}-\overline{\psi_1(z)}\right), \tag{5}$$

$$\varphi_2(z)+z\overline{\varphi_2'(z)}+\overline{\psi_2(z)} = \varphi_1(z)+z\overline{\varphi_1'(z)}+\overline{\psi_1(z)}, \tag{6}$$

for $z\in\Gamma$, where for definiteness, the condition

$$\varphi_2(z_0)-\overline{\varphi_2'(z_0)}+\overline{\psi_2(z_0)}-\varphi_1(z_0)+\overline{\varphi_1'(z_0)}-\overline{\psi_1(z_0)} = 0,$$

for $z_0\in\Gamma$ fixed, has been taken.

The presence of the inclusion give rise to disturbance of the remote fields. We introduce the potentials of these disturbances

$$\varphi(z) = \varphi_2(z)-\varphi_\infty(z), \psi(z) = \psi_2(z)-\psi_\infty(z). \tag{7}$$

Linear combinations of equations (5) and (6) give rise to the following equations

$$\varphi(z) = \frac{1+c}{b}\varphi_1(z)+\frac{c}{b}\left(z\overline{\varphi_1'(z)}+\overline{\psi_1(z)}\right)-\varphi_\infty(z), \tag{8}$$

$$\begin{aligned}\psi(z) = & -\frac{1+c-b}{b}\overline{\varphi_1(z)}+\frac{b-c}{b}\left(\overline{z}\varphi_1'(z)+\psi_1(z)\right)\\ & -\overline{z}\varphi'(z)-\overline{z}\varphi_\infty'(z)-\psi_\infty(z),\end{aligned} \tag{9}$$

for $z \in \Gamma$, where

$$b = \frac{\kappa_2 + 1}{\kappa_1 + 1}\frac{\mu_1}{\mu_2}, c = \frac{1}{\kappa_1 + 1}\frac{\mu_1 - \mu_2}{\mu_2}.$$

2. **Elliptic inclusion**

Consider an elliptical elastic inclusion with the boundary Γ described by the equation

$$x^2 + \frac{y^2}{\varepsilon^2} = 1.$$

where $0 < \varepsilon \leq 1$. Let ξ be a complex variable related to z by the transformation

$$z = \omega(\xi) = h\left(\xi + \lambda\xi^{-1}\right), \tag{10}$$

with $h = \frac{1+\varepsilon}{2}$ and $\lambda = \frac{1-\varepsilon}{1+\varepsilon}$. Equation (10) maps the unit circle $|\xi| = 1$ in the ξ-plane into the interface Γ in the z-plane and points exterior to the circle are mapped uniquely into points exterior to Γ.

In terms of ξ, equations (8)-(9) are

$$\widehat{\varphi}(\xi) = \frac{1+c}{b}\widehat{\varphi}_1(\xi) + \frac{c}{b}\frac{\omega(\xi)}{\overline{\omega'(\xi)}}\overline{\widehat{\varphi}'_1(\xi)} + \frac{c}{b}\overline{\widehat{\psi}_1(\xi)} - \widehat{\varphi}_\infty(\xi), \tag{11}$$

$$\begin{aligned}\widehat{\psi}(\xi) &= -\frac{1+c-b}{b}\overline{\widehat{\varphi}_1(\xi)} + \frac{b-c}{b}\widehat{\psi}_1(\xi) - \widehat{\psi}_\infty(\xi) \\ &+ \frac{\overline{\omega(\xi)}}{\omega'(\xi)}\left(\frac{b-c}{b}\widehat{\varphi}'_1(\xi) - \widehat{\varphi}'(\xi) - \widehat{\varphi}'_\infty(\xi)\right),\end{aligned} \tag{12}$$

where $\widehat{\phi}(\xi) = \phi(\omega(\xi))$.

