APPROXIMATE AMENABILITY FOR BANACH MODULES

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For a Banach algebra A, a Banach A-bimodule E and a bounded Banach A-bimodule homomorphism $\Delta : E \longrightarrow A$, the notions of approximate Δ-amenability and Δ-contractibility for E are introduced. The general theory is developed and some hereditary properties are given. In analogy with approximate amenability and contractibility for Banach algebras, it is shown that under some mild conditions approximate Δ-amenability and approximate Δ-contractibility are the same properties.

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1. **Introduction**

The concept of amenability for a Banach algebra was introduced by Johnson [9] in 1972, and it has been proved to be of enormous importance in Banach algebra theory (for example, [1], [2], [5] and [10]). The main example in [9] asserts that the group algebra $L^1(G)$ of a locally compact group G is amenable if and only if G is amenable. The definition of an amenable Banach algebra is strong enough to allow for the development of a rich general theory, but still weak enough to include a variety of interesting examples. For example, Johnson's result fails to be true for discrete semigroups. This failure is partially due to the fact that $l^1(S)$ is equipped with two algebraic structures. It is a Banach algebra and a Banach module over $l^1(E_S)$, where S is a discrete inverse semigroug with the set of idempotents E_S. There are many examples of Banach modules which do not have any natural algebra structure. One example is $L^p(G)$ which is a left $L^1(G)$ module, for a locally compact group G [4]. There is one thing in common in all of them, namely the existence of a module homomorphism, from the Banach module to the underlying Banach algebra. This consideration was the motivation to study the concept of module amenability (more precisely Δ-amenability) which is defined for a Banach module E over a Banach algebra A with a given module homomorphism $\Delta : E \longrightarrow A$. This notion was introduced by Ebrahimi Bagha and Amini in [6].

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The concepts of approximate amenability, contractibility and some other related concepts, were introduced and studied in [7] and further developed in [8]. In [7], the authors showed that the corresponding class of approximately amenable (contractible) Banach algebras is larger than that for the classical amenable algebras introduced by Johnson; for the module approximate amenability case refer to [11].

In this paper, we introduce the concepts of approximate Δ-amenability and Δ-contractibility, and indicate some basic properties of approximately Δ-amenable Banach modules. We also study the hereditary properties of approximate Δ-amenability for a Banach module. Finally, we provide some examples.

2. **Approximate amenability for Banach modules**

Let A be a Banach algebra and X be a Banach A-bimodule. A *derivation* from A to X is a linear map $D : A \longrightarrow X$ such that $D(ab) = D(a) \cdot b + a \cdot D(b)$ for $a, b \in A$. Also, D is said to be *inner* if there exists $x \in X$ such that $D(a) = a \cdot x - x \cdot a$ for every $a \in A$. In this case, we denote D by ad_x. Moreover, D is said to be *approximately inner* if there exists a net (x_i) in X such that for any $a \in A$, $D(a) = \lim_i (a \cdot x_i - x_i \cdot a) = \lim_i ad_{x_i}(a)$. A Banach algebra A is called *amenable* if every continuous derivation $D : A \longrightarrow X^*$, where X^* is the dual module of X, is inner for every Banach A-bimodule X and A is called *approximately amenable* if every continuous derivation $D : A \longrightarrow X^*$ is approximately inner, for all Banach A-bimodule X.

Throughout this paper, A is a Banach algebra, E is a Banach A-bimodule and $\Delta : E \longrightarrow A$ is a bounded Banach A-bimodule homomorphism, that is, a bounded linear map such that for any $a \in A$ and $x \in E$,

$$\Delta(a \cdot x) = a \cdot \Delta(x) \text{ and } \Delta(x \cdot a) = \Delta(x) \cdot a.$$

Let X be a Banach A-bimodule. A bounded linear map $D : A \longrightarrow X$ is called a module derivation (or more specifically Δ-derivation) if

$$D(\Delta(a \cdot x)) = a \cdot D(\Delta(x)) + D(a) \cdot \Delta(x), \; D(\Delta(x \cdot a)) = D(\Delta(x)) \cdot a + \Delta(x) \cdot D(a),$$

for all $a \in A$ and $x \in E$. Also, D is called Δ-inner if there is $f \in X$ such that for any $x \in E$

$$D(\Delta(x)) = \Delta(x) \cdot f - f \cdot \Delta(x) = ad_f(\Delta(x)).$$

An A-bimodule E is called module amenable (or more specifically Δ-amenable) if for each Banach A-bimodule X, all Δ-derivations from A to X^* are Δ-inner. It is clear that A is module amenable (with $\Delta = id$) if and only if it is amenable as a Banach algebra.

A (weak) right approximate identity of E is a net (a_α) in A such that for each $x \in E$, $\Delta(x) \cdot a_\alpha - \Delta(x) \to 0$ [$\Delta(x) \cdot a_\alpha - \Delta(x) \to 0$ in the weak topology]. The (weak) left and two sided approximate identities are defined similarly.

