ON A SPECIAL SYMMETRY IN THE DYNAMICS OF COMPLEX SYSTEMS IN A HOLOGRAPHIC-TYPE PERSPECTIVE

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By operating with the Scale Relativity Theory in the dynamics of complex systems, we can achieve a description of these complex systems through a holographic-type perspective. Then, gauge invariances of a Riccati-type become functional in complex system dynamics, which implies several consequences: conservation laws (in particular, for classical dynamics, the kinetic momentum conservation law), simultaneity and synchronization between the structural units (of a complex system) dynamics, and temporal patterns through harmonic mappings.

Keywords: scale relativity theory, Schrödinger-type scenario, Madelung-type scenario, SL(2R) group, harmonic mappings

1. Introduction

If, in the dynamics of complex systems, we operate with the Scale Relativity Theory [1,2], the description of said complex systems can be achieved through a holographic-type perspective. Indeed, as long as, in such a framework, the description of dynamics is performed through continuous and non-differentiable

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curves (fractal/multifractal curves), such a perspective becomes viable: the continuous and non-differentiable curves (fractal/multifractal curves) satisfy, in any one of their points, the self-similarity property (the part reflects the whole and the whole reflects the part – which would correspond to the holographic principle).

Now, two scenarios in the description of complex system dynamics become operational: one scenario is explained through a Schrödinger-type fractal/multifractal equation (Schrödinger-type scenario) and another through the hydrodynamic-type fractal/multifractal equation system (Madelung-type scenario).

The two scenarios for describing complex system dynamics are not mutually exclusive, moreover, they are complementary.

Taking into account the fact that, in any of the scenarios, symmetries are highlighted, in the current paper, several symmetries will be explained only in the Schrödinger-type scenario and their consequences will be discussed.

2. Conservation laws in complex systems dynamics as gauge invariances of a Riccati-type.

It is a known fact that the dynamics of complex systems in the Scale Relativity Theory (SRT) [1,2] can be described through a Schrödinger type multifractal scenario explained through the differential equation (nonstationary multifractal Schrödinger equation) [3]

$$\lambda^2 (dt)^{\left[\frac{4}{f(\alpha)}\right]-2} \partial_l \partial^l \Psi + i\lambda (dt)^{\left[\frac{2}{f(\alpha)}\right]-1} \partial_t \Psi = 0, \tag{1}$$

where

$$\partial_t = \frac{\partial}{\partial_t}, \partial_l = \frac{\partial}{\partial x^l}, \partial_l \partial^l = \frac{\partial^2}{\partial x_l^2}, \tag{2}$$

In the above relations Ψ is the states function, dt is the scale resolution, x^l is the multifractal spatial coordinate, t is the non-multifractal temporal coordinate with the role of an affine parameter of the motion curves (it is mentioned that in SRT, the dynamics of the entities belonging to any complex system are described through continuous and non-differentiable curves – multifractal curves), λ is a parameter associated to the multifractal-non-multifractal scale transition, $f(\alpha)$ is the singularity spectrum with a singularity index of order $\alpha = \alpha(D_F)$ and D_F is the fractal dimension of the motion curves [4,5].

The nonstationary multifractal Schrödinger equation admits, besides the clasical Galilei group proper, an extra set of symmetries [6] that, in general conditions, can be taken in a form involving just one space dimension and time, as a SL(2,R) type group in two variables with three parameters [7]. Limiting the general conditions, the space dimension can be chosen as the radial coordinate in a free fall, as in the case of Galilei kinematics, which can also be extended *as such* in general relativity [8,9], for instance in the case of free fall in a Schwarzschild field.

The essentials of the argument of Alicia Herrero's and Juan Antonio Morales' work just cited are delineated based on the fact that the radial motion in a Minkowski spacetime should be a conformal Killing field, which is a three-parameter realization of the SL(2R) algebra in time and the radial coordinate. This is a Riemannian manifold of the Bianchi type VIII (or even type IX, forcing the concepts a little) [10]. The bottom line here is that, as long as the general relativity is involved, the nonstationary Schrödinger equation *describes the continuity of matter*.

