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CONSTRUCTION OF CONTROLLED K-G-FUSION FRAMES IN HILBERT SPACES

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Considering the importance and application of dual of frames, especially fusion frames, which cannot be defined in the usual way, we try to investigate the concept of dual for controlled generalized K-fusion frames.

Keywords: K-g-fusion frame, controlled g-fusion frame, controlled K-g-fusion frame, Q-duality.

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1. Introduction and Preliminaries

Controlled frames have been recently introduced by Balazs et. al [1] in Hilbert spaces to improve the numerical efficiency of interactive algorithms for inverting the frame operator. Afterwards, this topic has been generalized for g-frames, fusion frames and K-frames, e. g. [9, 10, 11]. Generalized fusion frames or briefly g-fusion frames introduced by Sadri et al. [13] are obtained from the combination of fusion and g-frames also. In this note, we first introduce the concept of controlled K-g-fusion frames which are generalizations of controlled g-fusion frames in Hilbert spaces. After characterizing and constructing these frames by a bounded operator, we present the Q-dual of controlled K-g-fusion frames and we describe how to create the Q-dual of these frames. Finally, perturbation of these frames will be discussed.

Throughout this paper, H is a separable Hilbert spaces, $\mathcal{B}(H)$ is the collection of all bounded linear operators on H, $\mathcal{GL}(H)$ is the set of all bounded linear operators on H which have bounded inverses, $\mathcal{GL}^+(H)$ is the set of all positive operators in $\mathcal{GL}(H)$ and $K \in \mathcal{B}(H)$. Also, π_V is the orthogonal projection from H onto a closed subspace $V \subset H$ and $\{H_i\}_{i\in\mathbb{I}}$ is a sequence of Hilbert spaces, where $\mathbb{I}$ is a subset of $\mathbb{Z}$.

Lemma 1.1. [8] *Let $V \subseteq H$ be a closed subspace, and T be a linear bounded operator on H. Then*

$$\pi_V T^* = \pi_V T^* \pi_{\overline{TV}}.$$

If T is unitary (i.e. $T^*T = Id_H$), then

$$\pi_{\overline{TV}} T = T \pi_V.$$

If an operator U has closed range, then there exists a right-inverse operator $U^\dagger$ (pseudo-inverse of U) in the following sense (see [4]).

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Lemma 1.2. *Let $U \in \mathcal{B}(H_1, H_2)$ be a bounded operator with closed range $\mathcal{R}_U$. Then there exists a bounded operator $U^\dagger \in \mathcal{B}(H_2, H_1)$ such that*

$$UU^\dagger x = x, \quad x \in \mathcal{R}_U.$$

Lemma 1.3. [5] *Let $L_1 \in \mathcal{B}(H_1, H)$ and $L_2 \in \mathcal{B}(H_2, H)$ be operators on given Hilbert spaces. Then the following assertions are equivalent:*

(1) $\mathcal{R}(L_1) \subseteq \mathcal{R}(L_2)$;
(2) $L_1 L_1^* \leq \lambda^2 L_2 L_2^*$ *for some* $\lambda > 0$;
(3) *there exists a mapping* $U \in \mathcal{B}(H_1, H_2)$ *such that* $L_1 = L_2 U$.

Now, we review some definitions about K-g-fusion, (C, C')-controlled g-fusion.

Definition 1.1 (K-g-fusion frame)**.** [12] *Let $W = \{W_i\}_{i\in\mathbb{I}}$ be a collection of closed subspaces of H, $\{v_i\}_{i\in\mathbb{I}}$ be a family of weights, i.e. $v_i > 0$, $\Lambda_i \in \mathcal{B}(H, H_i)$ for each $i \in \mathbb{I}$ and $K \in \mathcal{B}(H)$. We say that $\Lambda := (W_i, \Lambda_i, v_i)$ is a K-g- fusion frame for H if there exists $0 < A \leq B < \infty$ such that for each $f \in H$*

$$A\|K^* f\|^2 \leq \sum_{i\in\mathbb{I}} v_i^2 \|\Lambda_i \pi_{W_i} f\|^2 \leq B\|f\|^2.$$

Corresponding to this frame, the representation space is defined by

$$\mathscr{H}_2 := \big\{\{f_i\}_{i\in\mathbb{I}} \ : \ f_i \in H_i, \ \sum_{i\in\mathbb{I}} \|f_i\|^2 < \infty\big\},$$

with the inner product defined by

$$\langle \{f_i\}, \{g_i\}\rangle = \sum_{i\in\mathbb{I}} \langle f_i, g_i\rangle.$$