Definition 2.1. *An A-bimodule E is called approximately module amenable (approximately Δ-amenale as an A-bimodule) if for each Banach A-bimodule X, all Δ-derivations from A to X^* are approximately Δ-inner. A Δ-derivation $D : A \longrightarrow X^*$*

is called approximately Δ*-inner if there is a net* $(f_\alpha) \subseteq X^*$ *such that* $D(\Delta(x)) = \lim_\alpha(\Delta(x) \cdot f_\alpha - f_\alpha \cdot \Delta(x))$ $(x \in E)$*.*

A point Δ-derivation d at a character ϕ of an algebra A is a linear functional d satisfying

$$d(\Delta(a \cdot x)) = d(a \cdot \Delta(x)) = d(a)\phi(\Delta(x)) + \phi(a)d(\Delta(x)),$$

$$d(\Delta(x \cdot a)) = d(\Delta(x) \cdot a) = d(\Delta(x))\phi(a) + \phi(\Delta(x))d(a) \quad (x \in E, a \in A).$$

Proposition 2.1. *Suppose that* A *admits a nonzero continuous point* Δ*-derivation* d *at a character* ϕ*. If* $\phi \circ \Delta \neq 0$*, then* E *is not approximately* Δ*-amenable.*

Proof. Let d be a non zero point Δ-derivation at a character ϕ, with $\phi \circ \Delta \neq 0$. Then the map $D : A \longrightarrow A^*$; $a \mapsto d(a)\phi$ is a Δ-derivation. We have

$$\begin{aligned} D(\Delta(a \cdot x)) = D(a \cdot \Delta(x)) &= (d(a)\phi(\Delta(x)) + \phi(a)d(\Delta(x))\phi \\ &= d(a)\phi(\Delta(x))\phi + \phi(a)d(\Delta(x))\phi \\ &= d(a)\phi \cdot \Delta(x) + a \cdot d(\Delta(x))\phi \end{aligned}$$

for all $a \in A$ and $x \in E$. The last equality holds, in fact, for each $b \in A$

$$\begin{aligned} (d(a)\phi \cdot \Delta(x) + a \cdot d(\Delta(x))\phi)(b) &= (d(a)\phi \cdot \Delta(x))(b) + (a \cdot d(\Delta(x))\phi)(b) \\ &= d(a)\phi(\Delta(x)b) + d(\Delta(x))\phi(ba) \\ &= d(a)\phi(\Delta(x))\phi(b) + d(\Delta(x))\phi(b)\phi(a) \\ &= d(a)\phi(\Delta(x))\phi(b) + \phi(a)d(\Delta(x))\phi(b) \\ &= (d(a)\phi(\Delta(x))\phi + \phi(a)d(\Delta(x))\phi)(b). \end{aligned}$$

Suppose the assertion of the proposition is false. Hence, there is a net (f_α) in A^* such that $(D \circ \Delta)(x) = \lim_\alpha(ad_{f_\alpha} \circ \Delta)(x)$ for all $x \in E$. Thus

$$\begin{aligned} (D(\Delta(x)))(\Delta(x)) = d(\Delta(x))\phi(\Delta(x)) &= \lim_\alpha(ad_{f_\alpha}(\Delta(x)))\Delta(x) \\ &= \lim_\alpha(\Delta(x) \cdot f_\alpha - f_\alpha \cdot \Delta(x))(\Delta(x)) \\ &= \lim_\alpha f_\alpha((\Delta(x))^2 - (\Delta(x))^2) = 0 \end{aligned}$$

for all $x \in E$. So, $d(\Delta(x))\phi(\Delta(x)) = 0$. This shows that $d \circ \Delta$ vanishes off $\ker(\phi \circ \Delta)$. On the other hand, if $z \notin \ker(\phi \circ \Delta)$ and $x \in \ker(\phi \circ \Delta)$, then $2x = (x + z) + (x - z)$ with $x + z$, $x - z \notin \ker(\phi \circ \Delta)$. So $d \circ \Delta(x) = 0$. Thus $d \circ \Delta = 0$. Now suppose that $d(a) \neq 0$ for some $a \in A$. Hence, for any $x \in E$ we have

$$d(\Delta(a \cdot x)) = d(a \cdot \Delta(x)) = d(a)\phi(\Delta(x)) + \phi(a)d(\Delta(x)).$$

Therefore, $d(a)\phi(\Delta(x)) = 0$. Since $d(a) \neq 0$, we have $\phi \circ \Delta = 0$ which is a contradiction. □

The proof of the following lemma is similar to the proof of [7, Lemma 2.1], so we do not include it.

Lemma 2.1. *Suppose that E has a weak left (right) approximate identity, then E has a left (right) approximate identity.*

Proposition 2.2. *Let E be approximately Δ-amenable. Then, E has left and right approximate identities. In particular, $\overline{\Delta(E)\cdot A} = \overline{A\cdot\Delta(E)} = \overline{\Delta(E)}$.*

Proof. Take A^{**} with usual left action and zero right action as an A-bimodule. Then, the natural injection $A \longrightarrow A^{**}$; $a \mapsto \widehat{a}$ is a Δ-derivation. Thus, there is a net $(f_\alpha) \subseteq A^{**}$ with $\Delta(x).f_\alpha \to \widehat{\Delta(x)}$ for each $x \in E$. Choose the finite sets $F \subseteq E$, $\Phi \subseteq A^*$ and $\epsilon > 0$. Let $H = \{\phi\cdot\Delta(x) : x \in F, \phi \in \Phi\}$, $K = max\{\|\psi\|, \|\phi\| : \psi \in H, \phi \in \Phi\}$. Similar to the proof of [7, Lemma 2.2], we can show that E has a weak left approximate identity. Now, apply Lemma 2.1. □

Proposition 2.3. *Suppose that E is approximately Δ-amenable (as an A-bimodule) and $\phi : A \longrightarrow B$ is a continuous epimorphism such that*

$$E \cdot \ker\phi = \ker\phi \cdot E = \{0\}.$$

If E is considered as a B-bimodule via $b\cdot x := a\cdot x, x\cdot b := x\cdot a$ $(b \in B, x \in E)$ where $a \in A$ with $b = \phi(a)$, then approximate Δ-amenability of E (as an A-bimodule) implies approximate $\phi\circ\Delta$-amenability of E (as a B-bimodule).