And since, as a universal instrument of knowledge, the nonstationary multifractal Schrödinger equation is referring to free particles, we need to show what kind of freedom is this in classical terms.

For our current necessities it is best to start with the finite equations of the specific SL(2,R) group, and build gradually upon these [11,12], in order to discover the connotations we are seeking for. Working in the variables (t, x) as above, the finite equations of this group are given by the transformations:

$$t \to \frac{\alpha t+\beta}{\gamma t+\delta};\ x \to \frac{x}{\gamma t+\delta} \tag{3}$$

This transformation is a realization of the SL(2,R) structure in variables (t,x), with three essential parameters (one of the four constants α, β, γ and δ is superfluous here). Every vector in the tangent space SL(2R) is a linear combination of three fundamental vectors, the infinitesimal generators:

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = t\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x}, \qquad X_3 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} \tag{4}$$

satisfying the basic structure equations:

$$[X_1, X_2] = X_1, \qquad [X_2, X_3] = X_3, \qquad [X_3, X_3] = -2X_2 \tag{5}$$

which we take as standard commutation relations for this type of algebraic structure, all along the present work. The group has an invariant function, which can be obtained as the solution of a partial differential equation:

$$\begin{gathered}(cX_1 + 2bX_2 + aX_3)f(t,x) = 0\\ (at^2 + 2bt + c)\frac{\partial f}{\partial t} + (at + b)x\frac{\partial f}{\partial x} = 0\end{gathered} \tag{6}$$

The general solution of this equation is a function of the constant values of the ratio:

$$\frac{x^2}{at^2 + 2bt + c} \tag{7}$$

which represents the different paths of transitivity of the action described by (4).

In order to draw some proper conclusions from these mathematical facts, let us go back to the transformation (3) and consider it from the point of view of classical physics.

First, comes the second of Kepler laws, viz. that law serving to Newton as a means to introduce the idea of a center of force: if, with respect to such a material point, a motion proceeds according to the second Kepler law, then the field of force should be Newtonian. The wave mechanics shows that this law means more than it was intended for initially, namely that it should have a statistical meaning, according to the idea of Planck's quantization [11,12]. Indeed, if 'x' denotes the distance of the moving material point from the center of force, we have

$$x^2 d\theta = \dot{a} dt \;\; \dot{a}\frac{dt}{d\theta} = x^2 \tag{8}$$

where θ is the central angle of the position vector of the moving material point with respect to the center of force. In this form the law usually serves as a transformation in the mathematical treatment the central motion. However, from the point of view of the physical content of time, the second equality in equation (8) tells us much more if we take the argument out of the mathematical form of the classical Kepler problem.

In such a context, if it is considered that (7) is constant

$$\frac{x^2}{at^2 + 2bt + c} = L = const. \tag{9}$$

then from (7) and (8), through the substitutions

$$\frac{dt}{d\theta} = \dot{w}, \qquad \frac{Lat^2}{\dot{a}} = \frac{1}{M}w^2, \qquad \frac{2Lbt}{\dot{a}} = -2\frac{R}{M}w, \qquad \frac{Lc}{\dot{a}} = K \tag{10}$$

the following Riccati-type differential equation is satisfied (i.e., we operate here with a Riccati-type gauge):

$$\dot{w} - \frac{1}{M}w^2 + 2\frac{R}{M}w - K = 0. \tag{11}$$

For obvious physical reasons, it is therefore important to find the most general solution of that equation. José Carineña et al. offer us a pass in short but modern and pertinent review of the integrability of Riccati's equation [13]. For our current needs it is enough to note that the complex numbers

$$w_0 \equiv R + iM\Omega, \; w_0^* \equiv R - iM\Omega; \; \Omega^2 = \frac{K}{M} - \left(\frac{R}{M}\right)^2 \tag{12}$$