Definition 1.2 ((C, C')-controlled g-fusion frame)**.** [14] *Let $W := \{W_i\}_{i\in\mathbb{I}}$ be a family of closed subspaces of H and $\{v_i\}_{i\in\mathbb{I}}$ be a family of weights i.e. $v_i > 0$ for all $i \in \mathbb{I}$. Let $\{H_i\}_{i\in\mathbb{I}}$ be a sequence of Hilbert spaces, $C, C' \in \mathcal{GL}(H)$ and $\Lambda_i \in \mathcal{B}(H, H_i)$. $\Lambda_{CC'} := (W_i, \Lambda_i, v_i)$ is a (C, C')-controlled g-fusion frame for H if there exist constants $0 < A \leq B < \infty$ such that for all $f \in H$*

$$A\|f\|^2 \leq \sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} C' f, \Lambda_i \pi_{W_i} C f\rangle \leq B\|f\|^2.$$

2. (C, C')-controlled K-g-fusion frames

In this section, we introduce the concept of (C, C')-controlled K-g-fusion frame on Hilbert spaces and present the corresponding operators. Throughout this paper, C and C' are invertible operators in $\mathcal{GL}(H)$.

Definition 2.1. *Let $W := \{W_i\}_{i\in\mathbb{I}}$ be a family of closed subspaces of H and $\{v_i\}_{i\in\mathbb{I}}$ be a family of weights. Suppose that $\{H_i\}_{i\in\mathbb{I}}$ is a sequence of Hilbert spaces and $\Lambda_i \in \mathcal{B}(H, H_i)$. We call $\Lambda_{CC'K} := (W_i, \Lambda_i, v_i)$ a (C, C')-controlled K-g-fusion frame (briefly CC'-KGF) for H if there exist constants $0 < A_{CC'} \leq B_{CC'} < \infty$ such that for each $f \in H$*

$$A_{CC'}\|K^* f\|^2 \leq \sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} C' f, \Lambda_i \pi_{W_i} C f\rangle \leq B_{CC'}\|f\|^2. \tag{1}$$

Throughout this paper, $\Lambda_{CC'K}$ will be a triple (W_i, Λ_i, v_i) with $i \in \mathbb{I}$ unless otherwise stated. We call $\Lambda_{CC'K}$ a Parseval CC'-KGF if $A_{CC'} = B_{CC'} = 1$ or, equivalently,

$$\sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} C' f, \Lambda_i \pi_{W_i} C f\rangle = \|K^* f\|^2.$$

When $K = Id_H$, we get a C, C'-controlled g-fusion frame for H. If only the second inequality (1) is required, $\Lambda_{CC'K}$ is called a (C, C')-controlled g-fusion Bessel sequence (briefly CC'-GBS) with bound $B_{CC'}$.

The synthesis and analysis operators are similar to those corresponding to controlled g-fusion frame ([14]). So, if $\Lambda_{CC'K}$ is a CC'-GBS, then

$$T_{CC'} : \mathscr{K}^2_{\Lambda_i} \to H,$$
$$T_{CC'}\Big(v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\Big) = \sum_{i\in\mathbb{I}} v_i^2 C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C'f,$$

and

$$T^*_{CC'} : H \to \mathscr{K}^2_{\Lambda_i},$$
$$T^*_{CC'}f = \{v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\}_{i\in\mathbb{I}},$$

where

$$\mathscr{K}^2_{\Lambda_i} := \{v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f \ : \ f \in H\}_{i\in\mathbb{I}} \subset (\bigoplus_{i\in\mathbb{I}} H)_{l^2}. \tag{2}$$

Therefore, the frame operator is given by

$$S_{CC'}f := T_{CC'}T^*_{CC'}f = \sum_{i\in\mathbb{I}} v_i^2 C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C'f,$$

and for each $f \in H$,

$$\begin{aligned}\langle S_{CC'}f, f\rangle &= \sum_{i\in\mathbb{I}} v_i^2 \langle C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C'f, f\rangle \\ &= \sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i\pi_{W_i}C'f, \Lambda_i\pi_{W_i}Cf\rangle.\end{aligned}$$

Hence

$$A_{CC'}KK^* \leq S_{CC'} \leq B_{CC'}Id_H.$$

Now, we conclude that that the following result holds.