Proof. Suppose that X is a B-bimodule and $D : B \longrightarrow X^*$ is a $\phi\circ\Delta$-derivation. Then X is naturally an A-bimodule via $a\cdot x = \phi(a)\cdot x, x\cdot a = x\cdot\phi(a)$ $(x \in X, a \in A)$. Thus $D\circ\phi : A \longrightarrow X^*$ is a Δ-derivation, so there is a net $(f_\alpha) \subseteq X^*$ such that

$$\begin{aligned} D\circ\phi(\Delta(x)) &= \lim_\alpha(\Delta(x)\cdot f_\alpha - f_\alpha\cdot\Delta(x)) \\ &= \lim_\alpha(\phi(\Delta(x))\cdot f_\alpha - f_\alpha\cdot\phi(\Delta(x))) \end{aligned}$$

for all $x \in E$. This shows that D is approximately $\phi\circ\Delta$-inner. □

The next corollary is a direct consequece of Proposition 2.3.

Corollary 2.1. *Let E be approximately Δ-amenable (as an A-bimodule) and J be a closed two-sided ideal of A such that $J\cdot E = E\cdot J = \{0\}$. If $\pi : A \longrightarrow A/J$ is the canonical map, then E is an A/J-bimodule which is approximately $\pi\circ\Delta$-amenable.*

Proposition 2.4. *Let E and E' be Banach A-bimodules with corresponding module homomorphisms $\Delta : E \longrightarrow A$ and $\Delta' : E' \longrightarrow A$, recpectively. If $\theta : E \longrightarrow E'$ is a bounded module epimorphism such that $\Delta'\circ\theta = \Delta$, then approximate Δ-amenability of E implies approximate Δ'-amenability of E'.*

Proof. Suppose that $D : A \longrightarrow X^*$ is a Δ'-derivation where X is a Banach A-bimodule. So $D : A \longrightarrow X^*$ is also a Δ-derivation because

$$\begin{aligned} D(\Delta(a\cdot x)) &= D(a\cdot\Delta(x)) = D(a\cdot(\Delta'\circ\theta)(x)) = D(a\cdot\Delta'(\theta(x)) \\ &= D(a)\cdot\Delta'(\theta(x)) + a\cdot D(\Delta'(\theta(x)) \\ &= D(a)\cdot\Delta(x) + a\cdot D(\Delta(x)) \end{aligned}$$

for all $a \in A$ and $x \in E$. Due to the approximate Δ-amenability of E, there is a net $(f_\alpha) \subseteq X^*$ such that $D(\Delta(x)) = \lim_\alpha \Delta(x) \cdot f_\alpha - f_\alpha \cdot \Delta(x)$ $(x \in E)$. Hence,

$$\begin{aligned} D(\Delta'(\theta(x)) = D(\Delta(x)) &= \lim_\alpha (\Delta(x) \cdot f_\alpha - f_\alpha \cdot \Delta(x)) \\ &= \lim_\alpha (\Delta'(\theta(x)) \cdot f_\alpha - f_\alpha \cdot \Delta'(\theta(x)) \end{aligned}$$

for all $x \in E$. Since θ is surjective, we conclude that D is approximately Δ'-inner and so E' is approximately Δ'-amenable. □

Proposition 2.5. *Let J be a closed submodule of E and I be the closed ideal of A generated by $\Delta(J)$, $q : A \longrightarrow A/I$ and $\tilde{q} : E \longrightarrow E/J$ be the corresponding quotient maps. Then, E is approximately Δ-amenable whenever J is $\Delta|_J$-amenable ($\Delta|_J : J \longrightarrow I$) and E/J is approximately $\tilde{\Delta}$-amenable whereas $\tilde{\Delta} : E/J \longrightarrow A/I$ is the unique A/I module map with $\tilde{\Delta} \circ \tilde{q} = q \circ \Delta$*

Proof. Let X be a Banach A-bimodule and let $D : A \longrightarrow X^*$ be a Δ-derivation. Then, $D|_I : I \to X^*$ is a $\Delta|_J$-derivation. Since J is $\Delta|_J$-amenable, there exists $\lambda_1 \in X^*$ with $D(\Delta(j)) = ad_{\lambda_1}(\Delta(j))$ $(j \in J)$. Replacing D by $D - ad_{\lambda_1}$, we may suppose that $D|_{\Delta(J)} = 0$ so $D|_I = 0$. Set $F = \overline{I \cdot X + X \cdot I}$. Then F is a closed A-submodule of X and X/F is clearly a Banach (A/I)-bimodule (indeed X/F is an A-bimodule such that $I(X/F) = (X/F)I = 0$). Also, $(X/F)^* \cong F^\perp = \{f \in X^* : f|_F = 0\}$ is a dual Banach (A/I)-bimodule. For each $a \in A$ and $b \in I$, we have $a \cdot D(b) = D(ab) = 0$ and so $D(a) \cdot b = 0$. Take $x \in X$. Then, $\langle b \cdot x, D(a) \rangle = \langle x, D(a) \cdot b \rangle = 0$ so $D(a)|_{I \cdot X} = 0$. Similarly, $D(a)|_{X \cdot I} = 0$ and so $D(a)|_F = 0$. Thus $D(A) \subseteq F^\perp$ and the map $D_I : A/I \to F^\perp$, $D_I(a + I) = D(a)$ is a continuous $\tilde{\Delta}$-derivation. By hypothesis, E/J is approximately $\tilde{\Delta}$-amenable. So, there exists a net $(f_\alpha) \subseteq F^\perp$ with $D_I(\tilde{\Delta}(e + J)) = \lim_\alpha ad_{f_\alpha}(\tilde{\Delta}(e + J))$. For each $e \in E$, we get

