TRANSVERSE VIBRATION OF A VISCOELASTIC EULER-BERNOULLI BEAM BASED ON EQUIVALENT VISCOELASTIC SPRING MODELS

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In this paper, the transverse vibration of the viscoelastic Euler-Bernoulli cracked beam is investigated. By Laplace transform and generalized Dirac delta functions, the equivalent stiffness of the viscoelastic cracked beam is derived with considering the transverse crack as a massless viscoelastic torsion spring. Utilizing the separation of variables method, the frequency equation of the viscoelastic cracked beam is established. By numerical examples, the effects of the crack location, crack depth, and number of cracks on the eigenfrequencies of the simple-supported viscoelastic cracked beam are discussed.

Keywords: viscoelastic; crack effect; natural frequency; decrement coefficient.

1. Introduction

Viscoelastic materials [1] are widely used in civil, mechanical, and aerospace engineering, etc. Up to now, there are a number of approaches to analyze the vibration characteristics of the viscoelastic beams reported in the literatures, i.e. complex modal approach [2], Finite element method [3], transfer matrix method [4], and et al. [5]. Supposing that the deflection mode shape of the simple-supported beam is $w(x,t)=\sin(n\pi x/L)e^{i\omega t}$, Lei et al. [5] presented the governing equations of motion for the viscoelastic Euler-Bernoulli and Timoshenko beams with the nonlocal theory models and analyzed the influences of velocity-dependent external damping on the dynamics characteristics of the beams. However, there are only a few published papers [6-7] concerned about the effects of cracks or defects on the vibration properties of the viscoelastic beams structures so far. Therefore, it is needed to discuss the vibration of a viscoelastic cracked beam.

With the standard linear solid constitutive equation, the main purpose in the present paper is to investigate the vibration properties of the viscoelastic Euler-Bernoulli cracked beam by using the exact analytical method (EAM). At first, the equivalent stiffness of the viscoelastic cracked beam is derived with

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regarding the transverse crack as a massless viscoelastic torsion spring. Then, the frequency equation of the viscoelastic cracked beam is established based on the separation of variables method and Laplace transform, and the exact analytical expressions are presented to analyze the viscoelastic cracked beam with open cracks. Finally, the effects of the crack location, crack depth, and number of cracks on the vibration properties of the viscoelastic cracked beams are numerically investigated.

2. Formulation of the problem

2.1. Equivalent bending stiffness of a viscoelastic beam

According to the constitutive equation of standard linear solid model, the relaxation modulus $Y(t)$ defined in time domain and Laplace domain are given as

$$Y(t) = q_0 + \left(\frac{q_1}{p_1} - q_0\right)\mathrm{e}^{-\frac{t}{p_1}}, \quad \overline{Y}(s) = \frac{q_0 + sq_1}{s(1+sp_1)}. \tag{1}$$

Here E_1 and E_2 are the elastic modulus of elastic elements, η_2 is the viscous coefficient of a viscous element, ν is the Poisson's ratio, and

$$p_1 = \frac{\eta_2}{E_1 + E_2}, \quad q_0 = \frac{E_1 E_2}{E_1 + E_2}, \quad q_1 = \frac{E_1 \eta_2}{E_1 + E_2}. \tag{2}$$

We consider a viscoelastic rectangular beam with length L (x axis), width b (y axis) and height h (z axis). Here $w(x,t)$ and $\varphi(x,t)$ denote the transverse deflection of the axial line and rotation angle of the beam cross section subjected to the distributed transverse load $q(x,t)$, respectively. According to the hypothesis of the Euler-Bernoulli beam theory, the axial normal strain, rotation angle, and normal stress of the cross section are given as

$$\varepsilon(x,z,t) = -y\frac{\partial\varphi(x,t)}{\partial x}, \quad \varphi(x,t) = \partial w(x,t)/\partial x, \quad \sigma(x,z,t) = Y(0)\varepsilon(x,t) + \dot{Y}(t) * \varepsilon(x,t). \tag{3}$$

Here $\dot{Y}(t)$ is the first derivative of $Y(t)$ with respect to the time t, and the asterisk * denotes the convolution, i.e. $f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau$.