roots of the quadratic polynomial on the right side of equation (11), are two solutions (constants, that's right) of the equation: being constants, their derivative is zero, being roots of the right-hand polynomial, it cancels. So, first we do the homographic transformation:

$$z = \frac{w - w_0}{w - w_0^*} \tag{13}$$

and now it can easily be seen by direct calculation that z is a solution of the linear and homogeneous equation of the first order

$$\dot{z} = 2i\Omega z \;\therefore\; z(t) = z(0)e^{2i\Omega t}. \tag{14}$$

Therefore, if we conveniently express the initial condition $z(0)$, we can give the general solution of the equation (11) by simply inverting the transformation (13), with the result

$$w = \frac{w_0 + re^{2i\Omega(t-t_r)}w_0^*}{1 + re^{2i\Omega(t-t_r)}} \tag{15}$$

where r and t_r are two real constants that characterize the solution. Using equation (12) we can put this solution in real terms, i.e.

$$\begin{aligned} z = \quad & R + M\Omega\left(\frac{2r\sin\left[2\Omega(t-t_r)\right]}{1 + r^2 + 2r\cos\left[2\Omega(t-t_r)\right]}\right. \\ & \left. + i\frac{1 - r^2}{1 + r^2 + 2r\cos\left[2\Omega(t-t_r)\right]}\right) \end{aligned} \tag{16}$$

which highlights a frequency modulation through what we would call a Stoler transformation [11,12] which leads us to a complex form of this parameter. More than that, if we make the notation

$$r \equiv \coth\,\tau, \tag{17}$$

equation (16) becomes

$$z = R + M\Omega h \tag{18}$$

where h is given by

$$h = -i\frac{\cosh\,\tau - e^{-2i\Omega(t-t_m)}\sinh\,\tau}{\cosh\,\tau + e^{-2i\Omega(t-t_m)}\sinh\,\tau}. \tag{19}$$

The meaning of this complex parameter will be clear a little later. For the moment, let's note that any dynamic process appears here as a frequency modulation process [13]. Moreover, by admitting a gauge invariance of a Riccati-type, in the Classical Theory of Motion, the kinetic momentum conservation law is obtained.

3. Simultaneity in complex systems dynamics as gauge invariances of a Riccati-type

Consider an *extended body* revolving in a central field (for example of Newtonian forces). It can be imagined as a swarm of classical material points, and such a swarm illustrates classical laws, provided it is considered as a swarm of free material points in the classical sense of the word [14]. In the first of equations (3) this requirement would mean that the material points are considered simultaneously [15]. Then each material point can be located in the swarm by four homogeneous coordinates $(\alpha,\beta,\gamma,\delta)$, or three nonhomogeneous coordinates, if the equations (3) represent the content of time and radial coordinate for the space region covered by this body. The simultaneity in the motion of the swarm of material points can be differentially characterized, giving a Riccati equation in pure differentials:

$$d\frac{\alpha t+\beta}{\gamma t+\delta}=0, \qquad dt=\omega^1 t^2+\omega^2 t+\omega^3 \tag{20}$$

Thus, for the description of the *extended body in motion as a succession of states of the ensemble of simultaneous material points*, it suffices to have three differential forms, representing a coframe of the **SL**(2R) algebra:

$$\omega^1=\frac{\alpha d\gamma-\gamma d\alpha}{\alpha\delta-\beta\gamma};\ \omega^2=\frac{\alpha d\delta-\delta d\alpha+\beta d\gamma-\gamma d\beta}{\alpha\delta-\beta\gamma};\ \omega^3=\frac{\beta d\delta-\delta d\beta}{\alpha\delta-\beta\gamma} \tag{21}$$

