INERTIAL BREGMAN EXTRAGRADIENT-LIKE ALGORITHMS FOR VARIATIONAL INEQUALITIES AND FIXED POINT PROBLEMS IN BANACH SPACES

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In this paper, we design an inertial Bregman extragradient-like algorithm with linesearch process for solving two pseudomonotone variational inequalities (VIPs) and the common fixed point problem (CFPP) of a Bregman relatively asymptotically nonexpansive mapping and finitely many Bregman relatively nonexpansive mappings in p-uniformly convex and uniformly smooth Banach spaces, which are more general than Hilbert spaces. Under mild conditions, we prove weak convergence of the suggested algorithm to a common solution of two pseudomonotone VIPs and the CFPP.

Keywords: inertial Bregman extragradient-like method, variational inequality, fixed point, Bregman projection.

MSC2020: 47H05, 47H10, 65K15, 65Y05, 68W25.

1. Introduction

Let C be a nonempty, convex and closed subset of a real Hilbert space H. Let $A : H \to H$ be an operator. Recall that the variational inequality problem (VIP) is to find a point $t \in C$ such that

$$\langle At, v - t\rangle \geq 0, \ \forall v \in C. \tag{1}$$

The solution set of the VIP is expressed as $\mathrm{VI}(C, A)$. The variational inequality problem (VIP) has been and will continue to be one of the important problems in optimization and nonlinear analysis. It contains, as special cases, such well-known problems in mathematical programming as: systems of nonlinear equations, optimization problems, complementarity problems, fixed point problems and so on. Some related works, please refer to [8–10, 12, 15, 18, 19, 25, 27–30, 32, 34–38, 42–44]. In the past few decades, the Korpelevich extragradient rule ([16]) put forward in 1976 is one of the most popular approaches for approximating an element of $\mathrm{VI}(C, A)$. Korpelevich extragradient method and its variant have been investigated extensively in the literature, see [2–6, 11, 14, 17, 24, 31, 39–41, 45]. Especially, Kraikaew and Saejung [17] proposed the Halpern subgradient extragradient algorithm for solving the VIP. Thong and Hieu [31] put forth the inertial subgradient extragradient algorithm for solving VIP. Ceng and Shang [5] introduced a hybrid inertial extragradient algorithm with linesearch process for solving the VIP with Lipschitz continuous pseudomonotone mapping A and the common fixed-point problem (CFPP) of asymptotically nonexpansive mapping S and finitely many nonexpansive mappings $\{S_i\}_{i=1}^N$ in H. Reich et al. [24] put forward two gradient-projection algorithms for solving the VIP for uniformly continuous

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pseudomonotone mapping. Eskandani et al. [11] proposed the hybrid projection rule with line-search process for finding a common solution of the VIP and the FPP of S.

In this paper, we use the VIP and CFPP to represent a variational inequality problem and the common fixed point problem (CFPP) of a Bregman relatively asymptotically nonexpansive mapping and finitely many Bregman relatively nonexpansive mappings in p-uniformly convex and uniformly smooth Banach space E. We design an inertial Bregman extragradient-like algorithm with linesearch process for solving the two pseudomonotone VIPs and the CFPP in E. Under mild conditions, we prove weak convergence of the suggested algorithm to a common solution of the two pseudomonotone VIPs and the CFPP. Our results improve and extend the corresponding results announced by some others, e.g., Ceng and Shang [5], Eskandani et al. [11] and Reich et al. [24].

2. **Preliminaries**

Let $(E, \|\cdot\|)$ be a real Banach space with the dual E^*. Let $p, q \in (1, \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$. The duality mapping $J_E^p : E \to E^*$ is defined by

$$J_E^p(t) = \{\psi \in E^* : \langle \psi, t \rangle = \|t\|^p \text{ and } \|\psi\| = \|t\|^{p-1}\}, \ \forall t \in E.$$

Let $f : E \to R$ be a Gâteaux differentiable convex function. The Bregman distance w.r.t. f is formulated as $D_f(t, x) := f(t) - f(x) - \langle \nabla f(x), t - x \rangle$, $\forall t, x \in E$. Note that

$$D_f(t, x) + D_f(x, y) = D_f(t, y) - \langle \nabla f(x) - \nabla f(y), t - x \rangle,$$

and

$$D_{f_p}(t, x) = (\|x\|^p - \|t\|^p)/q - \langle J_E^p(x) - J_E^p(t), t \rangle.$$

See [22] for more details on Bregman functions and distances.

In the smooth and p-uniformly convex Banach space E with $2 \le p < \infty$, there holds the following relation between the metric and Bregman distance:

$$\tau \|t - x\|^p \le D_{f_p}(t, x) \le \langle J_E^p(t) - J_E^p(x), t - x \rangle, \tag{2}$$

where $\tau > 0$ is some fixed number ([26]). From (2) it can be readily seen that for any bounded sequence $\{t_n\} \subset E$, the following holds:

$$t_n \to t \ \Leftrightarrow \ D_{f_p}(t_n, t) \to 0, \ n \to \infty.$$

Let C be a nonempty closed convex subset of reflexive, smooth and strictly convex Banach space E. The Bregman projection of $t \in E$ onto C w.r.t. f_p is the unique element $\Pi_C t \in C$ s.t. $D_{f_p}(\Pi_C t, t) = \min_{x \in C} D_{f_p}(x, t)$. Bregman projections can be characterized by the following inequality:

$$\langle J_E^p(t) - J_E^p(\Pi_C t), x - \Pi_C t \rangle \le 0, \ \forall x \in C, \tag{3}$$

which is equivalent to

$$D_{f_p}(x, \Pi_C t) + D_{f_p}(\Pi_C t, t) \le D_{f_p}(x, t), \quad \forall x \in C. \tag{4}$$

In case $p = 2$, the duality mapping J_E^p reduces to the normalized duality mapping and is denoted by J. The function $\phi : E^2 \to R$ is specified as $\phi(t, x) = \|t\|^2 - 2\langle Jx, t \rangle + \|x\|^2$, $\forall t, x \in E$, and $\Pi_C(t) = \arg\min_{x \in C} \phi(x, t)$, $\forall t \in E$. In terms of [11], the function $V_{f_p} : E \times E^* \to [0, \infty)$ associated with f_p is specified below

$$V_{f_p}(t, t^*) = \|t\|^p/p - \langle t^*, t \rangle + \|t^*\|^q/q, \quad \forall (t, t^*) \in E \times E^*. \tag{5}$$

So, $V_{f_p}(t, t^*) = D_{f_p}(t, J_{E^*}^q(t^*))$ $\forall (t, t^*) \in E \times E^*$. Moreover, by the subdifferential inequality, we obtain

$$V_{f_p}(t, t^*) + \langle x^*, J_{E^*}^q(t^*) - t \rangle \le V_{f_p}(t, t^* + x^*), \quad \forall t \in E, \ t^*, x^* \in E^*. \tag{6}$$

In addition, V_{f_p} is convex in the second variable. Thus one has

$$D_{f_p}(z, J^q_{E^*}(\sum_{i=1}^{n} \varsigma_i J^p_E(t_i))) \leq \sum_{i=1}^{n} \varsigma_i D_{f_p}(z, t_i),\ \forall z \in E, \{t_i\}_{i=1}^n \subset E, \tag{7}$$

where $\{\varsigma_i\}_{i=1}^n \subset [0,1]$ and $\sum_{i=1}^n \varsigma_i = 1$.