$$\tilde{\Delta}(e + J) = \tilde{\Delta} \circ \tilde{q}(e) = q \circ \Delta(e) = \Delta(e) + I.$$

Therefore

$$\begin{aligned} D(\Delta(e)) = D_I(\Delta(e) + I) = D_I(\tilde{\Delta}(e + J)) &= \lim_\alpha ad_{f_\alpha}(\tilde{\Delta}(e + J)) \\ &= \lim_\alpha ad_{f_\alpha}(\Delta(e) + I) = \lim_\alpha ad_{f_\alpha}(\Delta(e)). \end{aligned}$$

Consequently, D is the sum of a Δ-inner derivation ad_{λ_1} and approximately Δ-inner derivation $D - ad_{\lambda_1}$. □

We have the following lemmas which are analogous to [7, Lemma 2.3] and [7, Lemma 2.4], respectively. Since the proofs are similar, we omit them.

Lemma 2.2. *Let A be a unital Banach algebra with identity e, X be an A-bimodule, and let $D : A \longrightarrow X^*$ be a Δ-derivation. Then, there is a Δ-derivation $D_1 : A \longrightarrow e \cdot X^* \cdot e$ and $\eta \in X^*$ such that*

(i) $\|\eta\| \leq 2C\|D\|$ *(where C is a constant depending on X);*
(ii) $D(\Delta(x)) = D_1(\Delta(x)) + ad_\eta(\Delta(x)) \qquad (x \in E).$

Lemma 2.3. *Let A be a unital Banach algebra with identity e and E be approximately Δ-amenable, X be an A-bimodule and $D : A \longrightarrow X^*$ be a Δ-derivation. Then, there are a net $(f_\alpha) \subset e \cdot X^* \cdot e$ and $\eta \in X^*$ such that*

(i) $\|\eta\| \leq 2C\|D\|$;
(ii) $D(\Delta(x)) = ad_\eta(\Delta(x))) + \lim_\alpha(ad_{f_\alpha}(\Delta(x)) \qquad (x \in E)$.

Remark 2.1. *In the previous lemma if E is Δ-amenable then there are $f \in e \cdot X^* \cdot e$ and $\eta \in X^*$ such that*

(1) $\|\eta\| \leq 2C\|D\|$;
(2) $D(\Delta(x)) = ad_\eta(\Delta(x)) + ad_f(\Delta(x))$ $(x \in E)$.

Let A be a non-unital Banach algebra. Then, $A^\sharp = A \oplus \mathbb{C}$, the unitization of A, is a unital Banach algebra which contains A as a closed ideal. If E is a Banach A-bimodule and $\Delta : E \longrightarrow A$ is an A-bimodule homomorphism, then E is an $A^\sharp$-bimodule with the actions

$$(a, \lambda) \cdot x = a \cdot x + \lambda x, \;\; x \cdot (a, \lambda) = x \cdot a + \lambda x \quad (x \in E, \lambda \in \mathbb{C}, a \in A).$$

It is easy to check that $\Delta' : E \longrightarrow A^\sharp$ is an $A^\sharp$-bimodule homomorphism, where for any $x \in E, \Delta'(x) = \Delta(x)$.

Proposition 2.6. *E is approximately Δ-amenable (as an A-bimodule) if and only if E is approximately Δ'-amenable (as an $A^\sharp$-bimodule).*

Proof. Sufficient part: Let $D : A^\sharp \longrightarrow X^*$ be a Δ'-derivation, where X is an $A^\sharp$-bimodule. Clearly, X is an A-bimoule and $D|_A : A \longrightarrow X^*$ is a Δ-derivation. Since E is approxmiately Δ-amenable as an A-bimodule, there is a net $(f_\alpha) \subset X^*$ such that $D|_A(\Delta(x)) = \lim_\alpha ad_{f_\alpha}(\Delta(x))$. It follows from $\mathrm{Im}(\Delta') = \mathrm{Im}(\Delta) \subseteq \mathrm{A} \subseteq \mathrm{A}^\sharp$ that $D(\Delta'(x)) = D|_A(\Delta(x)) = \lim_\alpha ad_{f_\alpha}(\Delta(x)) = \lim_\alpha ad_{f_\alpha}(\Delta'(x))$. So, E is approximately Δ'-amenable as an $A^\sharp$-bimodule.