Proposition 2.1. *Let $\Lambda_{CC'K}$ be a CC'-GBS for H. Then $\Lambda_{CC'K}$ is a CC'-KGF if and only if there exists $A_{CC'} > 0$ such that $S_{CC'} \geq A_{CC'}KK^*$.*

For CC'-KGF, like for K-frames, the operator $S_{CC'}$ is not invertible and when K has closed range, $S_{CC'}$ is an invertible operator (for more details, we refer to [12]). Assume that K has closed range. Since $\mathcal{B}(H)$ is a C^*-algebra, then $S^{-1}_{CC'}$ is positive and self-adjoint. Now, for any $f \in S_{CC'}(\mathcal{R}(K))$ we have

$$\begin{aligned}\langle Kf, f\rangle &= \langle Kf, S_{CC'}S^{-1}_{CC'}f\rangle \\ &= \langle S_{CC'}(Kf), S^{-1}_{CC'}f\rangle \\ &= \langle \sum_{i\in\mathbb{I}} v_i^2 C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C'Kf, S^{-1}_{CC'}f\rangle \\ &= \sum_{i\in\mathbb{I}} v_i^2 \langle S^{-1}_{CC'}C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C'Kf, f\rangle.\end{aligned}$$

In the next results, we construct K-g-fusion frames by using a bounded linear operator.

Theorem 2.1. *Let $U \in \mathcal{B}(H)$ be an invertible operator on H such that U^* commutes with C, C' and let $\Lambda_{CC'K}$ be a CC'-KGF for H with bounds $A_{CC'}$ and $B_{CC'}$. Then, $\Gamma := (UW_i, \Lambda_i\pi_{W_i}U^*, v_i)$ is a CC'-$UKGF$ for H.*

Proof. Since U is invertible, UW_j is a closed subspace of H for each $i \in \mathbb{I}$. For $f \in H$, by applying Lemma 1.1 with U instead of T, we have

$$\begin{aligned}\sum_{i\in\mathbb{I}} v_i^2 \big\langle \Lambda_i \pi_{W_i} U^* \pi_{UW_i} C' f,\ \Lambda_i \pi_{W_i} U^* \pi_{UW_i} C f\big\rangle &= \sum_{i\in\mathbb{I}} v_i^2 \big\langle \Lambda_i \pi_{W_i} U^* C' f,\ \Lambda_i \pi_{W_i} U^* C f\big\rangle \\ &\leq B_{CC'} \|U^* f\|^2 \\ &\leq B_{CC'} \|U\|^2 \|f\|^2.\end{aligned}$$

So, Γ is a g-fusion Bessel sequence for H. On the other hand,

$$\begin{aligned}\sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} U^* \pi_{UW_i} C' f,\ \Lambda_i \pi_{W_i} U^* \pi_{UW_i} C f\rangle &= \sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} U^* C' f,\ \Lambda_i \pi_{W_i} U^* C f\rangle \\ &\geq A_{CC'} \|K^* U^* f\|^2 \\ &= A_{CC'} \|(UK)^* f\|^2,\end{aligned}$$

and the proof is completed. □

Corollary 2.1. *Let $U \in \mathcal{B}(H)$ be an invertible operator on H and U^* commutes with C, C' and K^*, furthermore, let $\Lambda_{CC'K}$ be a CC'-KGF for H. Then, $\Gamma = (UW_i, \Lambda_i \pi_{W_i} U^*, v_i)$ is a CC'-KGF for H.*

Theorem 2.2. *Let $U \in \mathcal{B}(H)$ be an unitary operator on H which commutes with C, C', and let $\Lambda_{CC'K}$ be a CC'-KGF for H with bounds $A_{CC'}$ and $B_{CC'}$. Then, $\Gamma = (UW_i, \Lambda_i U^{-1}, v_i)$ is a CC'-$(U^{-1})^* K$GF for H.*

Proof. Via Lemma 1.1, we can write for every $f \in H$,

$$A_{CC'} \|K^* U^{-1} f\|^2 \leq \sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i U^{-1} \pi_{UW_i} C' f, \Lambda_i U^{-1} \pi_{UW_i} C f\rangle \leq B_{CC'} \|U^{-1}\|^2 \|f\|^2.$$

□

Corollary 2.2. *Let $U \in \mathcal{B}(H)$ be an unitary operator on H which commutes with C, C' and K^*, furthermore $\Lambda_{CC'K}$ be a CC'-KGF for H with bounds $A_{CC'}$ and $B_{CC'}$. Then, $\Gamma = (UW_i, \Lambda_i U^{-1}, v_i)$ is a CC'-KGF for H.*