Lemma 2.1 ([1]). *Let E be a uniformly convex Banach space and $\{s_n\}, \{x_n\}$ be two sequences in E such that the first one is bounded. If $\lim_{n\to\infty} D_{f_p}(x_n, s_n) = 0$, then $\lim_{n\to\infty} \|x_n - s_n\| = 0$.*

Let $S : C \to C$ be a mapping. Let $\mathrm{Fix}(S)$ indicate the fixed-point set of S. A mapping $S : C \to C$ is referred to as being asymptotically nonexpansive if $\exists\{\theta_m\} \subset [0, \infty)$ s.t. $\lim_{m\to\infty} \theta_m = 0$ and $(\theta_m + 1)\|t - v\| \geq \|S^m t - S^m v\|,\ \forall t, v \in C,\ m \geq 1$. In particular, in the case of $\theta_m = 0,\ \forall m \geq 1$, S is called a nonexpansive mapping. A point $y^\dagger \in C$ is referred to as an asymptotic fixed point of S if $\exists\{y_n\} \subset C$ s.t. $y_n \rightharpoonup y^\dagger$ and $(I - S)y_n \to 0$. We denote by $\widehat{\mathrm{Fix}}(S)$ the set of asymptotic fixed points of S. The terminology of asymptotic fixed points was invented in Reich [23]. A mapping $S : C \to C$ is known as being Bregman relatively asymptotically nonexpansive w.r.t. f_p if $\mathrm{Fix}(S) = \widehat{\mathrm{Fix}}(S) \neq \emptyset$, and $\exists\{\theta_n\} \subset [0, \infty)$ s.t. $\lim_{n\to\infty} \theta_n = 0$ and $(\theta_n + 1)D_{f_p}(y, x) \geq D_{f_p}(y, S^n x),\ \forall y \in \mathrm{Fix}(S), x \in C,\ n \geq 1$. In particular, if $\theta_n = 0\ \forall n \geq 1$, then S reduces to a Bregman relatively nonexpansive mapping w.r.t. f_p, i.e., S is said to be Bregman relatively nonexpansive w.r.t. f_p if $\mathrm{Fix}(S) = \widehat{\mathrm{Fix}}(S) \neq \emptyset$ and $D_{f_p}(y, Sx) \leq D_{f_p}(y, x),\ \forall y \in \mathrm{Fix}(S), x \in C$. In addition, a mapping $A : C \to E^*$ is known as being

(i) monotone on C if $\langle Av - Ay, v - y\rangle \geq 0,\ \forall v, y \in C$;
(ii) pseudo-monotone if $\langle Ay, v - y\rangle \geq 0 \Rightarrow \langle Av, v - y\rangle \geq 0,\ \forall v, y \in C$;
(iii) L-Lipschitz continuous if $\exists L > 0$ s.t. $\|Av - Ay\| \leq L\|v - y\|, \forall v, y \in C$;
(iv) weakly sequentially continuous if $\forall\{t_n\} \subset C$, the relation holds: $t_n \rightharpoonup t \Rightarrow At_n \rightharpoonup At$.

Lemma 2.2 ([11]). *Let $r > 0$ be a constant and suppose that $f : E \to R$ is a uniformly convex function on bounded subsets of a Banach space E. Then*

$$f(\sum_{k=1}^{n} \alpha_k t_k) \leq \sum_{k=1}^{n} \alpha_k f(t_k) - \alpha_i \alpha_j \rho_r(\|t_i - t_j\|),$$

for all $i, j \in \{1, 2, ..., n\}$, $\{t_k\}_{k=1}^n \subset B(0, r)$ and $\{\alpha_k\}_{k=1}^n \subset (0, 1)$ with $\sum_{k=1}^n \alpha_k = 1$, where ρ_r is the gauge of uniform convexity of f.

Lemma 2.3 ([14]). *Let E_1 and E_2 be two Banach spaces. Suppose that $A : E_1 \to E_2$ is uniformly continuous on bounded subsets of E_1 and D is a bounded subset of E_1. Then $A(D)$ is bounded.*

Lemma 2.4 ([7]). *Let $\emptyset \neq C \subset E$ with C being closed and convex in a Banach space E and suppose $A : C \to E^*$ is pseudo-monotone and continuous. Then $t^\dagger \in C$ is a solution to the VIP $\langle At^\dagger, t - t^\dagger\rangle \geq 0,\ \forall t \in C$, if and only if $\langle At, t - t^\dagger\rangle \geq 0,\ \forall t \in C$.*

Lemma 2.5. *Let $2 \leq p < \infty$ and suppose that E is a smooth and p-uniformly convex Banach space with the weakly sequentially continuous duality mapping J^p_E. Let $\{t_n\} \subset E$ and $\emptyset \neq \Omega \subset E$. If $\{D_{f_p}(z, t_n)\}$ converges for each $z \in \Omega$, and $\omega_w(t_n) \subset \Omega$. Then $\{t_n\}$ converges weakly to a point in Ω.*

Proof. First, we have $\tau\|z - t_n\|^p \leq D_{f_p}(z, t_n),\ \forall z \in \Omega$ by (2). Hence we know that $\{t_n\}$ is bounded. So, from the reflexivity of E we get $\omega_w(t_n) \neq \emptyset$. Next let us show the weak convergence of $\{t_n\}$ to a point in Ω. Indeed, let $\bar{t}, \hat{t} \in \omega_w(t_n)$ with $\bar{t} \neq \hat{t}$. Then, $\exists\{t_{n_k}\} \subset \{t_n\}$ and $\exists\{t_{m_k}\} \subset \{t_n\}$ s.t. $t_{n_k} \rightharpoonup \bar{t}$ and $t_{m_k} \rightharpoonup \hat{t}$. By the weakly sequential continuity of J^p_E one obtains that $J^p_E(t_{n_k}) \rightharpoonup J^p_E\bar{t}$ and $J^p_E(t_{m_k}) \rightharpoonup J^p_E\hat{t}$. It is readily known that