That this coframe refers to such an algebra, can be checked by direct calculation of the Maurer-Cartan equations which are characteristic to this algebra:

$$d\wedge\omega^1-\omega^1\wedge\omega^2=0$$

$$d\wedge\omega^2+2(\omega^3\wedge\omega^1)=0 \tag{22}$$

$$d\wedge\omega^3-\omega^2\wedge\omega^3=0$$

Using these conditions one can prove that the right hand side of equation (9) is an exact differential [15], therefore it should always have an integral. The Cartan-Killing metric of this coframe is given by the quadratic form $(\omega^2/2)^2-\omega^1\omega^3$, so that a state of an extended orbiting body in the Kepler motion, can be organized as a metric phase space, a Riemannian three-dimensional space at that. The geodesics of this Riemannian space, are given by some conservation laws of equations

$$\omega^1=a^1(d\theta);\ \omega^2=2a^2(d\theta);\ \omega^3=a^3(d\theta) \tag{23}$$

where $a^{1,2,3}$ are constants and θ is the affine parameter of the geodesics, so that, along these geodesics the differential equation (9) is an ordinary differential equation of Riccati type:

$$\frac{dt}{d\theta}=a^1t^2+2a^2t+a^3 \tag{24}$$

This equation can be identified with (8), provided its right hand side is proportional to the square of a 'radial coordinate' of a free classical material point. Mathematically this requires an ensemble generated by a harmonic mapping between the positions in space and the material points, with the square of the radial coordinate 'x' measuring the variance characterizing the distribution of material points in space. Following the same line of thought previously presented, a solution of the same type as (19) can be highlighted [16-20].

4. Synchronization in complex systems dynamics as gauge invariances of a Riccati-type

According to the meanings of the state function Ψ from the Scale Relativity Theory, a physical significance is only attached to the density of state $\rho = \Psi\overline{\Psi}$. In such a context, if $\Psi = a + ib$, then the constant density of states can be localized inside the unity radius circle

$$x^2 + y^2 = 1 \tag{25}$$

where

$$\frac{a^2}{\rho} = x^2, \frac{b^2}{\rho} = y^2 \tag{26}$$

Now, the metric of the Lobachevsky plane can be produced as a Caylean metric of a Euclidean plane, for which the absoluteness is the circle (25). In this way, the Lobachevsky plane can be put into a biunivoc correspondence with the interior side of the circle. In such a conjecture, using the general procedure of metrization of a Caylean space, which implies the differential quadratic form [11,12]

$$-\frac{ds^2}{k^2} = \frac{\Omega(dX, dX)}{\Omega(X, X)} - \frac{\Omega^2(X, dX)}{\Omega^2(X, X)}, \tag{27}$$

where $\Omega(X, Y)$is the duplication of $\Omega(X, X)$ and k is a constant connected to the space curvature, it results:

$$\frac{ds^2}{k^2} = \frac{(1 - y^2)dx^2 + 2xydxdy + (1 - x^2)dy^2}{(1 - x^2 - y^2)^2}, \tag{28}$$

where

$$\begin{gathered} \Omega(X, X) = 1 - x^2 - y^2 \\ \Omega(X, dX) = -xdx - ydy \\ \Omega(dX, dX) = -dx^2 - dy^2 \end{gathered} \tag{29}$$

Now, performing the coordinate transformations

$$x = \frac{h\bar{h} - 1}{h\bar{h} + 1}, \qquad y = \frac{h + \bar{h}}{h\bar{h} + 1} \tag{30}$$

with

$$h = u + iv, \qquad \bar{h} = u - iv \tag{31}$$

the metric (28) takes the form of Poincaré metric of the superior complex plane

$$\frac{ds^2}{k^2} = 4\frac{dhd\bar{h}}{\left(h-\bar{h}\right)^2} = \frac{du^2+dv^2}{v^2}, \tag{32}$$

The metric (32) induces the simply transitive group in the quantities h and $\bar{h}$, whose actions are:

$$h \leftrightarrow \frac{ah+b}{ch+d}, \qquad \bar{h} \leftrightarrow \frac{a\bar{h}+b}{c\bar{h}+d}, \tag{33}$$