Necessary part: Let $D : A \longrightarrow X^*$ be a Δ-derivation where X is an A-bimodule. Then X is an $A^\sharp$-bimodule with the usual actions. Now, D can be extended to $\tilde{D} : A^\sharp \to X^*$ by $\tilde{D}(a, \lambda) = D(a)$. Then $\tilde{D}$ is a Δ'-derivation. In fact

$$\begin{aligned}\tilde{D}(\Delta'((a, \lambda) \cdot x))) &= \tilde{D}((a, \lambda) \cdot \Delta'(x)) = \tilde{D}((a \cdot \Delta(x) + \lambda\Delta(x)) \\ &= \tilde{D}((a \cdot \Delta(x)) + \tilde{D}(\lambda\Delta(x)) = D(a \cdot \Delta(x)) + \lambda D(\Delta(x)) \\ &= D(a) \cdot \Delta(x) + a \cdot D(\Delta(x)) + \lambda D(\Delta(x)) \\ &= \tilde{D}(a, \lambda) \cdot \Delta'(x) + (a, \lambda) \cdot \tilde{D}(\Delta'(x)).\end{aligned}$$

Then, $\tilde{D}$ is approximately Δ'-inner, whence D is approximately Δ-inner. □

Definition 2.2. *A Banach A-bimodule X is called right Δ-essential if for $x \in X$, there are $a \in \Delta(E)$ and $y \in X$ such that $x = y \cdot a$. The left Δ-essential and (two sided) Δ-essential modules are defined similarly.*

Theorem 2.1. *Suppose that Δ has a dense range and E has a bounded approximate identity. Then E is approximately Δ-amenable if and only if for each Δ-essential Banach A-bimodule X, all Δ-derivations from A to X^* are approximately Δ-inner.*

Proof. Let $(a_\alpha) \subseteq A$ be a bounded approximate identity for E. Let X be a Banach A-bimodule and $D : A \longrightarrow X^*$ be a Δ-derivation. Consider $T_\alpha : X^* \longrightarrow X^*$ defined by $T_\alpha(f) = a_\alpha \cdot f$, for all $f \in X^*$. Since (a_α) is bounded in A, $\{T_\alpha\}$ is bounded in $B(X^*)$. Hence, it has a w^*-cluster point, say T. We may assume that $T_\alpha \to T$ in w^*-topology. For each $e \in E, x \in X, f \in X^*$ we have

$$\begin{aligned}\langle x \cdot \Delta(e), Tf\rangle &= \lim_\alpha \langle x \cdot \Delta(e), T_\alpha f\rangle = \lim_\alpha \langle x \cdot \Delta(e), a_\alpha \cdot f\rangle \\ &= \lim_\alpha \langle x \cdot \Delta(e) a_\alpha, f\rangle = \langle x \cdot \Delta(e), f\rangle.\end{aligned}$$

Thus, $T - I : X^* \longrightarrow (X \cdot \Delta(E))^\perp$ is a bounded projection. Also, the following short exact sequence of Banach A-bimodules is admissible

$$0 \longrightarrow (X \cdot \Delta(E))^\perp \longrightarrow X^* \longrightarrow (X \cdot \Delta(E))^* \longrightarrow 0.$$

On the other hand, $\frac{X}{(X \cdot \Delta(E))} \cdot \Delta(E) = 0$. We have

$$X^* = (X \cdot \Delta(E))^* \oplus (X \cdot \Delta(E))^\perp.$$

This implies that TD and $(I - T)D$ are Δ-derivations, the latter being Δ-inner. It follows from $(X \cdot \Delta(E))^\perp \cong (\frac{X}{(X \cdot \Delta(E))})^*$ and $\frac{X}{(X \cdot \Delta(E))} \cdot \Delta(E) = 0$ that $\Delta(E) \cdot (\frac{X}{(X \cdot \Delta(E))})^* = 0$. Since Δ has dense range, $A \cdot (\frac{X}{(X \cdot \Delta(E))})^* = 0$ and so $A \cdot (X \cdot \Delta(E))^\perp = 0$. We now consider $\Delta(E) \cdot (X \cdot \Delta(E))$ and proceed as before to find that D is the sum of two Δ-inner derivations, plus a derivation mapping into the dual of the Δ-essential module $\Delta(E) \cdot (X \cdot \Delta(E))$. □

Lemma 2.4. *Let A be an approximately amenable Banach algebra. If B is another Banach algebra such that A is an ideal of B and $\Delta : A \longrightarrow B$ is the inclusion map, then A is approximately Δ-amenable (as a B-bimodule).*

Proof. Suppose that X is a B-bimodule and $D : B \longrightarrow X^*$ is a Δ-derivation. Then, X is also an A-bimodule and $D|_A : A \to X^*$ is a derivation. Since A ia approximately amenable there exists a net $(f_\alpha) \subset X^*$ such that $D|_A(a) = \lim_\alpha ad_{f_\alpha}(a)$, we have $D(\Delta(a)) = D(a) = D|_A(a) = \lim_\alpha ad_{f_\alpha}(a)$. □

It is shown in [6, Proposition 2.8] that if Δ has a dense range, then Δ-amenability of the A-module E is equivalent to amenability of the Banach algebra A. Also, it is known that the direct sum of two amenable Banach algebras is an amenable Banach algebra. Summing up:

Lemma 2.5. *Let E be a Banach A-bimodule and F be a Banach B-bimodule. If $\alpha : E \longrightarrow A$ and $\beta : F \longrightarrow B$ are bounded Banach A-bimodule homomorphism and B-bimodule homomorphism with dense ranges, respectively, then $E \oplus F$ is a Banach $A \oplus B$-bimodule with the natural action $(x, y) \cdot (a, b) = (xa, yb), (a, b) \cdot (x, y) = (ax, by)$ for $(a, b) \in A \oplus B, (x, y) \in E \oplus F$. Also, $(\alpha \oplus \beta) : E \oplus F \longrightarrow A \oplus B$ is defined by $(\alpha \oplus \beta)(x, y) = (\alpha(x), \beta(y))$ is a bounded Banach $A \oplus B$-bimodule homomorphism. In particular, if E is α-amenable (as an A-bimodule) and F is β-amenable (as a B-bimodule), then $E \oplus F$ is $\alpha \oplus \beta$-amenable as an $A \oplus B$-bimodule.*