Theorem 2.3. *Let $\Lambda_{CC'K}$ be a CC'-KGF for H with bounds $A_{CC'}$ and $B_{CC'}$ and let K be closed range. Assume that $U \in \mathcal{B}(H)$ is such that $\mathcal{R}(U) \subseteq \mathcal{R}(K)$ and also U^* commutes with C, C'. Then, $\Gamma = (\overline{UW_i}, \Lambda_i \pi_{W_i} U^*, v_i)$ is a CC'-KGF for H if and only if there exists $\delta > 0$ such that for every $f \in H$,*

$$\|U^* f\| \geq \delta \|K^* f\|.$$

Proof. Assume that $f \in H$ and Γ is a CC'-KGF for H with the lower bound D. So, by Lemma 1.1, we get

$$\begin{aligned}D\|K^* f\|^2 &\leq \sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} U^* \pi_{\overline{UW_i}} C' f, \Lambda_i \pi_{W_i} U^* \pi_{\overline{UW_i}} C f\rangle \\ &= \sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} U^* C' f, \Lambda_i \pi_{W_i} U^* C f\rangle \\ &\leq B_{CC'} \|U^* f\|^2.\end{aligned}$$

Thus, $\|U^* f\| \geq \sqrt{\frac{D}{B_{CC'}}} \|K^* f\|$. For the opposite implication, we can write for all $f \in H$,

$$\|U^* f\| = \|(K^\dagger)^* K^* U^* f\| \leq \|K^\dagger\| \|K^* U^* f\|.$$

Therefore,

$$\begin{aligned}A_{CC'}\delta^2\|K^\dagger\|^{-2}\|K^*f\|^2 &\leq A_{CC'}\|K^\dagger\|^{-2}\|U^*f\|^2\\ &\leq A_{CC'}\|K^*U^*f\|^2\\ &\leq \sum_{i\in\mathbb{I}} v_i^2\langle \Lambda_i\pi_{W_i}U^*C'f, \Lambda_i\pi_{W_i}U^*Cf\rangle\\ &= \sum_{i\in\mathbb{I}} v_i^2\langle \Lambda_i\pi_{W_i}U^*\pi_{\overline{UW_i}}C'f, \Lambda_i\pi_{W_i}U^*\pi_{\overline{UW_i}}Cf\rangle\\ &\leq B_{CC'}\|U\|^2\|f\|^2.\end{aligned}$$

The proof is completed. □

Theorem 2.4. *If $U\in\mathcal{B}(H)$, $\mathcal{R}(U)\subseteq\mathcal{R}(K)$ and $\Lambda_{CC'K}$ is a CC'-KGF for H with bounds $A_{CC'}$ and $B_{CC'}$, then $\Lambda_{CC'K}$ is a CC'-UGF for H.*

Proof. By Lemma 1.3, there exists $\lambda>0$ such that $UU^*\leq\lambda^2KK^*$. Thus, for each $f\in H$ we have

$$\|U^*f\|^2=\langle UU^*f,f\rangle\leq\lambda^2\langle KK^*f,f\rangle=\lambda^2\|K^*f\|^2.$$

It follows that

$$\frac{A_{CC'}}{\lambda^2}\|U^*f\|^2\leq\sum_{i\in\mathbb{I}} v_i^2\langle \Lambda_i\pi_{W_i}C'f, \Lambda_i\pi_{W_i}Cf\rangle\leq B_{CC'}\|f\|^2.$$

□

Theorem 2.5. *Let $\Lambda_{CC'}:=(W_j,\Lambda_j,v_j)$ and $\Theta_{CC'}:=(W_j,\Theta_j,v_j)$ be two CC'-GBS for H with bounds B_Λ and B_Θ, respectively. Suppose that T_Λ and T_Θ are their analysis operators such that $T_\Theta T_\Lambda^*=K^*$, where $K\in\mathcal{B}(H)$. Then, $\Lambda_{CC'}$ and $\Theta_{CC'}$ are CC'-KGF and CC'-K^*GF, respectively.*

Proof. For each $f\in H$ we have,

$$\begin{aligned}\|K^*f\|^4 &= \langle K^*f,K^*f\rangle^2\\ &= \langle T_\Lambda^*f, T_\Theta^*K^*f\rangle^2\\ &\leq \|T_\Lambda^*f\|^2\|T_\Theta^*K^*f\|^2\\ &= \Big(\sum_{i\in\mathbb{I}} v_i^2\langle \Lambda_i\pi_{W_i}C'f, \Lambda_i\pi_{W_i}Cf\rangle\Big)\Big(\sum_{i\in\mathbb{I}} v_i^2\langle \Theta_i\pi_{W_i}C'K^*f, \Theta_i\pi_{W_i}CK^*f\rangle\Big)\\ &\leq \Big(\sum_{i\in\mathbb{I}} v_i^2\langle \Lambda_i\pi_{W_i}C'f, \Lambda_i\pi_{W_i}Cf\rangle\Big)B_\Theta\|K^*f\|^2.\end{aligned}$$

Thus,

$$B_\Theta^{-1}\|K^*f\|^2\leq\sum_{i\in\mathbb{I}} v_i^2\langle \Lambda_i\pi_{W_i}C'f, \Lambda_i\pi_{W_i}Cf\rangle.$$