The structure of this group is typical of SL(2R), i.e.,

$$[B^1, B^2] = B^1, \quad [B^2, B^3] = B^3, \quad [B^3, B^1] = -2B^2 \tag{34}$$

where B^l are the infinitezimal generators of the group:

$$B^1 = \frac{\partial}{\partial h} + \frac{\partial}{\partial \bar{h}}, \quad B^2 = h\frac{\partial}{\partial h} + \bar{h}\frac{\partial}{\partial \bar{h}}, \quad B^3 = h^2\frac{\partial}{\partial h} + \bar{h}^2\frac{\partial}{\partial \bar{h}} \tag{35}$$

and admits the 2-form (32).

Since (32) is invariant with respect to the group $SL(2R)$[3,4,8], this group can be assimilated with a "synchronization" group between the different structural units of complex system. In this process, the amplitude of each of the structural units of any complex system participates, in the sense that they are correlated. Moreover, the phases of any entity of the complex system are also correlated [21-25].

5. Temporal patterns in complex systems dynamics through harmonic mappings

In the following, complex system dynamics will be generated through harmonic mappings. Indeed, let it be assumed that the complex system dynamics are described by the variables $\left(Y^j\right)$, for which the following multifractal metric was discovered:

$$h_{ij}dY^idY^j \tag{36}$$

in an ambient space of multifractal metric:

$$\gamma_{\alpha\beta}dX^\alpha dX^\beta \tag{37}$$

In this situation, the field equations of complex system dynamics are derived from a variational principle, connected to the multifractal Lagrangian:

$$L = \gamma^{\alpha\beta} h_{ij} \frac{dY^i dY^j}{\partial X^\alpha \partial X^\beta} \tag{38}$$

In the current case, (36) is given by (32), the field multifractal variables being h and $\bar{h}$ or, equivalently, the real and imaginary part of h. Therefore, if the variational principle:

$$\delta \int L\sqrt{\gamma} d^3x \tag{39}$$

is accepted as a starting point where $\gamma = |\gamma_{\alpha\beta}|$, the main purpose of the complex system dynamics research would be to produce fractal/multifractal metrics of the multifractal Lobachevski plane (or relate to them). In such a context, the multifractal Euler equations corresponding to the variational principle (39) are:

$$\begin{gathered}(h-\bar{h})\nabla(\nabla h) = 2(\nabla h)^2 \\ (h-\bar{h})\nabla(\nabla \bar{h}) = 2(\nabla \bar{h})^2\end{gathered} \tag{40}$$

which admits the solution:

$$h = \frac{\cosh(\Phi/_2) - \sinh(\Phi/_2) e^{-i\alpha}}{\cosh(\Phi/_2) + \sinh(\Phi/_2) e^{-i\alpha}}, \alpha \in \mathbb{R} \tag{41}$$

with α real and arbitrary, as long as $(\Phi/_2)$ is the solution of a multifractal Laplace equation for the free space, such that

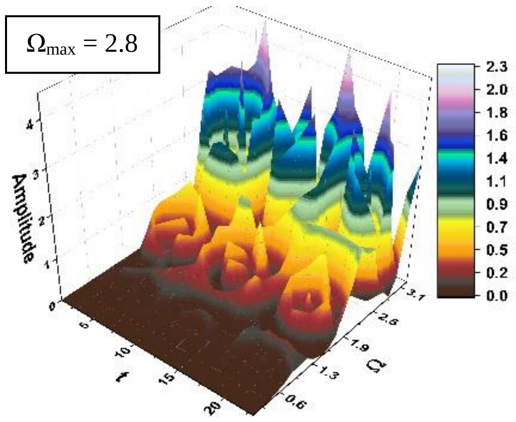
$$\nabla^2(\Phi/_2) = 0 \tag{42}$$

For a choice of the form $\alpha = 2\Omega t$, in which case a temporal dependency was introduced in the complex system dynamics, (41) becomes:

