G-DUALS OF CONTINUOUS FRAMES AND THEIR PERTURBATIONS IN HILBERT SPACES

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In this article we are going to define the concept of g-dual continuous frame. We actually extend the concept of g-duals from frame to continuous frame and show some of their properties. Also a perturbation result for g-dual continuous frames is investigated.

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1. **Introduction**

Frames for Hilbert space were formally defined by Duffin and Schaeffer [4] in 1952 for studying some problems in non harmonic Fourier series. Recall that for a Hilbert space $\mathcal{H}$ and a countable index set J, a collection $\{f_j\}_{j\in J} \subset \mathcal{H}$ is called a frame for the Hilbert space $\mathcal{H}$, if there exist two positive constants c, d, such that for all $f \in \mathcal{H}$

$$c\|f\|^2 \leq \sum_{j\in J} |\langle f, f_j\rangle|^2 \leq d\|f\|^2; \tag{1}$$

c and d are called the lower and upper frame bounds, respectively. If only the right-hand inequality in (1) is satisfied, we call $\{f_j\}_{j\in J}$ a Bessel sequence for $\mathcal{H}$ with Bessel bound d. For more information about frames see [2].

Two Bessel sequences $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are said to be duals for $\mathcal{H}$ if the following equalities hold

$$f = \sum_{j\in J}\langle f, f_j\rangle g_j = \sum_{j\in J}\langle f, g_j\rangle f_j, \text{ for all f} \in \mathcal{H}.$$

Dual frames are important in reconstructing vectors (or signals) in terms of the frame elements. Dehghan and Hasankhani Fard [3] introduced and characterized g-duals of a frame in a separable Hilbert space and Ramezani and Nazari [7] extended this concept for generalized frame.

A frame $\{g_j\}_{j\in J}$ is called a g-dual frame of the frame $\{f_j\}_{j\in J}$ for $\mathcal{H}$ if there exists an invertible operator $A \in \mathcal{B}(\mathcal{H})$ such that, for all $f \in \mathcal{H}$

$$f = \sum_{j\in J}\langle Af, g_j\rangle f_j,$$

where $\mathcal{B}(\mathcal{H})$ denotes the set of all bounded operators on $\mathcal{H}$. They showed that by applying g-duals as well, one can deduce further reconstruction formulas to obtain signals. Continuous frames were proposed by G. Kaiser [6] and independently by Ali, Antoine and Gazeau [1] to a family indexed by some locally compact space endowed with a Radon measure. Gabardo and Han [5] refer to these frames as frames associated with measurable spaces.

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2. **Preliminaries**

In this section, we briefly recall some definitions and basic properties of continuous frames in Hilbert spaces. See [1, 5] for details. Throughout this paper, (Ω,μ) is a measure space.

Definition 2.1. *A weakly-measurable mapping $F:\Omega\longrightarrow\mathcal{H}$ is called a continuous frame for $\mathcal{H}$ with respect to (μ,Ω) if there are two constants $0<A\le B<\infty$ such that*

$$A\|f\|^2\le\int_\Omega|\langle f,F(\omega)\rangle|^2d\mu(\omega)\le B\|f\|^2,\ \textit{for all } f\in\mathcal{H}. \tag{2}$$

A and B are called the lower and upper continuous frame bounds, respectively. If only the right-hand inequality of (2) is satisfied, F is called a continuous Bessel frame for $\mathcal{H}$ with respect to (μ,Ω) with continuous Bessel bound B. If $A=B=\lambda$, F is called a λ-tight continuous frame. Moreover, if $\lambda=1$, F is called a Parseval continuous frame.

The synthesis operator for a continuous Bessel frame F is defined as follows

$$\langle T_F\varphi,f\rangle=\int_\Omega\varphi(\omega)\langle F(\omega),f\rangle d\mu(\omega),\qquad\qquad \left(\varphi\in L^2(\mu,\Omega),f\in\mathcal{H}\right).$$