This means that $\Lambda_{CC'}$ is a CC'-KGF for H. Since $T_\Lambda T_\Theta^*=K$, then similarly $\Theta_{CC'}$ is a CC'-K^*GF with the lower bound B_Λ^{-1}. □

Theorem 2.6. *Let $\Lambda:=(W_j,\Lambda_j,v_j)$ be a K-g-fusion frame for H with the frame operator S_Λ moreover, suppose Λ is a CC'-GBS with its the frame operator $S_{CC'}$. Then Λ is a Parseval CC'-KGF for H if and only if $C=(S_\Lambda^{-p})^*U$ and $C'=S_\Lambda^{-q}V$, where U,V are two operators on H such that $U^*V=KK^*$ and $p+q=1$.*

Proof. Assume that Λ is a Parseval CC'-KGF for H. It is clear that $S_{CC'} = C^* S_\Lambda C'$ and $S_{CC'} = KK^*$. Therefore, for each $p, q \in \mathbb{R}$ such that $p + q = 1$, we obtain

$$KK^* = C^* S_\Lambda^p S_\Lambda^q C'.$$

We define $U := (S_\Lambda^p)^* C$ and $V := S_\Lambda^q C'$. So,

$$U^* V = C^* S_\Lambda^p S_\Lambda^q C' = KK^*.$$

Conversely, let U, V be two operators on H such that $U^* V = KK^*$. Let $C^* := U^* S_\Lambda^{-p}$ and $C' := S_\Lambda^{-q} V$ be two operators on H where $p, q \in R$ and $p + q = 1$. Since,

$$KK^* = U^* V = C^* S_\Lambda^p S_\Lambda^q C' = C^* S_\Lambda C' = S_{CC'},$$

for each $f \in H$, we have

$$\|K^* f\|^2 = \langle KK^* f, f\rangle = \sum_{i \in \mathbb{I}} v_i^2 \langle C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C' f, f\rangle,$$

showing that, $\Lambda_{CC'}$ is a Parseval CC'-KGF for H. □

3. Q-Duality of (C, C')-Controlled K-g-Fusion Frames

In this section, we shall define duality of (C, C')-KGF and present some properties of them.

Definition 3.1. *Let $\Lambda_{CC'} = (W_i, \Lambda_i, v_i)$ be a (C, C')-KGF for H with the synthesis operator T_Λ. A (C, C')-controlled g-fusion Bessel sequence $\Theta_{CC'} := (V_i, \Theta_i, w_i)$ is called Q-controlled dual K-g-fusion frame (or brevity Q-dual (C, C')-KGF) for $\Lambda_{CC'}$ if there exists a bounded linear operator $Q : \mathscr{K}_{\Lambda_j}^2 \longrightarrow \mathscr{K}_{\Theta_j}^2$ such that*

$$T_\Lambda Q^* T_\Theta^* = KCC'. \tag{3}$$

The following results present equivalent conditions of the duality with straightforward proofs.

Proposition 3.1. *Let $\Theta_{CC'}$ be a Q-dual (C, C')-KGF for $\Lambda_{CC'}$. The following conditions are equivalent:*

(1) $T_\Lambda Q^* T_\Theta^* = KCC'$;
(2) $T_\Theta Q T_\Lambda^* = C'^* C^* K^*$;
(3) *for each $f, f' \in H$, we have*

$$\langle KCf, C'^* f'\rangle = \langle T_\Theta^* f, Q T_\Lambda^* f'\rangle = \langle Q^* T_\Theta^* f, T_\Lambda^* f'\rangle.$$

Theorem 3.1. *If $\Theta_{CC'}$ is a Q-dual (C, C')-KGF for $\Lambda_{CC'}$ and $CC'K = KCC'$, then $\Theta_{CC'}$ is a C^2-K^*GF for H.*

Proof. Suppose that $f \in H$ and $B_{CC'}$ is an upper bound of $\Lambda_{CC'}$. Therefore,

$$\begin{aligned}
\|Kf\|^4 &= |\langle Kf, Kf\rangle|^2 \\
&= |\langle C'Kf, C^*(C^*)^{-1}(C'^*)^{-1}Kf\rangle|^2 \\
&= |\langle KCC'f, (C^*)^{-1}(C'^*)^{-1}Kf\rangle|^2 \\
&= |\langle T_\Lambda Q^* T_\Theta^* f, (C^*)^{-1}(C'^*)^{-1}Kf\rangle|^2 \\
&= |\langle T_\Theta^* f, Q T_\Lambda^* (C^*)^{-1}(C'^*)^{-1}Kf\rangle|^2 \\
&\le \|T_\Theta^* f\|^2 \|Q\|^2 \|T_\Lambda^*((C^*)^{-1}C'^*)^{-1}Kf\|^2 \\
&\le \|T_\Theta^* f\|^2 \|Q\|^2 B_{CC'} \|C^{-1}\|^2 \|C'^{-1}\|^2 \|Kf\|^2 \\
&= \|Q\|^2 B_{CC'} \|C^{-1}\|^2 \|C'^{-1}\|^2 \|Kf\|^2 \sum_{i \in I} w_i^2 \langle \Theta_i \pi_{V_i} Cf, \Theta_i \pi_{V_i} Cf\rangle,
\end{aligned}$$