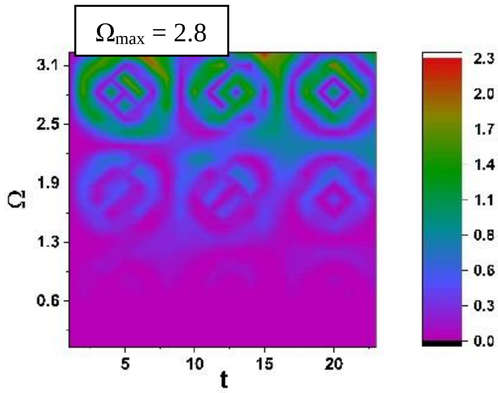
$$h = \frac{i[e^{2\Phi}\sin(2\Omega t) - \sin(2\Omega t) - 2i\, e^{\Phi}]}{e^{2\Phi}[\cos(2\Omega t) + 1] - \cos(2\Omega t) + 1} \tag{43}$$

The significance of this complex parameter must be linked to the harmonic mappings between the Euclidean space (i.e., measurement space) and the hyperbolic space (i.e., phase space, in which the "self-structuring" manifests – for details see [11,12]) For the moment, let it be noted that the measurement process is mimed here as a frequency modulation process. More precisely, this process is a calibration of the difference between the multifractal kinetic energy and the multifractal potential energy, which brings this quantity to a perfect squared form.

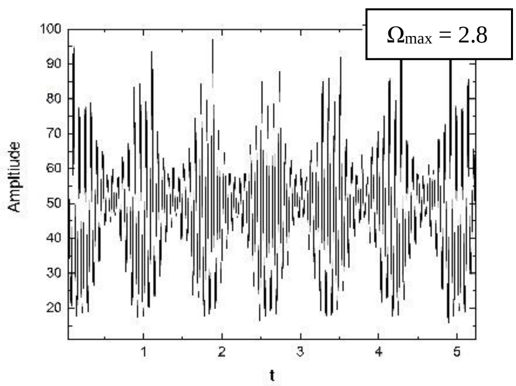
In Fig. 1 a-d the "self-structuring temporal pattern" of the structural units of complex systems in the form of quasi-periodicity are presented: 3D diagram, contour diagram, time series and reconstituted attractor for the scale resolutions given by $\Omega_{max} = 2.8$.



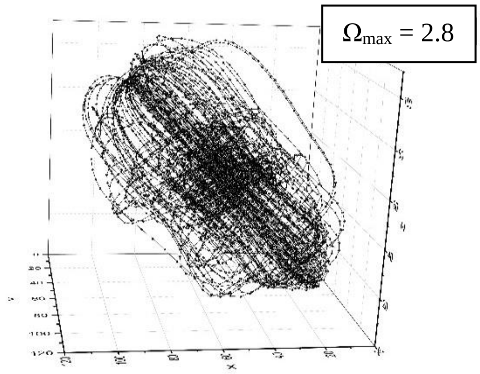
(a) 3D diagram



(b) contour plot



(c) time series



(d) reconstituted attractor

Fig.1. a–d. - "Self-structuring temporal pattern" of structural units of the complex system in the form of quasi-periodicity (3D diagram (a), contour diagram (b), time series (c) and reconstituted attractor (d) for scale resolution given by $\Omega_{max} = 2.8$). Such patterns are not singular. By employing the Multifractal Theory of Motion in the description of complex systems dynamics [26-31], several types of patterns can be highlighted: period doubling, intermittences, harmonized oscillations, damped oscillations etc.

6. Conclusions

In a Scrödinger-type scenario for the description of complex system dynamics, a SL(2R) symmetry is highlighted. The existence of such a symmetry has several consequences for the aforementioned dynamics: conservation laws as gauge invariances of a Riccati-type (in particular, for classical dynamics, the kinetic momentum conservation law); simultaneity as gauge invariances of a Riccati-type; synchronization as gauge invariances of a Riccati-type; temporal patterns through harmonic mappings.