The bending moment $M(x,t)$ of the beam cross section is

$$M(x,t) = -I\left[Y(0)\frac{\partial\varphi(x,t)}{\partial x} + \dot{Y}(t) * \frac{\partial\varphi(x,t)}{\partial x}\right]. \tag{4}$$

Here the moment of inertia of the neutral axis is given as $I = \iint_A y^2 \mathrm{d}y\mathrm{d}z$. Then, the Laplace transform of bending moment and axial bending curvature are given as

$$\overline{M}(x,s) = -s\overline{Y}(s)I\frac{\partial\overline{\varphi}(x,s)}{\partial x}, \quad \frac{\partial\overline{\varphi}(x,s)}{\partial x} = -\frac{\overline{M}(x,s)}{s\overline{Y}(s)I}. \tag{5}$$

Obviously, $s\overline{Y}(s)I$ is the bending stiffness of the viscoelastic intact beam in Laplace domain. The superscript $-$ denotes the Laplace transform of the function with respect to the time t, and s is the Laplace transform parameter.

In this paper, we suppose that the transverse crack $j\ (j=1,2,\cdots,N)$ is always open, which means the crack can be equivalent as a massless viscoelastic torsion spring [8]. Let us denote the bending moment and equivalent viscoelastic torsion spring of the crack j at the location $x=x_j$ by $M_j(t)$ and $k_j(t)$, respectively, and the rotation angle $\Delta_j(t)$ of the equivalent torsion spring in time domain and Laplace domain can be expressed as

$$M_j(t)=-\left[k_j(0)\Delta_j(t)+\dot{k}_j(t)*\Delta_j(t)\right],\quad \overline{\Delta}_j(s)=-\frac{\overline{M}_j(s)}{s\overline{k}_j(s)}. \tag{6}$$

Based on the crack effect and Laplace transform, the rotation angle of the cracked beam in time domain and Laplace domain can be expressed as, respectively

$$\phi(x,t)=\varphi(x,t)+\sum_{j=1}^{N}\Delta_j(t)H(x-x_j),\quad \overline{\phi}(x,s)=\overline{\varphi}(x,s)+\sum_{j=1}^{N}\overline{\Delta}_j(s)H(x-x_j). \tag{7}$$

Here $H(x)$ is the Heaviside function [9].

Denote the equivalent bending stiffness of a viscoelastic beam with open cracks by $(EI)_{\mathrm{e}}(x,t)$, the bending moment of the cracked beam in time domain and Laplace domain are given as, respectively

$$M(x,t)=-\left[(EI)_{\mathrm{e}}(x,0)\frac{\partial\phi(x,t)}{\partial x}+\dot{(EI)}_{\mathrm{e}}(x,t)*\frac{\partial\phi(x,t)}{\partial x}\right],\quad \overline{M}(x,s)=-s\overline{(EI)}_{\mathrm{e}}(x,s)\frac{\partial\overline{\phi}(x,s)}{\partial x}. \tag{8}$$

Utilizing the first derivative of the second equation of Eq. (7) with respect to the coordinate x, and then combining the second equation of Eq. (6) and Eq. (8), the equivalent bending stiffness of the viscoelastic cracked beam in Laplace domain can be written as

$$\frac{1}{\overline{(EI)}_{\mathrm{e}}(x,s)}=\frac{1}{\overline{Y}(s)I}+\sum_{j=1}^{N}\frac{1}{\overline{k}_j(s)}\delta(x-x_j). \tag{9}$$

Here $\delta(x)$ is the Dirac delta function [9].

2.2. Vibration of a viscoelastic cracked beam

According to the expression for the rectangular cross section beams by references [10-12], the equivalent stiffness of crack j ($j=1,\cdots,N$) in time domain and Laplace domain are given as, respectively,

$$k_j(t)=\mu_j IY(t),\quad \overline{k}_j(s)=\mu_j I\overline{Y}(s). \tag{10}$$

Here the parameter $\mu_j=(0.9/h)\left[(d_j/h)-1\right]^2\Big/\left\{(d_j/h)\left[2-(d_j/h)\right]\right\}$.

By substituting Eqs. (9), (10) and the second equation of Eq. (1) into the second equation of Eq. (8), and using the inverse Laplace transform,

$$\left(1+p_1\frac{\partial}{\partial t}\right)M(x,t)=-I\left[1+\sum_{j=1}^{N}\frac{1}{\mu_j}\delta(x-x_j)\right]^{-1}\left(q_0+q_1\frac{\partial}{\partial t}\right)\frac{\partial^2 w(x,t)}{\partial x^2}. \tag{11}$$

The free vibration equation of the Euler-Bernoulli beam [13] is

$$\rho A\frac{\partial^2 w(x,t)}{\partial t^2}-\frac{\partial^2 M(x,t)}{\partial x^2}=0. \tag{12}$$