$D_{f_p}(\bar{t},\hat{t})+D_{f_p}(\hat{t},t_n)=D_{f_p}(\bar{t},t_n)-\langle J_E^p\hat{t}-J_E^pt_n,\bar{t}-\hat{t}\rangle$. So, utilizing the convergence of the sequences $\{D_{f_p}(\bar{t},t_n)\}$ and $\{D_{f_p}(\hat{t},t_n)\}$, we deduce that

$$\begin{aligned}-\langle J_E^p\hat{t}-J_E^p\bar{t},\bar{t}-\hat{t}\rangle &= \lim_{n\to\infty}[D_{f_p}(\bar{t},\hat{t})+D_{f_p}(\hat{t},t_n)-D_{f_p}(\bar{t},t_n)]\\ &= \lim_{k\to\infty}[-\langle J_E^p\hat{t}-J_E^pt_{m_k},\bar{t}-\hat{t}\rangle]\\ &= -\langle J_E^p\hat{t}-J_E^p\hat{t},\bar{t}-\hat{t}\rangle=0,\end{aligned}$$

which hence yields $\langle J_E^p\bar{t}-J_E^p\hat{t},\bar{t}-\hat{t}\rangle=0$. From (2) we get $0<\tau\|\bar{t}-\hat{t}\|^p\le D_{f_p}(\bar{t},\hat{t})\le\langle J_E^p\bar{t}-J_E^p\hat{t},\bar{t}-\hat{t}\rangle=0$. This yields a contradiction. Accordingly, we get the weak convergence of $\{t_n\}$ to a point in Ω. □

Lemma 2.6 ([13]). *Let $\emptyset\ne C\subset E$ with C being closed and convex in a Banach space E. Suppose that $K:=\{x\in C:h(x)\le 0\}$ where h is a real-valued function on E. If $K\ne\emptyset$ and h is Lipschitz continuous on C with modulus $\theta>0$, then $\theta\operatorname{dist}(x,K)\ge\max\{h(x),0\}$, $\forall x\in C$, where $\operatorname{dist}(x,K)$ stands for the distance of x to K.*

Lemma 2.7 ([20]). *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that, $\exists\{\Gamma_{n_k}\}\subset\{\Gamma_n\}$ s.t. $\Gamma_{n_k}<\Gamma_{n_k+1}$, $\forall k\ge 1$. Let the sequence $\{\varphi(n)\}_{n\ge n_0}$ of integers be defined as $\varphi(n)=\max\{k\le n:\Gamma_k<\Gamma_{k+1}\}$, with integer $n_0\ge 1$ satisfying $\{k\le n_0:\Gamma_k<\Gamma_{k+1}\}\ne\emptyset$. Then, (i) $\varphi(n_0)\le\varphi(n_0+1)\le\cdots$ and $\varphi(n)\to\infty$ and (ii) $\Gamma_{\varphi(n)}\le\Gamma_{\varphi(n)+1}$ and $\Gamma_n\le\Gamma_{\varphi(n)+1}$, $\forall n\ge n_0$.*

Lemma 2.8 ([33]). *Let $\{a_n\}\subset[0,\infty)$ s.t. $a_{n+1}\le(1-\nu_n)a_n+\nu_nc_n,\forall n\ge 1$, where $\{\nu_n\}$ and $\{c_n\}$ both are real sequences, s.t. (i) $\{\nu_n\}\subset[0,1]$ and $\sum_{n=1}^\infty\nu_n=\infty$, and (ii) $\limsup_{n\to\infty}c_n\le 0$ or $\sum_{n=1}^\infty|\nu_nc_n|<\infty$. Then $\lim_{n\to\infty}a_n=0$.*

Lemma 2.9 ([21]). *Let $\{a_n\},\{b_n\}$ and $\{\mu_n\}$ be sequences of nonnegative real numbers s.t. $a_{n+1}\le(1+\mu_n)a_n+b_n$ $\forall n\ge 1$. If $\sum_{n=1}^\infty\mu_n<\infty$ and $\sum_{n=1}^\infty b_n<\infty$, then $\lim_{n\to\infty}a_n$ exists.*

3. Main results

Let E be a $p(2\le p<\infty)$-uniformly convex and uniformly smooth Banach space. Let C be a nonempty closed convex subset of E. We are now in a position to state and analyze our iterative algorithms for approximating a common solution of the two pseudomonotone VIPs and the CFPP of Bregman relatively asymptotically nonexpansive mapping and finitely many Bregman relatively nonexpansive mappings in E. Assume that the following conditions hold:

(C1): $S:C\to C$ is a uniformly continuous and Bregman relatively asymptotically nonexpansive mapping with a sequence $\{\theta_n\}$.
(C2): $S_i:C\to C$ is a uniformly continuous and Bregman relatively nonexpansive mapping for $i=1,...,N$.
(C3): For $i=1,2$, $A_i:E\to E^*$ is uniformly continuous and pseudomonotone on C, s.t. $\|A_it^\dagger\|\le\liminf_{n\to\infty}\|A_it_n\|$ $\forall\{t_n\}\subset C$ with $t_n\rightharpoonup t^\dagger$.
(C4): $S_0:=S$ and $\Omega:=(\bigcap_{i=1}^2\mathrm{VI}(C,A_i))\cap(\bigcap_{i=0}^N\mathrm{Fix}(S_i))\ne\emptyset$.

Algorithm 3.1. *Given $x_0,x_1\in C$ arbitrarily and let $\epsilon>0$, $\mu_i>0$, $\lambda_i\in(0,\frac{1}{\mu_i})$, $l_i\in(0,1)$ for $i=1,2$. Choose $\{\alpha_n\},\{\beta_n\}\subset(0,1)$ and $\{\ell_n\}\subset(0,\infty)$ s.t. $\liminf_{n\to\infty}\alpha_n(1-\alpha_n)>0$, $\liminf_{n\to\infty}\beta_n(1-\beta_n)>0$ and $\sum_{n=1}^\infty\ell_n<\infty$. Moreover, assume $\sum_{n=1}^\infty\theta_n<\infty$, and given the iterates x_{n-1} and x_n $(n\ge 1)$, choose ϵ_n s.t. $0\le\epsilon_n\le\overline{\epsilon_n}$, where*

$$\overline{\epsilon_n}=\begin{cases}\min\{\epsilon,\dfrac{\ell_n}{\|J_E^px_n-J_E^p(x_n+S_nx_n-S_nx_{n-1})\|}\}, & if\ S_nx_n\ne S_nx_{n-1},\\ \epsilon, & otherwise.\end{cases}$$

Iterative steps: Calculate x_{n+1} as follows:

Step 1. *Calculate* $g_n = J^q_{E^*}((1-\epsilon_n)J^p_E x_n + \epsilon_n J^p_E(x_n + S_n x_n - S_n x_{n-1}))$, $s_n = J^q_{E^*}(\beta_n J^p_E S_n x_n + (1-\beta_n)J^p_E g_n)$, $y_n = \Pi_C(J^q_{E^*}(J^p_E s_n - \lambda_1 A_1 s_n))$, $e_{\lambda_1}(s_n) := s_n - y_n$ *and* $t_n = s_n - \tau_n e_{\lambda_1}(s_n)$, *where* $\tau_n := l_1^{k_n}$ *and* k_n *is the smallest nonnegative integer* k *satisfying*

$$\langle A_1 s_n - A_1(s_n - l_1^k e_{\lambda_1}(s_n)), s_n - y_n\rangle \le \frac{\mu_1}{2} D_{f_p}(s_n, y_n). \tag{8}$$