Definition 2.3. *A Banach A-bimodule E is called approximately Δ-contractible if for any Banach A-bimodule X every Δ-derivation $D : A \longrightarrow X$ is approximately Δ-inner.*

Proposition 2.7. *Let E be a Banach A-bimodule, and $\Delta : E \longrightarrow A$ be a bounded Banach A-bimodule homomorphism. Let $\Delta \oplus \Delta : E \oplus E \longrightarrow A \oplus A$ be defined by $(\Delta \oplus \Delta)(x, y) = (\Delta(x), \Delta(y))$. If $E \oplus E$ is approximately $\Delta \oplus \Delta$ contractible (as an $A \oplus A$ bimodule), then A has an approximate identity.*

Proof. Let A be an $A \oplus A$-bimodule with the following actions

$$(a, b) \cdot x = ax, \; x \cdot (a, b) = xb \qquad (x \in A, a, b \in A).$$

Define $D : A \oplus A \to A$ by $D(a, b) = a - b$. Then, D is a derivation and hence a $(\Delta \oplus \Delta)$-derivation. So, there is a net $(a_i) \subset A$ such that

$$\begin{aligned} D(\Delta(x), \Delta(y)) = \Delta(x) - \Delta(y) &= \lim_i (\Delta(x), \Delta(y))a_i - a_i(\Delta(x), \Delta(y)) \\ &= \lim_i \Delta(x)a_i - a_i\Delta(y). \end{aligned}$$

for all $x, y \in E$. Therefore, $\lim_i \Delta(x)a_i = \Delta(x)$ and $\lim_i a_i\Delta(y) = \Delta(y)$. □

Theorem 2.2. *Let E be a Banach A-bimodule and $\Delta : E \longrightarrow A$ be a bounded Banach A-bimodule homomorphism. Suppose that $\Delta(E)$ is norm closed in A and $\Delta(E)$ has a bounded approximate identity. Then, E is approximately Δ-amenable if and only if it is approximately Δ-contractible.*

Proof. The sufficient part is clear. For the necessary part, assume that E is approximately Δ-amenable (as an A-bimodule). We claim that $\Delta(E)$ is an approximately amenable Banach algebra. So, by [8, Theorem 2.1] $\Delta(E)$ is an approximately contractible Banach algebra. Now, let $D : A \longrightarrow X$ be a Δ-derivation for some Banach A-bimoule X. Then, $D\,|_{\Delta(E)}: \Delta(E) \longrightarrow X$ is a derivation. Since $\Delta(E)$ is approximately contractible, $D\,|_{\Delta(E)}$ is approximately inner and hence D is approximately Δ-inner. Therefore E is approximately Δ-contractible.

To prove the claim suppose that $D : \Delta(E) \longrightarrow X^*$ is a derivation for a Banach $\Delta(E)$-bimodule X. Since $\Delta(E)$ is a norm closed ideal in A and $\Delta(E)$ has a bounded approximate identity, by [12, Proposition 2.1.6] we can extend D to a derivation $\bar{D} : A \longrightarrow X^*$. Due to the approximate Δ-amenability of E (as an A-bimodule), $\bar{D}$ is approximately Δ-inner and thus D is approximately inner. □

For a Banach algebra A, let $\pi : A \widehat{\bigotimes} A \longrightarrow A$ be the canonical map, that is, $\pi(a \otimes b) = ab$ for any $a, b \in A$.

Theorem 2.3. *Let A be a unital Banach algebra with identity e, E be a Banach A-bimodule and $\Delta : E \longrightarrow A$ be a bounded Banach A-bimodule homomorphism. Consider the following assertions:*

(i) *E is approximately Δ-amenable as a Banach A-bimodule;*

(ii) *There is a net $(M_v) \subseteq (A \widehat{\bigotimes} A)^{**}$ such that for each $x \in E$, $\Delta(x) \cdot M_v - M_v \cdot \Delta(x) \longrightarrow 0$ and $\pi^{**}(M_v) \longrightarrow e$;*

(iii) *There is a net* $(M'_v) \subseteq (A\widehat{\bigotimes}A)^{**}$ *such that for each* $x \in E$, $\Delta(x) \cdot M'_v - M'_v \cdot \Delta(x) \longrightarrow 0$ *and* $\pi^{**}(M'_v) = e$ *for every* v.

Then (i)⇒(iii)⇒ (ii), *and in the case that* Δ *has dense range,* (ii) *implies* (i).

Proof. (i)⇒ (iii). Let $D : A \longrightarrow (A\widehat{\bigotimes}A)^{**}$ be defined by $D(a) = a \cdot u - u \cdot a$ for any $a \in A$, where $u = e \otimes e$. Then, $\mathrm{Im(D)} \subseteq \ker(\pi^{**}) \cong (\ker \pi)^{**}$. Since E is approximately Δ-amenable, there is a net $(t_v) \subset \ker(\pi^{**})$ such that for any $x \in E$, $D(\Delta(x)) = \lim_v \Delta(x) \cdot t_v - t_v \cdot \Delta(x)$. Let $M'_v = u - t_v$. So, $\pi^{**}(M'_v) = e$ and

$$\Delta(x) \cdot M'_v - M'_v \cdot \Delta(x) = \Delta(x) \cdot u - u \cdot \Delta(x) - (\Delta(x) \cdot t_v - t_v \cdot \Delta(x)) \longrightarrow 0.$$

(iii)⇒(ii) It is trivial.