The operator T_F is well-defined and bounded, therefore, the operator T_F^* defined as

$$T_F^*:\mathcal{H}\longrightarrow L^2(\mu,\Omega),\qquad\qquad T_F^*(f)(\omega)=\langle f,F(\omega)\rangle,$$

is the adjoint of T_F and is called the analysis operator. The bounded linear operator S_F defined by $S_F=T_FT_F^*$, that is

$$S_F:\mathcal{H}\longrightarrow\mathcal{H},\qquad \langle S_Ff,g\rangle=\int_\Omega\langle f,F(\omega)\rangle\langle F(\omega),g\rangle d\mu(\omega),\quad\text{for all }\ f,g\in\mathcal{H},$$

is called the continuous frame operator of F. Clearly, S_F is self-adjoint. Moreover S_F is invertible. Indeed, using (2) we have

$$\|I-\frac{1}{B}S_F\|=\sup_{\|f\|=1}|\langle I-\frac{1}{B}S_F,f\rangle|\le\frac{B-A}{B}<1,$$

which shows that S_F is invertible.

Definition 2.2. *Let F and G be two Bessel mappings. We call G a dual of F if the following equality holds*

$$\langle f,g\rangle=\int_\Omega\langle f,F(\omega)\rangle\langle G(\omega),g\rangle d\mu(\omega).$$

In this case, (F,G) is called a dual pair for $\mathcal{H}$. For the continuous frame F, the mapping ${S_F}^{-1}F$ is a dual of F, since

$$\begin{aligned}\langle f,g\rangle&=\langle {S_F}^{-1}S_Ff,g\rangle\\&=\langle S_Ff,{S_F}^{-1}g\rangle\\&=\int_\Omega\langle f,F(\omega)\rangle\langle F(\omega),{S_F}^{-1}g\rangle d\mu(\omega)\\&=\int_\Omega\langle f,F(\omega)\rangle\langle {S_F}^{-1}F(\omega),g\rangle d\mu(\omega).\end{aligned}$$

It is called the standard dual of F. It is certainly possible for a continuous frame F to have only one dual.

3. **G-dual continuous frame**

In this section we define the concept of g-dual continuous frame by extending the concept of g-dual from frames to continuous frames. Then we show some properties of the g-dual continuous frames.

Definition 3.1. *Let F be a continuous frame for $\mathcal{H}$. A continuous frame G is called a generalized dual continuous frame or g-dual continuous frame of F for $\mathcal{H}$ if there exists an invertible operator $A \in \mathcal{B}(\mathcal{H})$ such that for all $f, g \in \mathcal{H}$*

$$\langle f, g\rangle = \int_\Omega \langle f, F(\omega)\rangle\langle G(\omega), Ag\rangle d\mu(\omega). \tag{3}$$

When $A = I$, G is an ordinary dual continuous frame of F. If S_F is the continuous frame operator of the continuous frame F, then for all $f, g \in \mathcal{H}$ we have

$$\langle f, g\rangle = \int_\Omega \langle f, F(\omega)\rangle\langle {S_F}^{-1}F(\omega), g\rangle d\mu(\omega) = \int_\Omega \langle f, F(\omega)\rangle\langle F(\omega), {S_F}^{-1}g\rangle d\mu(\omega),$$

and hence each continuous frame is a g-dual frame for itself. The operator A in equation (3) is unique, since for all $f, g \in \mathcal{H}$

$$\langle f, A^{-1}g\rangle = \int_\Omega \langle f, F(\omega)\rangle\langle G(\omega), g\rangle d\mu(\omega),$$

and hence $A^{-1} = T_F {T_G}^*$.

The following equivalent conditions for the Bessel mappings F and G may be useful. They can be proved straightforwardly from Definition 3.1.

Lemma 3.1. *For the Bessel mappings $F : \Omega \longrightarrow \mathcal{H}$ and $G : \Omega \longrightarrow \mathcal{H}$ the following statements are equivalent:*

(i) There exists an invertible operator $A \in \mathcal{B}(\mathcal{H})$ such that

$$\langle f, g\rangle = \int_\Omega \langle f, F(\omega)\rangle\langle G(\omega), Ag\rangle d\mu(\omega);$$

(ii) There exists an invertible operator $A \in \mathcal{B}(\mathcal{H})$ such that

$$\langle f, g\rangle = \int_\Omega \langle A^* f, F(\omega)\rangle\langle G(\omega), g\rangle d\mu(\omega).$$

In case that the equivalent conditions are satisfied, F and G are g-dual continuous frames.