Step 2. *Calculate* $v_n = \Pi_{C_n \cap Q_n}(s_n)$, *where* $Q_n := \{y \in C : D_{f_p}(y, g_n) \le D_{f_p}(y, x_n) + \epsilon_n \langle J^p_E x_n - J^p_E(x_n + S_n x_n - S_n x_{n-1}), y + S_n x_{n-1} - S_n x_n - x_n\rangle\}$, $C_n := \{y \in C : h_n(y) \le 0\}$ *and*

$$h_n(y) = \langle A_1 t_n, y - s_n\rangle + \frac{\tau_n}{2\lambda_1} D_{f_p}(s_n, y_n). \tag{9}$$

Step 3. *Calculate* $\bar{y}_n = \Pi_C(J^q_{E^*}(J^p_E v_n - \lambda_2 A_2 v_n))$, $e_{\lambda_2}(v_n) := v_n - \bar{y}_n$ *and* $\bar{t}_n = v_n - \bar{\tau}_n e_{\lambda_2}(v_n)$, *where* $\bar{\tau}_n := l_2^{j_n}$ *and* j_n *is the smallest nonnegative integer* j *satisfying*

$$\langle A_2 v_n - A_2(v_n - l_2^j e_{\lambda_2}(v_n)), v_n - \bar{y}_n\rangle \le \frac{\mu_2}{2} D_{f_p}(v_n, \bar{y}_n). \tag{10}$$

Step 4. *Calculate* $w_n = J^q_{E^*}(\alpha_n J^p_E v_n + (1-\alpha_n)J^p_E(S^n v_n))$ *and* $x_{n+1} = \Pi_{\bar{C}_n \cap \bar{Q}_n}(v_n)$, *where* $\bar{Q}_n := \{y \in C : D_{f_p}(y, w_n) \le (1+\theta_n) D_{f_p}(y, v_n)\}$, $\bar{C}_n := \{y \in C : \bar{h}_n(y) \le 0\}$ *and*

$$\bar{h}_n(y) = \langle A_2 \bar{t}_n, y - v_n\rangle + \frac{\bar{\tau}_n}{2\lambda_2} D_{f_p}(v_n, \bar{y}_n). \tag{11}$$

Set $n := n+1$ and go to Step 1.

Lemma 3.1. *Suppose that $\{x_n\}$ is the sequence constructed in Algorithm 3.1. Then the following hold:* $\frac{1}{\lambda_1} D_{f_p}(s_n, y_n) \le \langle A_1 s_n, e_{\lambda_1}(s_n)\rangle$ *and* $\frac{1}{\lambda_2} D_{f_p}(v_n, \bar{y}_n) \le \langle A_2 v_n, e_{\lambda_2}(v_n)\rangle$.

Proof. Note that the former inequality is similar to the latter. So it suffices to show that the latter holds. In fact, using the definition of $\bar{y}_n$ and properties of Π_C, one has

$$0 \ge \langle J^p_E v_n - \lambda_2 A_2 v_n - J^p_E \bar{y}_n, y - \bar{y}_n\rangle, \quad \forall y \in C.$$

Setting $y = v_n$ in the last inequality, from (2) we get

$$\lambda_2 \langle A_2 v_n, v_n - \bar{y}_n\rangle \ge \langle J^p_E v_n - J^p_E \bar{y}_n, v_n - \bar{y}_n\rangle \ge D_{f_p}(v_n, \bar{y}_n),$$

which completes the proof. □

Lemma 3.2. *The Armijo-type search rules* (8), (10) *and the sequence $\{x_n\}$ constructed in Algorithm 3.1 are well defined.*

Proof. Note that the rule (8) is similar to the one (10). So it suffices to show that the latter is valid. Using the uniform continuity of A_2 on C, from $l_2 \in (0,1)$ one gets $\lim_{j\to\infty}\langle A_2 v_n - A_2(v_n - l_2^j e_{\lambda_2}(v_n)), e_{\lambda_2}(v_n)\rangle = 0$. In the case of $e_{\lambda_2}(v_n) = 0$, it is explicit that $j_n = 0$. In the case of $e_{\lambda_2}(v_n) \ne 0$, we obtain that $\exists j_n \ge 0$ s.t. (10) holds.

It is not difficult to verify that for each $n \ge 1$, $\bar{C}_n$ and $\bar{Q}_n$ are convex and closed. Let us show that $\Omega \subset \bar{C}_n \cap \bar{Q}_n$. Take a fixed $z \in \Omega = (\bigcap_{i=1}^2 \mathrm{VI}(C, A_i)) \cap (\bigcap_{i=0}^N \mathrm{Fix}(S_i))$ arbitrarily. Using Lemma 2.2 and the Bregman relatively asymptotical nonexpansivity of S, we get

$$\begin{aligned} D_{f_p}(z, w_n) &\le \alpha_n D_{f_p}(z, v_n) + (1-\alpha_n) D_{f_p}(z, S^n v_n) - \alpha_n(1-\alpha_n)\rho^*_{b_{v_n}}\|J^P_E v_n - J^P_E S^n v_n\| \\ &\le (1+\theta_n) D_{f_p}(z, v_n) - \alpha_n(1-\alpha_n)\rho^*_{b_{v_n}}\|J^P_E v_n - J^P_E S^n v_n\| \\ &\le (1+\theta_n) D_{f_p}(z, v_n), \end{aligned}$$

which hence leads to $z \in \bar{Q}_n$. Meantime, from Lemma 2.4, we get $\langle A_2\bar{t}_n, \bar{t}_n - z\rangle \geq 0$. Thus,

$$\begin{aligned}\bar{h}_n(z) &= -\langle A_2\bar{t}_n, v_n - \bar{t}_n\rangle - \langle A_2\bar{t}_n, \bar{t}_n - z\rangle + \frac{\bar{\tau}_n}{2\lambda_2}D_{f_p}(v_n, \bar{y}_n) \\ &\leq -\bar{\tau}_n\langle A_2\bar{t}_n, e_{\lambda_2}(v_n)\rangle + \frac{\bar{\tau}_n}{2\lambda_2}D_{f_p}(v_n, \bar{y}_n).\end{aligned} \tag{12}$$

So it follows from (10) that $\frac{\mu_2}{2}D_{f_p}(v_n, \bar{y}_n) \geq \langle A_2v_n - A_2\bar{t}_n, e_{\lambda_2}(v_n)\rangle$. By Lemma 3.1, we have

$$(\frac{1}{\lambda_2} - \frac{\mu_2}{2})D_{f_p}(v_n, \bar{y}_n) \leq \langle A_2v_n, e_{\lambda_2}(v_n)\rangle - \frac{\mu_2}{2}D_{f_p}(v_n, \bar{y}_n) \leq \langle A_2\bar{t}_n, e_{\lambda_2}(v_n)\rangle,$$

which together with (12), leads to $\bar{h}_n(z) \leq -\frac{\bar{\tau}_n}{2}(\frac{1}{\lambda_2} - \mu_2)D_{f_p}(v_n, \bar{y}_n) \leq 0$. Therefore, $\Omega \subset \bar{C}_n \cap \bar{Q}_n$. As a result, the sequence $\{x_n\}$ is well defined. □

Lemma 3.3. *Suppose that $\{y_n\}$ and $\{\bar{y}_n\}$ are the sequences generated by Algorithm 3.1. If $\lim_{n\to\infty}\|s_n - y_n\| = 0$ and $\lim_{n\to\infty}\|v_n - \bar{y}_n\| = 0$, then $\omega_w(s_n) \subset \mathrm{VI}(C, A_1)$ and $\omega_w(v_n) \subset \mathrm{VI}(C, A_2)$.*

Proof. Note that the former inclusion is similar to the latter. So it suffices to show that the latter is valid. In fact, take a fixed $z \in \omega_w(v_n)$ arbitrarily. Then, $\exists\{v_{n_k}\} \subset \{v_n\}$, s.t. $v_{n_k} \rightharpoonup z$ and $\lim_{n\to\infty}\|v_{n_k} - \bar{y}_{n_k}\| = 0$. Hence, it is known that $\bar{y}_{n_k} \rightharpoonup z$. Since C is both closed and convex, from $\{\bar{y}_n\} \subset C$ and $\bar{y}_{n_k} \rightharpoonup z$ we get $z \in C$.