which completes the proof. □

Corollary 3.1. *Assume C_{op} and D_{op} are the optimal bounds of $\Theta_{CC'}$. Then*

$$C_{op} \geq B_{op}^{-1}\|Q\|^{-2}\|C'^{-1}\|^{-2}\|C^{-1}\|^{-2} \quad \text{and} \quad D_{op} \geq A_{op}^{-1}\|Q\|^{-2}\|C'^{-1}\|^{-2}\|C^{-1}\|^{-2},$$

where A_{op} and B_{op} are the optimal bounds of $\Lambda_{CC'}$.

Suppose that $\Lambda_{CC'}$ is a (C,C')-KGF for H. Since $S_{CC'} \geq A_{CC'}KK^*$, by Lemma (1.3) there exists an operator $U \in \mathcal{B}\big(H, \mathscr{K}^2_{\Lambda_j}\big)$ such that

$$T_{CC'}U = K. \tag{4}$$

Now, we define the j-th component of Uf by $U_jf = (Uf)_j$ for each $f \in H$. It is clear that $U_j \in \mathcal{B}(H, C^*(W_i))$. By this operator, we may construct a some Q-dual (C,C')-KGF for $\Lambda_{CC'}$.

Theorem 3.2. *Let $\Lambda_{CC'}$ be a (C,C')-KGF for H and $K \in \mathcal{GL}(H)$. If U is an operator as in (4) and $\widetilde{W}_i := C^*U_i^*C^*W_i$ is such that $\Theta_C := (\widetilde{W}_i, \Lambda_i, v_i)$ is a (C,C')-KGF for H, then $\Theta_C := (\widetilde{W}_i, \Lambda_i, v_i)$ is a Q-dual (C,C')-KGF for $\Lambda_{CC'}$.*

Proof. Define the mapping

$$\Psi_0 : \mathcal{R}(T^*_\Theta) \to \mathscr{K}^2_{\Lambda_j},$$
$$\Psi_0(T^*_\Theta f) = UCC'f.$$

Then Ψ_0 is well-defined, since $T^*_\Theta f$ is injective because $K \in \mathcal{GL}(H)$. Moreover,

$$\begin{aligned}
\|\Psi_0\| &= \sup_{f\neq 0} \frac{\|\Psi_0 T^*_\Theta f\|}{\|T^*_\Theta f\|} \\
&\leq \sup_{f\neq 0} \frac{\|UCC'f\|}{\sqrt{A_\Theta}\|K^*f\|} \\
&\leq \sup_{f\neq 0} \frac{\|U\|\|C\|\|C'\|\|f\|}{\sqrt{A_\Theta}\|K^{-1}\|^{-1}\|f\|} \\
&= \frac{\|U\|\|C\|\|C'\|}{\sqrt{A_\Theta}\|K^{-1}\|^{-1}} < \infty,
\end{aligned}$$

where A_Θ is a lower frame bound of $\Theta_{CC'}$. Therefore, Ψ_0 is a bounded operator. So, it has a unique linear extension (also denoted by Ψ_0) to $\overline{\mathcal{R}(T^*_\Theta)}$. Define

$$\Psi = \begin{cases} \Psi_0, & \text{on } \overline{\mathcal{R}(T^*_\Theta)}, \\ 0, & \text{on } \overline{\mathcal{R}(T^*_\Theta)}^\perp \end{cases}$$

and let $Q := \Psi^*$. This implies that $Q^* \in \mathcal{B}(\mathscr{K}^2_\Theta, \mathscr{K}^2_{\Lambda_j})$ and

$$T_{CC'}Q^*T^*_\Theta = T_{CC'}\Psi T^*_\Theta = T_{CC'}UCC' = KCC'.$$

□

Theorem 3.3. *Let $\Lambda_{CC'K}$ be a CC'-KGF with optimal bounds of A_{op} and B_{op}, and K with closed range. Then*

$$B_{op} = \|S_{CC'}\| = \|T_{CC'}\|^2\ , \qquad A_{op} = \|U_0\|^{-2},$$

where U_0 is the unique solution of equation (4).