Proof. Let (i) be satisfied and $f, g \in \mathcal{H}$. Then there exists $h \in \mathcal{H}$, such that $g = Ah$ and $\langle f, h\rangle = \int_\Omega \langle f, F(\omega)\rangle\langle G(\omega), Ah\rangle d\mu(\omega)$, so we have

$$\begin{aligned}\langle f, g\rangle = \langle f, Ah\rangle = \langle A^* f, h\rangle &= \int_\Omega \langle A^* f, F(\omega)\rangle\langle G(\omega), Ah\rangle d\mu(\omega)\\ &= \int_\Omega \langle A^* f, F(\omega)\rangle\langle G(\omega), g\rangle d\mu(\omega),\end{aligned}$$

and hence (ii) holds. A similar argument shows that (ii) implies (i). Next, if the conditions $(i), (ii)$ are satisfied for F and G, then

$$\begin{aligned}\|f\|^4 = |\langle f, f\rangle|^2 &= |\int_\Omega \langle f, F(\omega)\rangle\langle G(\omega), Af\rangle d\mu(\omega)|^2\\ &\leq \int_\Omega |\langle f, F(\omega)\rangle|^2 d\mu(\omega) \int_\Omega |\langle G(\omega), Af\rangle|^2 d\mu(\omega)\\ &\leq D\|f\|^2\|A\|^2 \int_\Omega |\langle f, F(\omega)\rangle|^2 d\mu(\omega),\end{aligned}$$

where D is the upper continuous frame bound for G. Accordingly

$$\frac{1}{D\|A\|^2}\|f\|^2 \leq \int_\Omega |\langle f, F(\omega)\rangle|^2 d\mu(\omega),$$

which shows that F is a continuous frame. Since (i) and (ii) are equivalent, G is also a continuous frame. □

Example 3.1. *Consider $\mathcal{H} = \mathbb{R}$ and let $\Omega := [0,1]$ and $\mu := \lambda$ be the Lebesgue measure. Define $F : [0,1] \longrightarrow \mathbb{R}$ and $G : [0,1] \longrightarrow \mathbb{R}$ such that*

$$F(\omega) = \begin{cases} 2 & \omega \in [0, \frac{1}{4}) \\ 0 & \omega \in [\frac{1}{4}, 1] \end{cases},$$

and

$$G(\omega) = \begin{cases} 1 & \omega \in [0, \frac{1}{4}) \\ \frac{\sqrt{3}}{3} & \omega \in [\frac{1}{4}, 1] \end{cases}.$$

F is a Parseval continuous frame and G is a $\frac{1}{2}$-tight continuous frame for $\mathbb{R}$ with respect to $([0,1], \lambda)$. Taking $A := 2x$, A is an invertible operator on $\mathbb{R}$ and for each $x, y \in \mathbb{R}$, we have

$$\begin{aligned}\int_\Omega \langle x, F(\omega)\rangle\langle G(\omega), Ay\rangle d\lambda(\omega) =& \int_0^1 \langle x, F(\omega)\rangle\langle G(\omega), 2y\rangle d\omega \\ =& \int_0^{\frac{1}{4}} \langle x, 2\rangle\langle 1, 2y\rangle d\omega + \int_{\frac{1}{4}}^1 \langle x, 0\rangle\langle \frac{\sqrt{3}}{3}, 2y\rangle d\omega \\ =& xy + 0 \\ =& \langle x, y\rangle,\end{aligned}$$

i.e., F and G are g-dual continuous frames for $\mathbb{R}$ with the invertible operator A.

The following propositions give a method to construct new g-dual continuous frames from given g-dual continuous frames.

Proposition 3.1. *Assume that G is a g-dual continuous frame of F for $\mathcal{H}$ with the invertible operator $A \in \mathcal{B}(\mathcal{H})$ and let α be a complex number. Then the mapping K defined by $K = \alpha G + (1-\alpha)(A^{-1})^* S_F^{-1} F$ is a g-dual continuous frame of F for $\mathcal{H}$ with the invertible operator A.*