Next, we deal with two aspects. If $A_2z = 0$, then $z \in \mathrm{VI}(C, A_2)$ because $\langle A_2z, y-z\rangle \geq 0$, $\forall y \in C$. If $A_2z \neq 0$, using the assumption on A_2, instead of the weakly sequential continuity of A_2, we get $0 < \|A_2z\| \leq \liminf_{k\to\infty}\|A_2v_{n_k}\|$. So, we might assume that $\|A_2v_{n_k}\| \neq 0$ $\forall k \geq 1$. From (3), we get $\langle J_E^p v_{n_k} - \lambda_2A_2v_{n_k} - J_E^p\bar{y}_{n_k}, y - \bar{y}_{n_k}\rangle \leq 0, \forall y \in C$ and hence

$$\frac{1}{\lambda_2}\langle J_E^p v_{n_k} - J_E^p\bar{y}_{n_k}, y - \bar{y}_{n_k}\rangle + \langle A_2v_{n_k}, \bar{y}_{n_k} - v_{n_k}\rangle \leq \langle A_2v_{n_k}, y - v_{n_k}\rangle, \quad \forall y \in C. \tag{13}$$

According to the uniform continuity of A_2, one knows that $\{A_2v_{n_k}\}$ is bounded by Lemma 2.3. Note that $\{\bar{y}_{n_k}\}$ is bounded as well. So, using the uniform continuity of J_E^p on bounded subsets of E, from (13) we have

$$\liminf_{k\to\infty}\langle A_2v_{n_k}, y - v_{n_k}\rangle \geq 0, \quad \forall y \in C. \tag{14}$$

To show that $z \in \mathrm{VI}(C, A_2)$, we now select a sequence $\{\kappa_k\} \subset (0,1)$ s.t. $\kappa_k \downarrow 0$ as $k \to \infty$. For each $k \geq 1$, we denote by m_k the smallest positive integer such that

$$\langle A_2v_{n_j}, y - v_{n_j}\rangle + \kappa_k \geq 0, \quad \forall j \geq m_k. \tag{15}$$

Because $\{\kappa_k\}$ is decreasing, it is easily known that $\{m_k\}$ is increasing. For convenience, we still denote $\{A_2v_{n_{m_k}}\}$ by $\{A_2v_{m_k}\}$. Note that $A_2v_{m_k} \neq 0$ $\forall k \geq 1$ (due to $\{A_2v_{m_k}\} \subset \{A_2v_{n_k}\}$). Then, putting $\bar{g}_{m_k} = \frac{A_2v_{m_k}}{\|A_2v_{m_k}\|^{\frac{q}{q-1}}}$, one gets $\langle A_2v_{m_k}, J_{E^*}^q\bar{g}_{m_k}\rangle = 1$, $\forall k \geq 1$. In fact, it is evident that $\langle A_2v_{m_k}, J_{E^*}^q\bar{g}_{m_k}\rangle = (\frac{1}{\|A_2v_{m_k}\|^{\frac{q}{q-1}}})^{q-1}\|A_2v_{m_k}\|^q = 1$, $\forall k \geq 1$. So, by (15) one has $\langle A_2v_{m_k}, y + \kappa_kJ_{E^*}^q\bar{g}_{m_k} - v_{m_k}\rangle \geq 0$, $\forall k \geq 1$. Again from the pseudomonotonicity of A_2 one has

$$\langle A_2(y + \kappa_kJ_{E^*}^q\bar{g}_{m_k}), y + \kappa_kJ_{E^*}^q\bar{g}_{m_k} - v_{m_k}\rangle \geq 0, \quad \forall y \in C. \tag{16}$$

Let us show that $\lim_{k\to\infty}\kappa_kJ_{E^*}^q\bar{g}_{m_k} = 0$. In fact, because $\{v_{m_k}\} \subset \{v_{n_k}\}$ and $\kappa_k \downarrow 0$ as $k \to \infty$, it follows that $0 \leq \limsup_{k\to\infty}\|\kappa_kJ_{E^*}^q\bar{g}_{m_k}\| = \limsup_{k\to\infty}\frac{\kappa_k}{\|A_2v_{m_k}\|} \leq \frac{\limsup_{k\to\infty}\kappa_k}{\liminf_{k\to\infty}\|A_2v_{n_k}\|} = 0$. Hence one gets $\kappa_kJ_{E^*}^q\bar{g}_{m_k} \to 0$ as $k \to \infty$. Thus, taking the limit as $k \to \infty$ in (3.9), by condition (C3) we have $\langle A_2y, y - z\rangle \geq 0$, $\forall y \in C$. By Lemma 2.4 one obtains $z \in \mathrm{VI}(C, A_2)$. □

Lemma 3.4. *Let* $\{y_n\}$ *and* $\{\bar{y}_n\}$ *be the sequences generated by Algorithm 3.1. Then, (i)* $\lim_{n\to\infty}\tau_n D_{f_p}(s_n, y_n) = 0 \;\Rightarrow\; \lim_{n\to\infty} D_{f_p}(s_n, y_n) = 0$ *and (ii)* $\lim_{n\to\infty}\bar{\tau}_n D_{f_p}(v_n, \bar{y}_n) = 0 \;\Rightarrow\; \lim_{n\to\infty} D_{f_p}(v_n, \bar{y}_n) = 0$.

Proof. Note that the claim (i) is similar to the one (ii). So it suffices to show that the second is valid. To verify the second, we discuss two cases. In case $\liminf_{n\to\infty}\bar{\tau}_n > 0$, we might assume that $\exists\bar{\tau} > 0$ s.t. $\bar{\tau}_n \geq \bar{\tau} > 0,\ \forall n \geq 1$, which immediately leads to

$$D_{f_p}(v_n, \bar{y}_n) = \frac{1}{\bar{\tau}_n}\bar{\tau}_n D_{f_p}(v_n, \bar{y}_n) \leq \frac{1}{\bar{\tau}} \cdot \bar{\tau}_n D_{f_p}(v_n, \bar{y}_n). \tag{17}$$

This together with $\lim_{n\to\infty}\bar{\tau}_n D_{f_p}(v_n, \bar{y}_n) = 0$, arrives at $\lim_{n\to\infty} D_{f_p}(v_n, \bar{y}_n) = 0$.