Because $\lim_{|\xi|\to\infty}\widehat{\varphi}(\xi) = \lim_{|\xi|\to\infty}\widehat{\psi}(\xi) = 0$ and $\widehat{\varphi}(\xi), \widehat{\psi}(\xi)$ are analytic in $|\xi| > 1$, we write their Laurent expansions

$$\widehat{\varphi}(\xi) = \sum_{k=1}^{\infty}\frac{f_k}{(2h)^k}\xi^{-k}, \widehat{\psi}(\xi) = \sum_{k=1}^{\infty}\frac{d_k}{(2h)^k}\xi^{-k}. \tag{13}$$

For the interior potentials we assume

$$\varphi_1(z) = A_0 + A_1 z + A_2 z^2, \psi_1(z) = B_0 + B_1 z + B_2 z^2, \tag{14}$$

where $A_j = A_{j1} + iA_{j2}$ and $B_j = B_{j1} + iB_{j2}$ are complex constants.

When the expresions (4), (13), (14) are substituted in (11) and the coefficients of the same powers of ξ are equated we find

$$(1+c)A_2 + 2\lambda c\overline{A_2} + \lambda^2 c\overline{B_2} = bC_2 \tag{15}$$

$$(1+c)A_1 + c\overline{A_1} + \lambda c\overline{B_1} = bC_1 \tag{16}$$

$$(1+c)A_0 + c\overline{B_0} = -2c\varepsilon\left(\overline{A_2} + \lambda\overline{B_2}\right) \tag{17}$$

$$f_1 = 2\frac{c}{b}\varepsilon\overline{B_1} \tag{18}$$

$$f_2 = 2\frac{c}{b}\varepsilon\left(\overline{B_2} + \overline{A_2}\right) + 2\frac{c}{b}\varepsilon^3\left(\overline{B_2} - \overline{A_2}\right) \tag{19}$$

and $f_k = 0$ for $k = 3, 4, \ldots$.

Hence we get the expresion of the disturbance potential

$$\varphi(z) = f_1 \frac{z - \sqrt{z^2 - 1 + \varepsilon^2}}{1 - \varepsilon^2} + f_2 \left(\frac{z - \sqrt{z^2 - 1 + \varepsilon^2}}{1 - \varepsilon^2} \right)^2 . \tag{20}$$

Similarly, when the expresions (4), (13), (14) and (20) are substituted in (12) and the coefficients of the same powers of ξ are equated we find

$$2\lambda(b-c)A_2 - \lambda^2(1+c-b)\overline{A_2} + (b-c)B_2 = 2\lambda b C_2 + bG_2 \tag{21}$$

$$\lambda(b-c)A_1 - \lambda(1+c-b)\overline{A_1} + (b-c)B_1 = \lambda b C_1 + bG_1 \tag{22}$$

$$-(1+c-b)\overline{A_0} + (b-c)B_0 = 2\varepsilon(1+2c-b)\left(A_2 + \lambda\overline{A_2}\right) + 2\varepsilon c\lambda\left(\overline{A_2} + \lambda\overline{B_2}\right) \tag{23}$$

$$d_1 = -2\varepsilon \frac{1+2c-b}{b}\left(A_1 + \overline{A_1}\right) \tag{24}$$

$$\begin{aligned} d_2 &= -2\varepsilon\left(\frac{1+2c-b}{b}\left(A_2 + \overline{A_2}\right) - \frac{c}{b}\left(\overline{B_2} + \overline{A_2}\right)\right) \\ &\quad + 2\varepsilon^3\left(\frac{1+2c-b}{b}\left(A_2 - \overline{A_2}\right) - \frac{c}{b}\left(\overline{B_2} - \overline{A_2}\right)\right) \end{aligned} \tag{25}$$

$$\sum_{k=3}^{\infty} \frac{d_k}{(2h)^k}\xi^{-k} = \left(f_1 \frac{1}{2h\xi} + 2f_2 \frac{1}{4h^2\xi^2}\right)\frac{1+\lambda^2}{\xi^2 - \lambda} \tag{26}$$