Proof. For all $f, g \in \mathcal{H}$, we have

$$\begin{aligned}&\int_\Omega \langle f, F(\omega)\rangle\langle \alpha G(\omega) + (1-\alpha)(A^{-1})^* S_F^{-1} F(\omega), Ag\rangle d\mu(\omega) \\ =&\ \alpha\int_\Omega \langle f, F(\omega)\rangle\langle G(\omega), Ag\rangle d\mu(\omega) + (1-\alpha)\int_\Omega \langle f, F(\omega)\rangle\langle S_F^{-1} F(\omega), g\rangle d\mu(\omega) \\ =&\ \alpha\langle f, g\rangle + (1-\alpha)\langle f, g\rangle = \langle f, g\rangle,\end{aligned}$$

as asserted. □

Proposition 3.2. *Assume that G and K are g-dual continuous frames for F with the invertible operators A and B, respectively. Then for any $\alpha \in \mathbb{C}$, $\alpha A^* G + (1-\alpha) B^* K$ is an ordinary dual for the continuous frame F.*

Proof. By Lemma 3.1 we have

$$\begin{aligned}\int_\Omega \langle f, F(\omega)\rangle\langle \alpha A^*G(\omega)+&(1-\alpha)B^*K(\omega), g\rangle d\mu(\omega)\\ &= \alpha\int_\Omega \langle f, F(\omega)\rangle\langle A^*G(\omega), g\rangle d\mu(\omega)\\ &+(1-\alpha)\int_\Omega \langle f, F(\omega)\rangle\langle B^*K(\omega), g\rangle d\mu(\omega)\\ &= \alpha\int_\Omega \langle f, F(\omega)\rangle\langle G(\omega), Ag\rangle d\mu(\omega)\\ &+(1-\alpha)\int_\Omega \langle f, F(\omega)\rangle\langle K(\omega), Bg\rangle d\mu(\omega)\\ &= \alpha\langle f,g\rangle + (1-\alpha)\langle f,g\rangle\\ &=\langle f,g\rangle.\end{aligned}$$

□

Proposition 3.3. *Let F be a continuous frame for $\mathcal{H}$ with the continuous frame operator S_F and let G be a g-dual continuous frame of F for $\mathcal{V}=\overline{Range\,G}$ with the invertible operator $B\in\mathcal{B}(\mathcal{V})$. Then the mapping $K=B^*G+S^{-1}F$ is a g-dual frame of F for $\mathcal{H}$.*

Proof. The operator B can be extended to the operator B_1 on $\mathcal{H}$ defined by $B_1=BP+Q$, where P and Q are the orthogonal projections onto $\mathcal{V}$ and $\mathcal{V}^\perp$, respectively, of $\mathcal{H}$. By Proposition 2.3 from [3], $B_1(\mathcal{V}^\perp)\subseteq\mathcal{V}^\perp$ and $B_1^*=B^*$. Now let $A=I-\frac{1}{2}P$, where I denotes the identity operator on $\mathcal{H}$. Since $\|I-A\|\le 1$, the operator A is invertible. Then, for $g\in\mathcal{H}$, there exist unique vectors $u\in\mathcal{V}$ and $v\in\mathcal{V}^\perp$ such that $g=u+v$. So, with $f\in\mathcal{H}$, we have

$$\begin{aligned}\int_\Omega\langle f,F(\omega)\rangle\langle K(\omega),Ag\rangle d\mu(\omega)&=\int_\Omega\langle f,F(\omega)\rangle\langle B^*G(\omega)+S^{-1}F(\omega),\frac{1}{2}u+v\rangle d\mu(\omega)\\ &=\int_\Omega\langle f,F(\omega)\rangle\langle G(\omega),B_1(v)\rangle d\mu(\omega)\\ &+\int_\Omega\langle f,F(\omega)\rangle\langle G(\omega),B(\frac{1}{2}u)\rangle d\mu(\omega)\\ &+\int_\Omega\langle f,F(\omega)\rangle\langle S^{-1}F(\omega),\frac{1}{2}u+v\rangle d\mu(\omega)\\ &=\langle f,0+\frac{1}{2}u+\frac{1}{2}u+v\rangle\\ &=\langle f,g\rangle,\end{aligned}$$

and this marks the end of the proof. □

Corollary 3.1. *Let F be a continuous frame for $\mathcal{H}$ with the continuous frame operator S_F and let G be a dual continuous frame of F for $\mathcal{V}=\overline{Range\,G}$. Then the mapping $K=G+S^{-1}F$ is a g-dual continuous frame of F.*

Example 3.2. *Let $G_1,G_2,G_3,\cdots G_m$ be m dual continuous frames of F in $\mathcal{H}$, so there are g-dual continuous frames with the invertible operator I and put $K:=\sum_{i=1}^m G_i$. Then*