Moreover, the existence of such a symmetry implies, through a Poincaré metric of the hyperbolic space, that holography can be associated with deep learning.

R E F E R E N C E S

[1]. *L. Nottale*, "Scale Relativity and Fractal Space-Time: A New Approach to Unifying Relativity and Quantum Mechanics", Imperial College Press, London, UK, 2011.

[2]. *I. Merches, M. Agop*, "Differentiability and fractality in dynamics of physical systems, World Scientific", New Jersey, 2016.

[3]. *M. A. Paun, M. R. N. Avanaki, G. Dobre et al*., "Wavefront aberration correction in single mode fibre systems", in Journal of Optoelectronics and Advanced Materials, **vol. 11**, no. 11, 2009, pp. 1681-1685.

[4]. *B. B. Mandelbrot*, "Fractal Geometry of Nature", San Francisco: W.H. Freeman and Co., 1982.

[5]. *E. A. Jackson*, "Perspectives of Nonlinear Dynamics", Vol. 1, Cambridge University Press, New York, 1993.

[6]. *U. Niederer*, "The Maximal Kinematical Invariance Group of the Free Schrödinger Equation", Helvetica Physica Acta, **vol. 45**, 1972, pp. 802-810.

[7]. *V. Alfaro, S. de Fubini, G. Furlan*, "Conformal Invariance in Quantum Mechanics", Il Nuovo Cimento A, **vol. 34**, 1976, pp. 569-611.

[8]. *A. Herrero, J. A. Morales*, "Radial Conformal Motions in Minkowski Space-Time, Journal of Mathematical Physics", **vol. 40(6)**, 1999, pp. 3499-3508.

[9]. *A. Herrero, J. A. Morales*, "Painleve-Gullstrand Synchronizations in Spherical Symmetry, Classical and Quantum Gravity", **vol. 27**, 2010, 175007.

[10]. *L. Bianchi*, "On Three-Dimensional Spaces Which Admit a Continuous Group of Motions, General Relativity and Gravitation", **vol. 33**, 2001, pp. 2171-2253. (This is a translation of the original Italian from year 1897, by R. T. Jantzen under Golden Oldies initiative of the periodical).

[11]. *N. Mazilu, M. Agop*, "At the Crossroads of Theories. Between Newton and Einstein - The Barbilian Universe" (In Romanian), Ars Longa Publishing House, Iasi, 2010.

[12]. *N. Mazilu, M. Agop*, "Skyrmions: A Great Finishing Touch to Classical Newtonian Philosophy", World Philosophy Series, Nova, New York, 2012.

[13] *J. F. Carinena, J. Clemente-Gallardo, G. Marmo*, "Reduction Procedures in Classical and Quantum Mechanics", International Journal of Geometrical Methods in Modern Physics, **vol. 4**, 2007, pp. 1363 – 1403. arXiv:math-ph/0709.2366.

[14]. *J. W. Larmor*, "On the Statistical Dynamics of Gas Theory as Illustrated by Meteor Swarms and Optical Rays", Nature, Volume **63(1624),** 1900, pp. 168 - 169; British Association Report, September 1900, pp. 632-634.

[15]. *E. Cartan*, "La Théorie des Groupes Finis et Continus et la Géométrie Différentielle Traitées par la Méthode du Repère Mobile", Gauthier Villars, Paris, 1951.

[16]. *N. D. Tesloianu, I. Nedelciuc, V. Ghizdovat, M. Agop, and C. Lupascu-Ursulescu*, "A fractal physics explanation for acute thrombotic occlusion in an apparently healthy coronary artery", Anatolian Journal of Cardiology, **18(2)**, 2017, p. 155.