From these relations we deduce the expresion of the other disturbance potential

$$\psi(z) = d_1 \frac{z - \sqrt{z^2 - 1 + \varepsilon^2}}{1 - \varepsilon^2} + d_2 \left(\frac{z - \sqrt{z^2 - 1 + \varepsilon^2}}{1 - \varepsilon^2} \right)^2$$

$$+ f_1 \frac{1+\varepsilon^2}{(1-\varepsilon^2)^2} \frac{\left(z - \sqrt{z^2 - 1 + \varepsilon^2}\right)^2}{\sqrt{z^2 - 1 + \varepsilon^2}} + 2f_2 \frac{1+\varepsilon^2}{(1-\varepsilon^2)^3} \frac{\left(z - \sqrt{z^2 - 1 + \varepsilon^2}\right)^3}{\sqrt{z^2 - 1 + \varepsilon^2}}$$

3. **Exact solution**

The system of equations is not full coupled. Solving (15), (21) we obtain A_2, B_2 and, next, f_2 and d_2 results using (19), (25). These constants depend only on C_2 and G_2. Next, the constants A_0 and B_0 can be obtained solving (17), (23).

From (16), (22) we find the constants A_1, B_1 and substituting them in (18), (24) we get f_1 and d_1, both of them using only C_1 and G_1. This procedure show that we can analyse separately constant stresses and linear stresses imposed at infinity.

3.1. Elliptic inclusion disturbing a constant stress field

If constant stresses $\sigma_{xx}^{\infty}, \sigma_{yy}^{\infty}, \sigma_{xy}^{\infty}$ are imposed at infinity, we have

$$C_{11} = \frac{1}{4}\left(\sigma_{xx}^{\infty} + \sigma_{yy}^{\infty}\right), G_1 = \frac{1}{2}\left(\sigma_{yy}^{\infty} - \sigma_{xx}^{\infty}\right) + i\sigma_{xy}^{\infty}.$$

Equating real and imaginar parts in (16), (22) we obtain two systems which yelds

$$\begin{aligned}
\Delta_{11}A_{11} &= \left((b-2c) + 2b\varepsilon + (b-2c)\,\varepsilon^2\right) C_{11} - c\left(1-\varepsilon^2\right) G_{11},\\
\Delta_{11}B_{11} &= 2\left(1-\varepsilon^2\right)(1+2c-b)\,C_{11} + (1+\varepsilon)^2 (1+2c)\,G_{11},\\
\Delta_{11} &= (1+\varepsilon)^2 - 4\frac{c}{b}(1+2c-2b)\,\varepsilon,
\end{aligned}$$

and

$$\Delta_{12}A_{12} = c\left(1-\varepsilon^2\right) G_{12}, \Delta_{12}B_{12} = (1+\varepsilon)^2 G_{12}, \Delta_{12} = (1+\varepsilon)^2 - 4\frac{c}{b}\varepsilon.$$

These results show that if shear modulus are different, $\mu_1 \neq \mu_2$, or the inclusion is not circular, $\varepsilon < 1$, then shearing imposed at infinity, $\sigma_{xy}^{\infty} = G_{12} \neq 0$, implies also a rigid rotation ($A_{12} \neq 0$) of the inclusion, wich was not present in the solution done by Sendeckyj [4]. Our results agree with those obtained for this problem of plane deformation using the method of Hardiman [1] for generalized plane stress.

3.2. Elliptic inclusion disturbing a linear stress field

Equating real and imaginar parts in (15), (21) we obtain two systems which yields

$$\begin{aligned}
\Delta_{21}A_{21} &= \left(\left((b-c)(1+\varepsilon)^4 - 2c\left(1-\varepsilon^2\right)(1-\varepsilon)^2\right) C_{21} - c\left(1-\varepsilon^2\right)^2 G_{21}\right)\\
\Delta_{21}B_{21} &= \left(2(1+2c-b)\left(1-\varepsilon^2\right) + (1+5c-b)(1-\varepsilon)^2\right)(1+\varepsilon)^2 C_{21}\\
&\quad + \left((1+c)(1+\varepsilon)^2 + 2c\left(1-\varepsilon^2\right)\right)(1+\varepsilon)^2 G_{21}\\
\Delta_{21} &= (1+\varepsilon)^4 - 8\frac{c}{b}\varepsilon^3 - 8\frac{c}{b}(1+2c-2b)\,\varepsilon
\end{aligned}$$