In case $\liminf_{n\to\infty}\bar{\tau}_n = 0$, we assume that $\limsup_{n\to\infty} D_{f_p}(v_n, \bar{y}_n) = \bar{a} > 0$. Then we deduce that $\exists\{m_j\} \subset \{n\}$ s.t. $\lim_{j\to\infty}\bar{\tau}_{m_j} = 0$ and $\lim_{j\to\infty} D_{f_p}(v_{m_j}, \bar{y}_{m_j}) = \bar{a} > 0$. We define $\widehat{t_{m_j}} = \frac{1}{l_2}\bar{\tau}_{m_j}\bar{y}_{m_j} + (1 - \frac{1}{l_2}\bar{\tau}_{m_j})v_{m_j},\ \forall j \geq 1$. Noticing $\lim_{j\to\infty}\bar{\tau}_{m_j} D_{f_p}(v_{m_j}, \bar{y}_{m_j}) = 0$, From (2) we get $\lim_{j\to\infty}\bar{\tau}_{m_j}\|v_{m_j} - \bar{y}_{m_j}\|^p = 0$ and hence

$$\lim_{j\to\infty}\|\widehat{t_{m_j}} - v_{m_j}\|^p = \lim_{j\to\infty}\frac{\bar{\tau}_{m_j}^{p-1}}{l_2^p} \cdot \bar{\tau}_{m_j}\|v_{m_j} - \bar{y}_{m_j}\|^p = 0. \tag{18}$$

Because A_2 is uniformly continuous on bounded subsets of C, we obtain

$$\lim_{j\to\infty}\|A_2 v_{m_j} - A_2\widehat{t_{m_j}}\| = 0. \tag{19}$$

From the step size rule (10) and the definition of $\widehat{t_{m_j}}$, it follows that

$$\langle A_2 v_{m_j} - A_2\widehat{t_{m_j}}, v_{m_j} - \bar{y}_{m_j}\rangle > \frac{\mu_2}{2} D_{f_p}(v_{m_j}, \bar{y}_{m_j}). \tag{20}$$

Now, taking the limit as $j \to \infty$, from (19) we have $\lim_{j\to\infty} D_{f_p}(v_{m_j}, \bar{y}_{m_j}) = 0$. This arrives at a contradiction. Therefore, $\lim_{n\to\infty} D_{f_p}(v_n, \bar{y}_n) = 0$. □

Now, we are ready to show the weak convergence theorem.

Theorem 3.1. *Let* E *be a* p*-uniformly convex and uniformly smooth Banach space with the weakly sequentially continuous duality mapping* J_E^p*. If* $\{x_n\}$ *is the sequence generated by Algorithm 3.1, then* $x_n \rightharpoonup z \in \Omega \Leftrightarrow \sup_{n\geq 0}\|x_n\| < \infty$ *provided* $S^{n+1}v_n - S^n v_n \to 0$.

Proof. Note that that the necessity of Theorem 3.1 is valid. So it suffices to show that the sufficiency is valid. Assume that $\sup_{n\geq 0}\|x_n\| < \infty$. Take a fixed $z \in \Omega$ arbitrarily. It is clear that $S_n x_n \neq S_n x_{n-1} \;\Leftrightarrow\; J_E^p x_n \neq J_E^p(x_n + S_n x_n - S_n x_{n-1})$. Using the definition of ϵ_n, we get $\epsilon_n\|J_E^p x_n - J_E^p(x_n + S_n x_n - S_n x_{n-1})\| \leq \ell_n,\ \forall n \geq 1$. From (2), (7) and the three point identity of D_{f_p} we get

$$\begin{aligned}
D_{f_p}(z, g_n) &\leq (1 - \epsilon_n)D_{f_p}(z, x_n) + \epsilon_n D_{f_p}(z, x_n + S_n x_n - S_n x_{n-1})\\
&\leq D_{f_p}(z, x_n) + \epsilon_n\langle J_E^p x_n - J_E^p(x_n + S_n x_n - S_n x_{n-1}), z + S_n x_{n-1} - S_n x_n - x_n\rangle\\
&\leq D_{f_p}(z, x_n) + \ell_n M,
\end{aligned}$$

where $\sup_{n\geq 1}\|z + S_n x_{n-1} - S_n x_n - x_n\| \leq M$ for some $M > 0$. By Lemma 2.2 we have

$$\begin{aligned}
D_{f_p}(z, s_n) &= V_{f_p}(z, \beta_n J_E^p S_n x_n + (1 - \beta_n)J_E^p g_n)\\
&\leq \frac{1}{p}\|z\|^p - \beta_n\langle J_E^p S_n x_n, z\rangle - (1 - \beta_n)\langle J_E^p g_n, z\rangle + \frac{\beta_n}{q}\|J_E^p S_n x_n\|^q\\
&\quad + \frac{(1 - \beta_n)}{q}\|J_E^p g_n\|^q - \beta_n(1 - \beta_n)\rho_b^*\|J_E^p S_n x_n - J_E^p g_n\|\\
&= \beta_n D_{f_p}(z, S_n x_n) + (1 - \beta_n)D_{f_p}(z, g_n) - \beta_n(1 - \beta_n)\rho_b^*\|J_E^p S_n x_n - J_E^p g_n\|\\
&\leq D_{f_p}(z, x_n) + \ell_n M - \beta_n(1 - \beta_n)\rho_b^*\|J_E^p S_n x_n - J_E^p g_n\|.
\end{aligned}$$

Noticing $v_n = \Pi_{C_n} s_n$, by (2) and (4) we get

$$\begin{aligned} D_{f_p}(z, v_n) &\leq D_{f_p}(z, s_n) - D_{f_p}(v_n, s_n) \\ &\leq D_{f_p}(z, s_n) - \tau[\mathrm{dist}(C_n, s_n)]^p. \end{aligned}$$

Because $x_{n+1} = \Pi_{\bar{C}_n \cap \bar{Q}_n} v_n$, by (2) and (4) we have

$$\begin{aligned} D_{f_p}(z, x_{n+1}) &\leq D_{f_p}(z, v_n) - D_{f_p}(x_{n+1}, v_n) \\ &\leq D_{f_p}(z, v_n) - \tau\|P_{\bar{C}_n} v_n - v_n\|^p \\ &= D_{f_p}(z, v_n) - \tau[\mathrm{dist}(\bar{C}_n, v_n)]^p. \end{aligned}$$