(ii)⇒(i). Let $D : A \longrightarrow X^*$ be a Δ-derivation for some Δ-essential Banach A-bimodule X. For each $x \in X$, let $\mu_x \in (A\widehat{\bigotimes}A)^*$ defined by $\mu_x(a \otimes b) = \langle aD(b), x\rangle$ for all $a, b \in A$. Now, for each v, put $f_v(x) = M_v(\mu_x)$ for any $x \in X$. We show that for any $y \in E$, $D(\Delta(y)) = \lim_v ad_{f_v}(\Delta(y))$ and hence by Theorem 2.1 E is approximately Δ-amenable. It is easy to check that for $x \in X$, $m \in A\widehat{\bigotimes}A$

$$\mu_{x\cdot\Delta(y)-\Delta(y)\cdot x}(m) = (\mu_x \cdot \Delta(y) - \Delta(y) \cdot \mu_x)(m) + (\pi(m)Da)(x).$$

There is a net (m_v^α) in $A\widehat{\bigotimes}A$ such that $M_v = w^* - \lim_\alpha m_v^\alpha$. So,

$$\begin{aligned}
(\Delta(y) \cdot f_v - f_v \cdot \Delta(y))(x) &= f_v(x \cdot \Delta(y) - \Delta(y) \cdot x) \\
&= M_v(\mu_{\Delta(y)\cdot x - x\cdot\Delta(y)}) = \lim_\alpha (\mu_{\Delta(y)\cdot x - x\cdot\Delta(y)})(m_v^\alpha) \\
&= M_v(\mu_x \cdot \Delta(y) - \Delta(y) \cdot \mu_x) + \lim_\alpha (\pi(m_v^\alpha)D(\Delta(y))(x) \\
&= (\Delta(y) \cdot M_v - M_v \cdot \Delta(y))(\mu_x) + (\pi^{**}(M_v)D(\Delta(y)))(x).
\end{aligned}$$

Thus

$$\begin{aligned}
&\|(\Delta(y) \cdot f_v - f_v \cdot \Delta(y))(x) - D(\Delta(y))(x)\| \\
&\qquad\leqslant \|\Delta(y) \cdot M_v - M_v \cdot \Delta(y)\| \cdot \|D\| \cdot \|x\| \\
&\qquad+ \|x\| \cdot \|\pi^{**}(M_v) - e\| \cdot \|D(\Delta(y))\|
\end{aligned}$$

Therefore, $D(\Delta(y)) = \lim_v ad_{f_v}(\Delta(y))$ as required. □

Given a sequence $\{A_n\}$ of Banach algebras, define their l^∞ direct sum as

$$l^\infty(A_n) = \{(x_n) : x_n \in A_n, \|(x_n)\| = \sup \|x_n\| < \infty\}$$

and

$$c_0(A_n) = \{(x_n) \in l^\infty(A_n) : \|x_n\| \to 0\}.$$

We finish the paper by four examples.

Example 2.1. We present an approximately Δ-amenable module which is not a Δ-amenable Banach module. Consider the algebra M_n of $n \times n$ matrices with norm $\|a_{ij}\|_2 = (\sum_{i,j} |a_{ij}|^2)^{1/2}$. Then $\|AB\|_2 \leq \|A\|_2\|B\|_2$ for any $A, B \in M_n$. One should remember that the duality between M_n and M_n^* is, $\langle A, E\rangle = \sum_{i,j} a_{ij}e_{ij}$. Also, the map $M_n \to M_n^* : A \longmapsto A$, is isometric. Let $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ as an element of M_2^*. Inductively, define $P_{n+1} = \begin{bmatrix} 0 & -P_n \\ P_n & 0 \end{bmatrix}$ so that $P_n \in M_{2^n}$. Let $A_n = M_{2^n}^\sharp$. By [7, Example 6.2], $c_0(A_n)$ is an approximately amenable Banach algebra which is not amenable, since $c_0(A_n)$ is an ideal of $l^\infty(A_n)$, by Theorem 2.1 $c_0(A_n)$ is approximately Δ-amenable as an $l^\infty(A_n)$-bimodule, where $\Delta : c_0(A_n) \longrightarrow l^\infty(A_n)$ is the inclusion map. Now we define $D : l^\infty(A_n) \longrightarrow l^1(A_n^*)$ by $D((x_n)) = \left(\frac{ad_{P_n}(x_n)}{n^2}\right)$. As in [7, Example 6.2] D cannot be Δ-inner. Thus, $c_0(A_n)$ is not Δ-amenable as an $l^\infty(A_n)$-bimodule.

Example 2.2. Let A be an approximately amenable Banach algebra and let $\pi : A\widehat{\otimes}A \to A$ be the canonical map. Then $A\widehat{\otimes}A$ is approximately π-amenable (as an A-bimodule). It is easy to see that π is a Banach A-bimodule homomorphism. Since A is an approximately amenable Banach algebra, A has left approximate identity. Therefore, π has a dense range. Let $D : A \longrightarrow X^*$ be a π-derivation where X is a Banach A-bimodule. Since π has a dense range, D is a derivation. Due to the approximate amenability of A an Banach algebra, there exists a net $(f_\alpha) \subset X^*$ such that $D(a) = \lim_\alpha ad_{f_\alpha}(a)$, So $D(\pi(x)) = \lim_\alpha ad_{f_\alpha}(\pi(x))$ for all $x \in A\widehat{\otimes}A$.