$Af=\frac{1}{m}f$ defines a bounded invertible operator on $\mathcal{H}$ and

$$\begin{aligned}\int_\Omega \langle f, F(\omega)\rangle\langle K(\omega), Ag\rangle d\mu(\omega) &= \int_\Omega \langle f, F(\omega)\rangle\langle \sum_{i=1}^m G_i(\omega), \frac{1}{m}g\rangle d\mu(\omega)\\ &= \frac{1}{m}(m\langle f,g\rangle)\\ &=\langle f,g\rangle,\end{aligned}$$

and therefore K is a g-dual continuous frame for F with the invertible operator $Af=\frac{1}{m}f$.

This example shows that the sum of many dual continuous frames can be a g-dual continuous frame. The following proposition states that the sum of some two g-dual continuous frames is a g-dual continuous frame.

Proposition 3.4. *Let G and K be two g-dual continuous frames of F with corresponding invertible operators A and B, respectively. If $A^{-1}+B^{-1}$ is an invertible operator, then $G+K$ is a g-dual continuous frame for F.*

Proof. Let $T\in\mathcal{B}(\mathcal{H})$ be the inverse operator of $A^{-1}+B^{-1}$. We have

$$\begin{aligned}\int_\Omega \langle f, F(\omega)\rangle\langle (G+K)(\omega), Tg\rangle d\mu(\omega) &= \int_\Omega \langle f, F(\omega)\rangle\langle G(\omega), Tg\rangle d\mu(\omega)\\ &+\int_\Omega \langle f, F(\omega)\rangle\langle K(\omega), Tg\rangle d\mu(\omega)\\ &=\langle f, A^{-1}Tg\rangle+\langle f, B^{-1}Tg\rangle\\ &=\langle f,(A^{-1}+B^{-1})Tg\rangle\\ &=\langle f,g\rangle,\end{aligned}$$

for all $f,g\in\mathcal{H}$. □

Proposition 3.5. *Let F, $G:\Omega\longrightarrow\mathcal{H}$ and let $U,V\in\mathcal{B}(\mathcal{H})$ be two invertible operators on $\mathcal{H}$. Then F and G are g-dual continuous frames for $\mathcal{H}$ if and only if UF and VG are g-dual continuous frames for $\mathcal{H}$.*

Proof. Let F and G be g-dual continuous frames for $\mathcal{H}$. Then there exists an invertible operator $A\in\mathcal{B}(\mathcal{H})$ such that $\langle f,g\rangle=\int_\Omega\langle f,F(\omega)\rangle\langle G(\omega),Ag\rangle d\mu(\omega)$ for all $f,g\in\mathcal{H}$ and hence

$$\begin{aligned}\langle f,g\rangle=\langle f, UU^{-1}g\rangle &=\langle U^*f, U^{-1}g\rangle\\ &=\int_\Omega\langle U^*f, F(\omega)\rangle\langle G(\omega), AU^{-1}g\rangle d\mu(\omega)\\ &=\int_\Omega\langle f, UF(\omega)\rangle\langle VG(\omega), (V^{-1})^*AU^{-1}g\rangle d\mu(\omega).\end{aligned}$$

This shows that UF and VG are g-dual continuous frames for $\mathcal{H}$ with the invertible operator $(V^{-1})^*AU^{-1}\in\mathcal{B}(\mathcal{H})$. The converse is obtained by applying the operators U^{-1} and V^{-1} to the g-dual continuous frames UF and VG. □

4. Perturbations of g-dual continuous frames

The stability of frames is of great importance in frame theory, and it is studied widely by many authors [2, 8]. In this section we show that, under some conditions, approximately g-dual continuous frames are stable under some perturbations. Moreover, a perturbation result for g-dual continuous frames is investigated.