Introduce the following dimensionless variables and parameters

$$\begin{cases} w^*=\dfrac{w}{L},\ \xi=\dfrac{x}{L},\ \xi_j=\dfrac{x_j}{L},\ \mu_j^*=\mu_j L,\ I^*=\dfrac{I}{L^4},\ A^*=\dfrac{A}{L^2},\ \rho^*=\dfrac{\rho L^2}{E_1T^2},\ t^*=\dfrac{t}{T}, \\ m^*=\dfrac{M}{E_1L^3},\ E_2^*=\dfrac{E_2}{E_1},\ \eta_2^*=\dfrac{\eta_2}{E_1T},\ p_1^*=\dfrac{\eta_2^*}{1+E_2^*},\ q_0^*=\dfrac{E_2^*}{1+E_2^*},\ q_1^*=\dfrac{\eta_2^*}{1+E_2^*}. \end{cases} \tag{13}$$

Combining the dimensionless forms of Eqs. (11) and (12)

$$\rho^*A^*\left(1+p_1^*\frac{\partial}{\partial t^*}\right)\frac{\partial^2 w^*(\xi,t^*)}{\partial t^{*2}}=-I^*\left(q_0^*+q_1^*\frac{\partial}{\partial t^*}\right)\frac{\partial^2}{\partial\xi^2}\left\{\left[1+\sum_{j=1}^{N}\frac{1}{\mu_j^*}\delta(\xi-\xi_j)\right]^{-1}\frac{\partial^2 w^*(\xi,t^*)}{\partial\xi^2}\right\}. \tag{14}$$

3. Solutions

Based on the separation of variables method [13], the vibration solutions can be assumed as

$$w^*(\xi,t^*)=W^*(\xi)T(t^*),\ \ m^*(\xi,t^*)=M^*(\xi)T(t^*). \tag{15}$$

Here $W^*(\xi)$ and $M^*(\xi)$ are the dimensionless mode functions of the transverse displacement and bending moment for the cracked beam, $T(t^*)$ is the function dependent with time t^*.

Eq. (14) can be rewritten as

$$\frac{\left(1+p_1^*\dfrac{\mathrm{d}}{\mathrm{d}t^*}\right)\dfrac{\mathrm{d}^2T(t^*)}{\mathrm{d}t^{*2}}}{\left(q_0^*+q_1^*\dfrac{\mathrm{d}}{\mathrm{d}t^*}\right)T(t^*)}=-\frac{I^*\dfrac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left\{\left[1+\displaystyle\sum_{j=1}^{N}\frac{1}{\mu_j^*}\delta(\xi-\xi_j)\right]^{-1}\dfrac{\mathrm{d}^2W^*(\xi)}{\mathrm{d}\xi^2}\right\}}{\rho^*A^*W^*(\xi)}. \tag{16}$$

The left side and right side of Eq. (16) are independent with the dimensionless coordinate ξ and time t^*, respectively, so the above equation is equal to a constant [13], which can be defined as $-Y^4$, and

$$\left(1+p_1^*\frac{\mathrm{d}}{\mathrm{d}t^*}\right)\frac{\mathrm{d}^2T(t^*)}{\mathrm{d}t^{*2}}=-Y^4\left(q_0^*+q_1^*\frac{\mathrm{d}}{\mathrm{d}t^*}\right)T(t^*). \tag{17}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left\{\left[1+\sum_{j=1}^{N}\frac{1}{\mu_j^*}\delta(\xi-\xi_j)\right]^{-1}\frac{\mathrm{d}^2W^*(\xi)}{\mathrm{d}\xi^2}\right\}=Y^4\frac{\rho^*A^*}{I^*}W^*(\xi). \tag{18}$$

Considering free vibration of the viscoelastic beam, the time function [4] can be expressed as

$$T(t^*)=\mathrm{e}^{\mathrm{i}\omega t^*}. \tag{19}$$

Here $\mathrm{i}=\sqrt{-1}$, ω is the complex eigenfrequency, and the real part and imaginary part of ω are the natural frequency and decrement coefficient [2,4,14], respectively.

Substituting Eqs. (15) and (19) into Eqs. (17) , (18) and the dimensionless form of Eq. (11), respectively

$$\left(1+\mathrm{i}\omega p_1^*\right)(\mathrm{i}\omega)^2=-Y^4\left(q_0^*+\mathrm{i}\omega q_1^*\right). \tag{20}$$

$$\frac{\mathrm{d}^2F^*(\xi)}{\mathrm{d}\xi^2}-\beta^4W^*(\xi)=0. \tag{21}$$

$$M^*(\xi)=-I^*\frac{q_0^*+\mathrm{i}\omega q_1^*}{1+\mathrm{i}\omega p_1^*}F^*(\xi). \tag{22}$$