and

$$\begin{aligned}
\Delta_{22}A_{22} &= \left((b-c)(1+\varepsilon)^4 + 2c\left(1-\varepsilon^2\right)(1-\varepsilon)^2\right) C_{22} + c\left(1-\varepsilon^2\right)^2 G_{22}\\
\Delta_{22}B_{22} &= \left(2(1+2c-b)\left(1-\varepsilon^2\right) - (1+5c-b)(1-\varepsilon)^2\right)(1+\varepsilon)^2 C_{22}\\
&\quad + \left((1+c)(1+\varepsilon)^2 - 2c\left(1-\varepsilon^2\right)\right)(1+\varepsilon)^2 G_{22}\\
\Delta_{22} &= (1+\varepsilon)^4 - 8\frac{c}{b}\varepsilon - 8\frac{c}{b}(1+2c-2b)\,\varepsilon^3
\end{aligned}$$

From (17), (23) we get

$$A_0 = -2\varepsilon c\left(\lambda\frac{1+3c-b}{b}A_2+\frac{1+c}{b}\overline{A_2}+\lambda^2\frac{c}{b}B_2+\lambda\frac{b-c}{b}\overline{B_2}\right) \quad (27)$$

$$\begin{aligned} B_0 &= 2\varepsilon\frac{1+c-b}{b}\left(A_2+(1+c)\,\lambda\overline{A_2}-\lambda cB_2\right) \\ &\quad +2\varepsilon\frac{c}{b}\,(1+c)\left(A_2+2\lambda\overline{A_2}+\lambda^2\overline{B_2}\right) \end{aligned} \quad (28)$$

which yields a rigid translation in the inclusion

$$\begin{aligned} \frac{\kappa_1 A_0-\overline{B_0}}{2\mu_1} &= -\varepsilon\frac{1}{\mu_2}\left(2\frac{c}{b}\lambda A_2+\frac{1+c-b}{b}\left(\lambda A_2+\overline{A_2}\right)+\lambda\frac{c}{b}\left(\lambda B_2-\overline{B_2}\right)\right) \\ &\quad +\varepsilon\frac{\mu_2-\mu_1}{\mu_2\mu_1}\left(\overline{A_2}+\lambda\overline{B_2}\right) \end{aligned}$$

If bending stresses are imposed at infinity $\sigma_{xy}^\infty = 0$, $\sigma_{yy}^\infty = 8Sx\cos t$, $\sigma_{xx}^\infty = -8Sy\sin t$, then $C_{21} = G_{21} = S\cos t$ and $C_{22} = -G_{22} = S\sin t$ in the above formulas. For $t = 0$ we find $A_{22} = B_{22} = 0$ and

$$\begin{aligned} \frac{\kappa_1 A_0-\overline{B_0}}{2\mu_1} &= -\varepsilon\frac{1}{\mu_2}\left(2\frac{c}{b}\lambda A_{21}+\frac{1+c-b}{b}\,(\lambda+1)\,A_{21}+\lambda\frac{c}{b}\,(\lambda-1)\,B_{21}\right) \\ &\quad +\varepsilon\frac{\mu_2-\mu_1}{\mu_2\mu_1}\,(A_{21}+\lambda B_{21}) \end{aligned}$$

wich represents the rigid translation of the inclusion. On observe that this translation is vanishing if $\mu_1 = \mu_2$ and $\kappa_1 = \kappa_2$. For $\mu_1 \neq \mu_2$ or $\kappa_1 \neq \kappa_2$ the translation can be present even if the inclusion is circular, $\lambda = 0$. In conclusion, bending field at infinity give rise also to a rigid translation of the inclusion. This fact is omitted in the solution given by Sendeckyj [4].

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