Proof. By Lemma (1.3), the equation (4) has a unique solution, U_0, such that

$$\begin{aligned}\|U_0\|^2 &= \inf\{\alpha > 0 \mid KK^* \leq \alpha T_{CC'}T^*_{CC'}\}\\ &= \inf\{\alpha > 0 \mid \|K^*f\|^2 \leq \alpha\|T^*_{CC'}f\|^2\ ,\ f \in H\}.\end{aligned}$$

Now, we have

$$\begin{aligned}A_{op} &= \sup\{A > 0 \mid A\|K^*f\|^2 \leq \|T^*_{CC'}f\|^2\ ,\ f \in H\}\\ &= \Big(\inf\{\alpha > 0 \mid \|K^*f\|^2 \leq \alpha\|T^*_{CC'}f\|^2\ ,\ f \in H\}\Big)^{-1}\\ &= \|U_0\|^{-2}.\end{aligned}$$

□

4. Perturbation of controlled K-g-fusion frame

Perturbation of frames has been firstly discussed by Cazassa and Christensen in [2]. Recently, for K-g-fusion frames, Sadri et al. studied it in [12]. Now, we present some perturbation of CC'-KGF.

Theorem 4.1. *Let $\Lambda_{CC'K}$ be a CC'-KGF for H with bounds $A_{CC'}$ and $B_{CC'}$ and $\{\Theta_i \in \mathcal{B}(H, H_i)\}_{i\in\mathbb{I}}$ be a sequence of operators such that for each $f \in H$ and $i \in \mathbb{I}$,*

$$\begin{aligned}\|v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C' - C^*\pi_{W_i}\Theta_i^*\Theta_i\pi_{W_i}C')^{\frac{1}{2}}f\| &\leq \lambda_1\|v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\|+\\ &+ \lambda_2\|v_i(C^*\pi_{W_i}\Theta_i^*\Theta_i\pi_{W_i}C')^{\frac{1}{2}}f\| + c_i\|K^*f\|,\end{aligned}$$

where, $\{c_i\}_{i\in\mathbb{I}}$ is a sequence of positive numbers such that $\beta := \sum_{i\in\mathbb{I}} c_i^2 < \infty$ and $0 \leq \lambda_1, \lambda_2 < 1$. Then $\Theta_{CC'}$ is a CC'-KGF for H with bounds:

$$\Big(\frac{(1-\lambda_1)\sqrt{A_{CC'}} - \beta}{1+\lambda_2}\Big)^2 \quad , \quad \Big(\frac{(1+\lambda_1)\sqrt{B_{CC'}} + \beta\|K\|}{1-\lambda_2}\Big)^2.$$

Proof. Let $f \in H$ be arbitrary. We have

$$\begin{aligned}\|v_i(C^*\pi_{W_i}\Theta_i^*\Theta_i\pi_{W_i}C')^{\frac{1}{2}}f\| &= \|v_i(C^*\pi_{W_i}\Theta_i^*\Theta_i\pi_{W_i}C' - C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\\ &\quad + v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\|\\ &\leq \|v_i(C^*\pi_{W_i}\Theta_i^*\Theta_i\pi_{W_i}C' - C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\|\\ &\quad + \|v_i(C^*\pi_{w_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\|\\ &\leq \lambda_1\|v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C'f\| + \lambda_2\|v_i(C^*\pi_{w_i}\Theta_i^*\Theta_i\pi_{\widetilde{W}_i}C')^{\frac{1}{2}}f\|\\ &\quad + c_i\|K^*f\| + \|v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\|.\end{aligned}$$

Hence,

$$(1-\lambda_2)\|(v_i(C^*\pi_{W_i}\Theta_i^*\Theta_i\pi_{W_i}C')^{\frac{1}{2}}f\| \leq (1+\lambda_1)\|v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{w_i}C')^{\frac{1}{2}}f\| + c_i\|K^*f\|.$$

Since $\Lambda_{CC'K}$ is a CC'-KGF, it follows that

$$\begin{aligned}\|T^*_{CC'}f\|^2 &= \|v_i(C^*\pi_{w_i}\Lambda_i^*\Lambda_i\pi_{w_i}c')^{\frac{1}{2}}f\|^2\\ &= \sum_{i\in\mathbb{I}} v_i^2\langle\Lambda_i\pi_{W_i}C'f, \Lambda_i\pi_{W_i}Cf\rangle\\ &\leq B_{CC'}\|f\|^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\|v_i(C^*\pi_{W_i}\Theta_i^*\Theta_i\pi_{W_i}C')^{\frac{1}{2}}f\| &\leq \frac{(1+\lambda_1)\|v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\|+c_i\|K^*f\|}{1-\lambda_2}\\ &\leq \Big(\frac{(1+\lambda_1)\sqrt{B_{CC'}}+\beta\|K\|}{1-\lambda_2}\Big)\|f\|.\end{aligned}$$