Example 2.3. Let G be a locally compact group. We know that $L^1(G)$ is a closed two sided ideal in $M(G)$. We can consider $L^1(G)$ as a Banach $M(G)$-bimodule. Let $i : L^1(G) \longrightarrow M(G)$ be the inclusion map. If G is a non discrete amenable group then $M(G)$ is not an approximately amenable Banach algebra [7]. Let $D : M(G) \longrightarrow X^*$ be an i-derivation where X is a Banach $M(G)$-bimodule. Then, X is also an $L^1(G)$-bimodule and $D|_{L^1(G)} : L^1(G) \longrightarrow X^*$ is a derivation. Since G is amenable, $D|_{L^1(G)}$ is inner and hence $D|_{L^1(G)}$ is an approximately inner derivation. Consequently, $L^1(G)$ is an approximately i-amenable $M(G)$-bimodule.

Example 2.4. Let G be an abelian compact group. Then, $L^p(G)$ $(1 < p < \infty)$ is a Banach $L^1(G)$-bimodule. If $1/p + 1/q = 1$ and $f \in L^q(G)$, then define

$$\Delta_f : L^p(G) \to L^1(G)$$

by $\Delta_f(g) = g * f$. Since G is an abelian compact group, Δ_f has dense range. If G is amenable so $L^1(G)$ is an amenable Banach algebra and so $L^p(G)$ is Δ_f-amenable. Therefore, $L^p(G)$ is approximately Δ_f-amenable.

The idea of the next example is motivated by [7, Example 6.1].

Example 2.5. For each $n \in N$, let A_n be a unital Banach algebra with identity e_n. Let M_n be an A_n-bimodule such that there exists $k_n > 0$ such that for each $x \in M_n$ and $a \in A_n$, we have $\|a \cdot x\| \leq k_n\|a\|\|x\|, \|x \cdot a\| \leq k_n\|x\|\|a\|$. Let $\Delta_n : M_n \longrightarrow A_n$ be a bounded Banach A_n-bimodule homomorphism with dense range. Suppose that M_n is Δ_n amenable as an A_n-bimodule. Let $M = c_0(M_n)$ and $A = c_0(A_n)$. If $\sup\{k_n : n \in N\} < \infty$, then M is a Banach A-bimodule. Consider the mapping $\Delta : M \longrightarrow A$ defined through $(\Delta(m_n)) = (\Delta_n(m_n))$ and $\sup\{\|\Delta_n\| : n \in N\} < \infty$.

Then Δ is a bounded Banach A-bimodule homomorphism and M is approximately Δ-amenable (as an A-bimodule). It is easy to see that M is a Banach A-bimodule and Δ is a bounded Banach A-bimodule homomorphism. Let X be an A-bimodule and $D : A \longrightarrow X^*$ be a Δ-derivation. Put

$$B_k = \{(x_n) \in c_0(A_n) : x_n = 0 \text{ for } n > k\}$$

and

$$C_k = \{(m_n) \in c_0(M_n) : m_n = 0 \text{ for } n > k\}.$$

Set $E_n = (e_1, e_2, e_3, ..., e_n, 0, ...)$. Then, (E_n) is a central approximate identity for A. Restricting D to some B_n we have a $\Delta|_{C_n}$-derivation $D_n : B_n \longrightarrow X^*$. Since B_n is unital, by Lemma 2.5 and Remark 2.1, there exists $f_n \in E_n \cdot X^* \cdot E_n$ and $\eta_n \in X^*(\|\eta_n\| \leqslant 2C\|D\|)$ such that $D_n(\Delta|_{C_n}(x)) = ad_{f_n}(\Delta|_{C_n}(x)) + ad_{\eta_n}(\Delta|_{C_n}(x))$, for any $x \in C_n$. Note that for each $x \in M$, $\|ad_{\eta_n}(E_n\Delta(x) - \Delta(x))\| \to 0$, because (E_n) is an approximate identity and (η_n) is bounded. Since $f_n \in E_n \cdot X^* \cdot E_n$, we have $aE_n \cdot f_n = a \cdot f_n$ and $f_n \cdot E_n a = f_n \cdot a$ for any $a \in A$. As (E_n) is central, for any $x \in M$, we get

$$\begin{aligned} D(\Delta(x)) &= \lim_n D(E_n\Delta(x)) = \lim_n((\Delta(x)E_n) \cdot f_n - f_n \cdot (E_n\Delta(x)) + ad_{\eta_n}(E_n\Delta(x)) \\ &= \lim_n(\Delta(x) \cdot f_n - f_n \cdot \Delta(x) + ad_{\eta_n}(E_n\Delta(x)) \\ &= \lim_n(ad_{f_n}(\Delta(x)) + ad_{\eta_n}E_n\Delta(x)) + ad_{\eta_n}(\Delta(x) - (E_n\Delta(x)) \\ &= \lim_n(ad_{f_n}(\Delta(x)) + ad_{\eta_n}(\Delta(x))) = \lim_n ad_{(f_n+\eta_n)}(\Delta r(x)). \end{aligned}$$

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