Here

$$F^*(\xi)=\left[1+\sum_{j=1}^{N}\frac{1}{\mu_j^*}\delta(\xi-\xi_j)\right]^{-1}\frac{\mathrm{d}^2W^*(\xi)}{\mathrm{d}\xi^2},\quad \beta^4=Y^4\frac{\rho^*A^*}{I^*}. \tag{23}$$

By the Laplace transformation of Eq. (21) and the first equation of Eq. (23), one obtain

$$s^2\overline{F}^*(s)-sC_1-C_2=\beta^4\overline{W}^*(s). \tag{24}$$

$$\overline{F}^*(s)+\sum_{j=1}^{N}\frac{1}{\mu_j^*}F^*(\xi_j)\mathrm{e}^{-s\xi_j}=s^2\overline{W}^*(s)-sC_3-C_4. \tag{25}$$

Here $C_m(m=1,2,3,4)$ are the undetermined functions, and

$$C_1=F^*(0),\ C_2=\left.\frac{\mathrm{d}F^*(\xi)}{\mathrm{d}\xi}\right|_{\xi=0},\ C_3=W^*(0),\ C_4=\left.\frac{\mathrm{d}W^*(\xi)}{\mathrm{d}\xi}\right|_{\xi=0}. \tag{26}$$

Combining Eqs. (24) and (25), and utilizing the inverse Laplace transform, we obtain

$$W^*(\xi)=\frac{\cosh(\beta\xi)-\cos(\beta\xi)}{2\beta^2}C_1+\frac{\sinh(\beta\xi)-\sin(\beta\xi)}{2\beta^3}C_2+\frac{\cosh(\beta\xi)+\cos(\beta\xi)}{2}C_3+$$
$$\frac{\sinh(\beta\xi)+\sin(\beta\xi)}{2\beta}C_4+\sum_{j=1}^{N}\frac{F^*(\xi_j)}{\mu_j^*}\frac{\sinh\left[\beta\left(\xi-\xi_j\right)\right]+\sin\left[\beta\left(\xi-\xi_j\right)\right]}{2\beta}H\left(\xi-\xi_j\right). \tag{27}$$

$$F^*(\xi)=\frac{\cosh(\beta\xi)+\cos(\beta\xi)}{2}C_1+\frac{\sinh(\beta\xi)+\sin(\beta\xi)}{2\beta}C_2+\frac{\cosh(\beta\xi)-\cos(\beta\xi)}{2}\beta^2C_3+$$
$$\frac{\sinh(\beta\xi)-\sin(\beta\xi)}{2}\beta C_4+\beta\sum_{j=1}^{N}\frac{F^*(\xi_j)}{\mu_j^*}\frac{\sinh\left[\beta\left(\xi-\xi_j\right)\right]-\sin\left[\beta\left(\xi-\xi_j\right)\right]}{2}H\left(\xi-\xi_j\right). \quad (28)$$

If $0<\xi_1<\cdots<\xi_j<\cdots<\xi_N<1$, and $\xi=\xi_m$, Eq. (28) can be rewritten as

$$F^*(\xi_m)=\mathrm{X}_mC_1+\Pi_mC_2+\Lambda_mC_3+\Gamma_mC_4. \quad (m=1,2,3,\cdots,N) \quad (29)$$

Here

$$\begin{cases}\mathrm{X}_m=\Omega_3(\xi_m)+\beta\sum_{j=1}^{m-1}\frac{\mathrm{X}_j}{\mu_j^*}\Omega_2(\xi_m-\xi_j), & \Pi_m=\frac{\Omega_1(\xi_m)}{\beta}+\beta\sum_{j=1}^{m-1}\frac{\Pi_j}{\mu_j^*}\Omega_2(\xi_m-\xi_j),\\ \Lambda_m=\Omega_4(\xi_m)\beta^2+\beta\sum_{j=1}^{m-1}\frac{\Lambda_j}{\mu_j^*}\Omega_2(\xi_m-\xi_j), & \Gamma_m=\Omega_2(\xi_m)\beta+\beta\sum_{j=1}^{m-1}\frac{\Gamma_j}{\mu_j^*}\Omega_2(\xi_m-\xi_j).\end{cases} \quad (30)$$

$$\begin{cases}\Omega_1(\xi)=\frac{\sinh(\beta\xi)+\sin(\beta\xi)}{2}, & \Omega_2(\xi)=\frac{\sinh(\beta\xi)-\sin(\beta\xi)}{2},\\ \Omega_3(\xi)=\frac{\cosh(\beta\xi)+\cos(\beta\xi)}{2}, & \Omega_4(\xi)=\frac{\cosh(\beta\xi)-\cos(\beta\xi)}{2}.\end{cases} \quad (31)$$