[17]. *N. Mazilu, M. Agop, I. Gatu, et al*., "The classical theory of light colors: a paradigm for description of particle interactions", International Journal of Theoretical Physics, **55(6)**, 2016, pp. 2773-2793.

[18]. *N. Mazilu, M. Agop, I. Gatu, et al*., "From Kepler problem to skyrmions", Modern Physics Letters B, **30(13)**, 2016, p.1650153.

[19]. *M. Agop, S. Irimiciuc, D. Dimitriu, et al.* "Novel approach for EKG signals analysis based on Markovian and non-Markovian fractalization type in scale relativity theory", Symmetry, **13(3)**, 2021, p. 456.

[20]. *N. Cimpoeşu, S. Stanciu, D. Tesloianu, R. Cimpoeşu, R. F. Popa and E. Moraru*, "A study of the damping capacity of mechanically processed cu–9.2 Al–5.3 Mn–0.6 Fe shape memory alloys", Metal Science and Heat Treatment, **58(11)**, 2017, pp. 729-733.
[21]. *R. Stana, I. C. Botez, V.P. Paun, M. Agop*, "New Model for Heat Transfer in Nanostructures", Journal of Computational and Theoretical Nanoscience, **9(1),** 2012, pp. 55-66.
[22]. *V. P. Paun*, "Relaxation model for polymeric materials in the hereditary theory of elasticity", Materiale Plastice, **40(2)**, 2003, pp. 81-82.
[23]. *Z. Borsos, V.P. Paun, I.C. Botez, C.M. Stoica, P. Vizureanu, and M. Agop*, "Structural Conductivity of Carbon Nanotubes", Revista de Chimie, **vol. 59**, no. 10, 2008, pp. 1169-1171.
[24]. *V. P. Paun*, "An estimation of the polymer translocation time through membrane", Materiale Plastice, **43(1)**, 2006, pp. 57-58.
[25]. *V. P. Paun*, "Creep model for polymeric materials", Materiale Plastice, **40(1),** 2003, pp.25-26.
[26]. *P. Postolache, Z. Borsos, V.A. Paun, and V.P. Paun*, "New Way in Fractal Analysis of Pulmonary Medical Images", in University Politehnica of Bucharest Scientific Bulletin-Series A-Applied Mathematics and Physics, **vol. 80,** no. 1, 2018, pp. 313-322.
[27]. *M.V. Nichita, M.A. Paun, V.A. Paun, and V.P. Paun*, "Fractal Analysis of Brain Glial Cells. Fractal Dimension and Lacunarity", in University Politehnica of Bucharest Scientific Bulletin-Series A-Applied Mathematics and Physics, **vol. 81**, no. 1, 2019, pp. 273-284.
[28]. *V. P. Paun*, "Fractal surface analysis of Zircaloy-4 SEM micrographs using the time-series method", 6th International Conference on Solid State Surfaces and Interfaces, CENTRAL EUROPEAN JOURNAL OF PHYSICS, **7(2)**, 2009, pp. 264-269.
[29]. *D. Bordescu, M.A. Paun, V.A. Paun, and V.P. Paun*, "Fractal analysis of Neuroimagistic. Lacunarity degree, a precious indicator in the detection of Alzheimer's disease", in University POLITEHNICA of Bucharest Scientific Bulletin, Series A-Applied Mathematics and Physics, **vol. 80**, no. 4, 2018, pp. 309-320.
[30]. *M. A. Paun, and V. A. Paun*, High-Frequency 3-D Model for the Study of Antennas in Cochlear Implants, 7th IEEE International Workshop on Advances in Sensors and Interfaces (IWASI), IEEE Transactions on Components Packaging and Manufacturing Technology, **8(7)**, 2018, pp.1135-1140.
[31]. *V. A. Paun, L. Ochiuz, M. Hortolomei, et al.*, In Vitro Release Kinetics Evaluation of Erythromycin in Microemulsions for Dermal Applications, Materiale Plastice, **53(4)**, 2016, pp. 699-702.