Theorem 4.1. *Let G be a g-dual continuous frame of K for $\mathcal{H}$ with the invertible operator $A \in \mathcal{B}(\mathcal{H})$ and let $F : \Omega \longrightarrow \mathcal{H}$ be a μ-measurable mapping. Assume that there exist constants $\lambda, \gamma \geq 0$, such that*

$$\int_\Omega \varphi\langle f, (K - F)(\omega)\rangle d\mu(\omega) \leq \lambda\left| \int_\Omega \varphi\langle f, K(\omega)\rangle d\mu(\omega)\right| + \gamma \int_\Omega |\varphi(\omega)|^2 d\mu(\omega),$$

for all $\varphi \in \mathcal{L}^2(\Omega, \mu)$.
(1) If $\lambda + \gamma C\|A\| < 1$, where C is an upper continuous frame bound for G, then F and G are g-dual continuous frames for $\mathcal{H}$.
(2) If $\lambda + \gamma D\|S_K^{-1}\| < 1$, where D is an upper continuous frame bound for K, then F and K are g-dual continuous frames for $\mathcal{H}$.

Proof.

$$\begin{aligned}
\left|\langle f, f\rangle - \int_\Omega \langle f, F(\omega)\rangle\langle G(\omega), Af\rangle d\mu(\omega)\right| =& \Big| \int_\Omega \langle f, K(\omega)\rangle\langle G(\omega), Af\rangle d\mu(\omega) \\
&- \int_\Omega \langle f, F(\omega)\rangle\langle G(\omega), Af\rangle d\mu(\omega)\Big| \\
=&\left|\int_\Omega \langle f, (K - F)(\omega)\rangle\langle G(\omega), Af\rangle d\mu(\omega)\right| \\
\leq&\lambda\left| \int_\Omega \langle G(\omega), Af\rangle\langle f, K(\omega)\rangle d\mu(\omega)\right| \\
&+\gamma \int_\Omega |\langle G(\omega), Af\rangle|^2 d\mu(\omega) \\
\leq&\lambda\langle f, f\rangle + \gamma C\langle Af, Af\rangle \\
=&(\lambda + \gamma C\|A\|^2)\|f\|^2 \\
\leq&\|f\|^2.
\end{aligned}$$

So $\|I - A^*T_G {T_F}^*\| \leq 1$ and therefore $T_G {T_F}^*$ is invertible and we have

$$\begin{aligned}
\langle f, g\rangle =&\langle f, T_G {T_F}^*(T_G {T_F}^*)^{-1} g\rangle \\
=&\langle T_F {T_G}^* f, (T_G {T_F}^*)^{-1} g\rangle \\
=& \int_\Omega \langle f, G(\omega)\rangle\langle F(\omega), (T_G {T_F}^*)^{-1} g\rangle d\mu(\omega).
\end{aligned}$$

Hence F and G are g-dual continuous frames for $\mathcal{H}$. If $\lambda + \gamma D\|S_K^{-1}\| < 1$, then by using a similar argument we can show that

$$\left|\langle f, f\rangle - \int_\Omega \langle f, F(\omega)\rangle\langle G(\omega), S^{-1} f\rangle d\mu(\omega)\right| \leq \|f\|^2,$$

and hence F and K are g-dual continuous frames for $\mathcal{H}$. i.e., (2) holds. □

Corollary 4.1. *Let G and K be g-dual continuous frames for $\mathcal{H}$ and let $F : \Omega \longrightarrow \mathcal{H}$ be measurable. If there exists a constant $\lambda \in [0, 1)$, such that*

$$\int_\Omega \varphi\langle f, (K - F)(\omega)\rangle d\mu(\omega) \leq \lambda\left| \int_\Omega \varphi\langle f, K(\omega)\rangle d\mu(\omega)\right|,$$

for all $\varphi \in \mathcal{L}^2(\Omega, \mu)$, then F and G are g-dual continuous frames for $\mathcal{H}$. The same is true for F and K.

Corollary 4.2. *Let G be a continuous frame for $\mathcal{H}$ with the upper continuous frame bound C. Given $F : \Omega \longrightarrow \mathcal{H}$, assume that there exist constants $\lambda, \gamma \geq 0$, such that*

$$\int_\Omega \varphi\langle f, (F-G)(\omega)\rangle d\mu(\omega) \leq \lambda \left| \int_\Omega \varphi\langle f, G(\omega)\rangle d\mu(\omega)\right| + \gamma \int_\Omega |\varphi(\omega)|^2 d\mu(\omega),$$

for all $\varphi \in \mathcal{L}^2(\Omega, \mu)$. If $\lambda + \gamma C\|S_K^{-1}\| < 1$, then F and K are g-dual continuous frames for $\mathcal{H}$.

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