Combining (20) and the last two inequalities, we obtain

$$\begin{aligned} D_{f_p}(z, x_{n+1}) &\leq D_{f_p}(z, v_n) - D_{f_p}(x_{n+1}, v_n) \\ &\leq D_{f_p}(z, x_n) + \ell_n M - \beta_n(1-\beta_n)\rho_b^*\|J_E^p S_n x_n - J_E^p g_n\| \\ &\quad - \tau[\mathrm{dist}(C_n, s_n)]^p - \tau[\mathrm{dist}(\bar{C}_n, v_n)]^p, \end{aligned} \tag{21}$$

which hence leads to $D_{f_p}(z, x_{n+1}) \leq D_{f_p}(z, x_n) + \ell_n M$. Since $\sum_{n=1}^\infty \ell_n < \infty$, by Lemma 2.9 we deduce that $\lim_{n\to\infty} D_{f_p}(z, x_n)$ exists. In addition, by the boundedness of $\{x_n\}$, we conclude that $\{g_n\}, \{s_n\}, \{v_n\}, \{w_n\}, \{y_n\}, \{\bar{y}_n\}$, $\{t_n\}, \{\bar{t}_n\}, \{S_n x_n\}$ and $\{S^n v_n\}$ are also bounded. From (21) we obtain

$$\begin{aligned} D_{f_p}(v_n, s_n) + D_{f_p}(x_{n+1}, v_n) &\leq D_{f_p}(z, x_n) + \ell_n M - \beta_n(1-\beta_n)\rho_b^*\|J_E^p S_n x_n - J_E^p g_n\| \\ &\quad - D_{f_p}(z, x_{n+1}), \end{aligned}$$

which immediately yields

$$\begin{aligned} &D_{f_p}(v_n, s_n) + D_{f_p}(x_{n+1}, v_n) + \beta_n(1-\beta_n)\rho_b^*\|J_E^p S_n x_n - J_E^p g_n\| \\ &\leq D_{f_p}(z, x_n) - D_{f_p}(z, x_{n+1}) + \ell_n M. \end{aligned}$$

Since $\lim_{n\to\infty} \ell_n = 0$, $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ and $\lim_{n\to\infty} D_{f_p}(z, x_n)$ exists, it follows that $\lim_{n\to\infty} D_{f_p}(v_n, s_n) = 0$, $\lim_{n\to\infty} D_{f_p}(x_{n+1}, v_n) = 0$, and $\lim_{n\to\infty} \rho_b^*\|J_E^p S_n x_n - J_E^p g_n\| = 0$, which hence yields $\lim_{n\to\infty} \|J_E^p S_n x_n - J_E^p g_n\| = 0$. From $s_n = J_{E^*}^q(\beta_n J_E^p S_n x_n + (1-\beta_n) J_E^p g_n)$, it is readily known that $\lim_{n\to\infty} \|J_E^p s_n - J_E^p S_n x_n\| = 0$. Noticing $g_n = J_{E^*}^q((1-\epsilon_n)J_E^p x_n + \epsilon_n J_E^p(x_n + S_n x_n - S_n x_{n-1}))$, we obtain from $\lim_{n\to\infty} \ell_n = 0$ and the definition of ϵ_n that

$$\|J_E^p g_n - J_E^p x_n\| = \epsilon_n \|J_E^p(x_n + S_n x_n - S_n x_{n-1}) - J_E^p x_n\| \leq \ell_n \to 0 \quad (n \to \infty).$$

Hence, using (2) and uniform continuity of $J_{E^*}^q$ on bounded subsets of E^*, we conclude that $\lim_{n\to\infty} \|g_n - x_n\| = 0$ and

$$\lim_{n\to\infty} \|v_n - s_n\| = \lim_{n\to\infty} \|x_{n+1} - v_n\| = \lim_{n\to\infty} \|x_n - S_n x_n\| = \lim_{n\to\infty} \|s_n - x_n\| = 0. \tag{22}$$

Since $\{x_n\}$ is bounded and E is reflexive, we know that $\omega_w(x_n) \neq \emptyset$. In what follows, we claim that $\omega_w(x_n) \subset \Omega$. Take a fixed $z \in \omega_w(x_n)$ arbitrarily. Then, $\exists \{x_{n_k}\} \subset \{x_n\}$ s.t. $x_{n_k} \rightharpoonup z$. From (22) one gets $v_{n_k} \rightharpoonup z$. Since $\{A_1 t_n\}$ is bounded, we know that $\exists L_1 > 0$ s.t. $\|A_1 t_n\| \leq L_1$. This ensures that for all $x, y \in C_n$,

$$|h_n(x) - h_n(y)| = |\langle A_1 t_n, x - y\rangle| \leq \|A_1 t_n\|\|x - y\| \leq L_1\|x - y\|,$$

which implies that $h_n(y)$ is L_1-Lipschitz continuous on C_n. Using Lemma 2.6, we get

$$\mathrm{dist}(C_n, s_n) \geq \frac{1}{L_1} h_n(s_n) = \frac{\tau_n}{2\lambda_1 L_1} D_{f_p}(s_n, y_n). \tag{23}$$

Noticing $x_{n+1} \in \bar{Q}_n$, from the definition of Q_n and (21), we have

$$\begin{aligned} D_{f_p}(x_{n+1}, w_n) &\leq (1+\theta_n)[D_{f_p}(z, v_n) - D_{f_p}(z, x_{n+1})] \\ &\leq (1+\theta_n)[D_{f_p}(z, x_n) - D_{f_p}(z, x_{n+1}) + \ell_n M]. \end{aligned}$$

It follows that $\lim_{n\to\infty}\|x_{n+1}-w_n\|=0$. This together with (22), arrives at

$$\lim_{n\to\infty}\|v_n-w_n\|=0. \tag{24}$$

On the other hand, using Lemma 2.2, we get

$$\begin{aligned}
D_{f_p}(z,w_n) &= V_{f_p}(z,\alpha_n J_E^p v_n+(1-\alpha_n)J_E^p S^n v_n)\\
&\le \frac{1}{p}\|z\|^p-\alpha_n\langle J_E^p v_n,z\rangle-(1-\alpha_n)\langle J_E^p S^n v_n,z\rangle+\frac{\alpha_n}{q}\|J_E^p v_n\|^q\\
&\quad+\frac{(1-\alpha_n)}{q}\|J_E^p S^n v_n\|^q-\alpha_n(1-\alpha_n)\rho_b^*\|J_E^p v_n-J_E^p S^n v_n\|\\
&\le \alpha_n D_{f_p}(z,v_n)+(1-\alpha_n)(1+\theta_n)D_{f_p}(z,v_n)-\alpha_n(1-\alpha_n)\rho_b^*\|J_E^p v_n-J_E^p S^n v_n\|\\
&\le (1+\theta_n)D_{f_p}(z,v_n)-\alpha_n(1-\alpha_n)\rho_b^*\|J_E^p v_n-J_E^p S^n v_n\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\alpha_n(1-\alpha_n)\rho_b^*\|J_E^p v_n-J_E^p S^n v_n\| &\le (1+\theta_n)D_{f_p}(z,v_n)-D_{f_p}(z,w_n)\\
&\le \langle J_E^p w_n-J_E^p v_n,z-v_n\rangle+\theta_n D_{f_p}(z,v_n).
\end{aligned}$$