Now, for the lower bound, we get

$$\begin{aligned}\|v_i(C^*\pi_{W_i}\Theta_i^*\Theta_i\pi_{W_i}C')^{\frac{1}{2}}f\| &= \|v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f - v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C'\\ &\quad - C^*\pi_{W_i}\Theta_i^*\Theta_i\pi_{W_i}C')^{\frac{1}{2}}f\|\\ &\geq \|v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\|\\ &\quad - \|v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C' - C^*\pi_{W_i}\Theta_i^*\Theta_i\pi_{W_i}C')^{\frac{1}{2}}f\|\\ &\geq \|v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\| - \lambda_1\|v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\|\\ &\quad - \lambda_2\|v_i(C^*\pi_{W_i}\Theta_i^*\Theta_i\pi_{W_i}C')^{\frac{1}{2}}f\| - c_i\|K^*f\|.\end{aligned}$$

Therefore

$$\begin{aligned}&(1+\lambda_2)\|v_i(C^*\pi_{W_i}\Theta_i^*\Theta_i\pi_{W_i}C')^{\frac{1}{2}}f\|\\ &\quad\geq (1-\lambda_1)\|v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\| - c_i\|K^*f\|,\end{aligned}$$

or

$$\|v_i(C^*\pi_{W_i}\Theta_i^*\Theta_i\pi_{W_i}C')^{\frac{1}{2}}f\| \geq \frac{(1-\lambda_1)\|v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\| - c_i\|K^*f\|}{1+\lambda_2}.$$

Since,

$$\|T_{CC'}^*f\|^2 = \|v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\|^2 \geq A_{CC'}\|K^*f\|^2,$$

it follows that

$$\|v_i(C^*\pi_{W_i}\Theta_i^*\Theta_i\pi_{W_i}C')^{\frac{1}{2}}f\| \geq \Big(\frac{(1-\lambda_1)\sqrt{A_{CC'}}-\beta}{1+\lambda_2}\Big)\|K^*f\|.$$

This completes the proof. □

Proposition 4.1. *Let $\Lambda_{CC'K}$ be a CC'-KGF for H with bounds $A_{CC'}$ and $B_{CC'}$ and $\{\Theta_i \in \mathcal{B}(H,H_i)\}_{i\in\mathbb{I}}$ be a sequence of operators such that for each $f \in H$ and $i \in \mathbb{I}$,*

$$\|v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C' - C^*\pi_{W_i}\Theta_i^*\Theta_i\pi_{W_i}C')^{\frac{1}{2}}f\| \leq c_i\|K^*f\|,$$

where, $\{c_i\}_{i\in\mathbb{I}}$ is a sequence of positive numbers such that $\beta := \sum_{i\in\mathbb{I}} c_i^2 < \infty$. Then $\Theta_{CC'}$ is a CC'-KGF for H with bounds:

$$(\sqrt{A_{CC'}}-\beta))^2 \quad , \quad (\beta\|K\|+\sqrt{B_{CC'}})^2.$$

Proof. For$f \in H$ we have

$$\begin{aligned}\|v_i(C^*\pi_{W_i}\Theta_i^*\Theta_i\pi_{W_i}C')^{\frac{1}{2}}f\| &= \|v_i(C^*\pi_{W_i}\Theta_i^*\Theta_i\pi_{W_i}C' - C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\\ &\quad + v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\|\\ &\leq \|v_i(C^*\pi_{W_i}\Theta_i^*\Theta_i\pi_{W_i}C' - C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\|\\ &\quad + \|v_i(C^*\pi_{w_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\|\\ &\leq (\beta\|K\|+\sqrt{B_{CC'}})\|f\|.\end{aligned}$$

Therefore, $\Theta_{CC'}$ is a CC'-GBS for H. On the other hand,

$$\begin{aligned}\|v_i(C^*\pi_{W_i}\Theta_i^*\Theta_i\pi_{W_i}C')^{\frac{1}{2}}f\| &\geq \|v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\| \\ &\quad - \|v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C' - C^*\pi_{W_i}\Theta_i^*\Theta_i\pi_{W_i}C')^{\frac{1}{2}}f\| \\ &\geq (\sqrt{A_{CC'}} - \beta)\|K^*f\|,\end{aligned}$$

and the proof is completed. □

Conclusions

The study of the controlled k-g-fusion frames shows that the emphasis on the Hilbert spaces introduces a new idea.
Especially, the topic of the dual of frames which is important for frame applications has been specified completely for those frames.

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