Taking the limit in the last inequality as $n\to\infty$, and using uniform continuity of J_E^p on bounded subsets of E, (24) and $\liminf_{n\to\infty}\alpha_n(1-\alpha_n)>0$, we get $\lim_{n\to\infty}\rho_b^*\|J_E^p v_n-J_E^p S^n v_n\|=0$ and hence $\lim_{n\to\infty}\|J_E^p v_n-J_E^p S^n v_n\|=0$. This together with uniform continuity of $J_{E^*}^q$ on bounded subsets of E^* implies that

$$\lim_{n\to\infty}\|v_n-S^n v_n\|=0. \tag{25}$$

Now let us show that $z\in\bigcap_{i=1}^2 \mathrm{VI}(C,A_i)$. Since $\{A_2\bar{t}_n\}$ is bounded, we know that $\exists L_2>0$ s.t. $\|A_2\bar{t}_n\|\le L_2$. This ensures that for all $x,y\in\bar{C}_n$,

$$|\bar{h}_n(x)-\bar{h}_n(y)|=|\langle A_2\bar{t}_n,x-y\rangle|\le\|A_2\bar{t}_n\|\|x-y\|\le L_2\|x-y\|,$$

which guarantees that $\bar{h}_n(y)$ is L_2-Lipschitz continuous on $\bar{C}_n$. By Lemma 2.6, we get

$$\mathrm{dist}(\bar{C}_n,v_n)\ge\frac{1}{L_2}\bar{h}_n(v_n)=\frac{\bar{\tau}_n}{2\lambda_2 L_2}D_{f_p}(v_n,\bar{y}_n). \tag{26}$$

Combining (21), (23) and (26), we have

$$\begin{aligned}
D_{f_p}(z,x_n)-D_{f_p}(z,x_{n+1})+\ell_n M &\ge D_{f_p}(z,s_n)-D_{f_p}(z,x_{n+1})\\
&\ge \tau[\frac{\tau_n}{2\lambda_1 L_1}D_{f_p}(s_n,y_n)]^p+\tau[\frac{\bar{\tau}_n}{2\lambda_2 L_2}D_{f_p}(v_n,\bar{y}_n)]^p.
\end{aligned} \tag{27}$$

Thus, $\lim_{n\to\infty}\tau_n D_{f_p}(s_n,y_n)=\lim_{n\to\infty}\bar{\tau}_n D_{f_p}(v_n,\bar{y}_n)=0$. By Lemma 3.4, we get $\lim_{n\to\infty}\|s_n-y_n\|=\lim_{n\to\infty}\|v_n-\bar{y}_n\|=0$. Besides, combining (22) and $x_{n_k}\rightharpoonup z$ guarantees that $s_{n_k}\rightharpoonup z$ and $v_{n_k}\rightharpoonup z$. By Lemma 3.3 we deduce that $z\in\omega_w(s_n)\subset\mathrm{VI}(C,A_1)$ and $z\in\omega_w(v_n)\subset\mathrm{VI}(C,A_2)$. Consequently,

$$z\in\bigcap_{i=1}^2\mathrm{VI}(C,A_i).$$

Next we claim that $z\in\bigcap_{i=0}^N\mathrm{Fix}(S_i)$ with $S_0:=S$. In fact, by (22) we immediately get $\|x_{n+1}-x_n\|\le\|x_{n+1}-v_n\|+\|v_n-s_n\|+\|s_n-x_n\|\to 0\quad(n\to\infty)$. We first claim that $\lim_{n\to\infty}\|v_n-Sv_n\|=0$ and $\lim_{n\to\infty}\|x_n-S_i x_n\|=0\ \forall i\in\{1,2,...,N\}$. We first show that $\lim_{n\to\infty}\|x_n-S_r x_n\|=0$ for $r=1,...,N$. Actually, according to the definition of S_n, we obtain that $S_n \in \{S_1, ..., S_N\}$ $\forall n \geq 1$, which hence leads to $S_{n+i} \in \{S_1, ..., S_N\}$ $\forall n \geq 1, i = 1, \cdots, N$. Note that for $i = 1, \cdots, N$,

$$\begin{aligned}\|x_n - S_{n+i}x_n\| &\leq \|x_n - x_{n+i}\| + \|x_{n+i} - S_{n+i}x_{n+i}\| + \|S_{n+i}x_{n+i} - S_{n+i}x_n\| \\ &\leq \|x_n - x_{n+i}\| + \|x_{n+i} - S_{n+i}x_{n+i}\| + \sum_{j=1}^{N}\|S_jx_{n+i} - S_jx_n\|.\end{aligned}$$

Noticing (22) and the uniform continuity of each S_j on C, we deduce that $x_{n+i}-S_{n+i}x_{n+i} \to 0$ and $S_jx_{n+i} - S_jx_n \to 0$ for $i, j = 1, \cdots, N$. Thus, we get $\lim_{n\to\infty} \|x_n - S_{n+i}x_n\| = 0$ for $i = 1, \cdots, N$. This immediately implies that $\lim_{n\to\infty} \|x_n - S_rx_n\| = 0$ for $r = 1, \cdots, N$. So it follows from $x_{n_k} \rightharpoonup z$ that $z \in \widehat{\mathrm{Fix}}(S_r) = \mathrm{Fix}(S_r)$ for $r = 1, \cdots, N$. Therefore, $z \in \bigcap_{i=1}^{N} \mathrm{Fix}(S_i)$. In addition, observe also that

$$\|v_n - Sv_n\| \leq \|v_n - S^nv_n\| + \|S^nv_n - S^{n+1}v_n\| + \|S^{n+1}v_n - Sv_n\|. \tag{28}$$

Noticing the uniform continuity of S on C, we conclude from (25) that $Sv_n - S^{n+1}v_n \to 0$. Thus, using the assumption $S^nv_n - S^{n+1}v_n \to 0$, from (28) we get $\lim_{n\to\infty} \|v_n - Sv_n\| = 0$. Again from (22) and $x_{n_k} \rightharpoonup z$, one has that $v_{n_k} \rightharpoonup z$. Hence, we obtain $z \in \widehat{\mathrm{Fix}}(S) = \mathrm{Fix}(S)$. Consequently, $z \in \bigcap_{i=0}^{N} \mathrm{Fix}(S_i)$, and hence $z \in \Omega = (\bigcap_{i=1}^{2} \mathrm{VI}(C, A_i)) \cap (\bigcap_{i=0}^{N} \mathrm{Fix}(S_i))$. This means that $\omega_w(x_n) \subset \Omega$. As a result, applying Lemma 2.5 we conclude that $x_n \rightharpoonup z$. This completes the proof. □

4. **Conclusion**

The projection method is a powerful tool for solving VIP in Hilbert spaces. In this paper, we investigate VIP in Banach spaces by using Bregman projection. We construct an inertial Bregman extragradient-like algorithm with linesearch process for solving two pseudomonotone variational inequalities and the common fixed point problem of a Bregman relatively asymptotically nonexpansive mapping and finitely many Bregman relatively nonexpansive mappings in p-uniformly convex and uniformly smooth Banach spaces, which are more general than Hilbert spaces. We demonstrate convergence analysis of the suggested algorithm to a common solution of the considered problems under standard assumptions.

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