Substituting Eq. (29) into Eqs. (27) and (28), respectively, the dimensionless functions of $W^*(\xi)$ and $F^*(\xi)$ are expressed as

$$F^*(\xi)=C_1\left[\Omega_3(\xi)+\beta\sum_{j=1}^{N}\frac{\mathrm{X}_j}{\mu_j^*}\Omega_2(\xi-\xi_j)H(\xi-\xi_j)\right]+C_2\left[\frac{\Omega_1(\xi)}{\beta}+\beta\sum_{j=1}^{N}\frac{\Pi_j}{\mu_j^*}\Omega_2(\xi-\xi_j)H(\xi-\xi_j)\right]+$$
$$C_3\left[\Omega_4(\xi)\beta^2+\beta\sum_{j=1}^{N}\frac{\Lambda_j}{\mu_j^*}\Omega_2(\xi-\xi_j)H(\xi-\xi_j)\right]+C_4\left[\Omega_2(\xi)\beta+\beta\sum_{j=1}^{N}\frac{\Gamma_j}{\mu_j^*}\Omega_2(\xi-\xi_j)H(\xi-\xi_j)\right]. \quad (32)$$

$$W^*(\xi)=C_1\left[\frac{\Omega_4(\xi)}{\beta^2}+\sum_{j=1}^{N}\frac{\mathrm{X}_j}{\mu_j^*}\frac{\Omega_1(\xi-\xi_j)}{\beta}H(\xi-\xi_j)\right]+C_2\left[\frac{\Omega_2(\xi)}{\beta^3}+\sum_{j=1}^{N}\frac{\Pi_j}{\mu_j^*}\frac{\Omega_1(\xi-\xi_j)}{\beta}H(\xi-\xi_j)\right]+$$
$$C_3\left[\Omega_3(\xi)+\sum_{j=1}^{N}\frac{\Lambda_j}{\mu_j^*}\frac{\Omega_1(\xi-\xi_j)}{\beta}H(\xi-\xi_j)\right]+C_4\left[\frac{\Omega_1(\xi)}{\beta}+\sum_{j=1}^{N}\frac{\Gamma_j}{\mu_j^*}\frac{\Omega_1(\xi-\xi_j)}{\beta}H(\xi-\xi_j)\right]. \quad (33)$$

Utilizing the first derivative of Eq. (33) with respect to the variable ξ ,

$$\Phi^*(\xi)=C_1\left[\frac{\Omega_1(\xi)}{\beta}+\sum_{j=1}^{N}\frac{\mathrm{X}_j}{\mu_j^*}\Omega_3(\xi-\xi_j)H(\xi-\xi_j)\right]+C_2\left[\frac{\Omega_4(\xi)}{\beta^2}+\sum_{j=1}^{N}\frac{\Pi_j}{\mu_j^*}\Omega_3(\xi-\xi_j)H(\xi-\xi_j)\right]+$$
$$C_3\left[\beta\Omega_2(\xi)+\sum_{j=1}^{N}\frac{\Lambda_j}{\mu_j^*}\Omega_3(\xi-\xi_j)H(\xi-\xi_j)\right]+C_4\left[\Omega_3(\xi)+\sum_{j=1}^{N}\frac{\Gamma_j}{\mu_j^*}\Omega_3(\xi-\xi_j)H(\xi-\xi_j)\right]. \quad (34)$$

Then, by substituting Eq. (32) into Eq. (22), and applying the first derivative with respect to the variable ξ, the dimensionless mode functions of the bending moment and shearing force can be derived (due to the space limitation, the exact expressions are not given at all).

By the boundary conditions, the set of linear equations is derived to determine the functions $\{\boldsymbol{C}\}$

$$[\boldsymbol{A}]\{\boldsymbol{C}\} = \mathbf{0}. \tag{35}$$

Here $[\boldsymbol{A}]$ is a 4×4 coefficient vector, and $\{\boldsymbol{C}\} = \{C_1, C_2, C_3, C_4\}^{\mathrm{T}}$.

If there exists a nonzero solution of $\{\boldsymbol{C}\}$, the determinant of the coefficients vector is zero, i.e.

$$\det[\boldsymbol{A}] = 0. \tag{36}$$

By utilizing Matlab programs, the complex eigenfrequency ω can be obtained with the different boundary conditions.

The dimensionless boundary conditions of a simply-supported viscoelastic beam with an arbitrary number of cracks are given as

$$W^*(0) = 0,\ \ W^*(1) = 0,\ \ M^*(0) = 0,\ \ M^*(1) = 0. \tag{37}$$

Then, one obtain

$$C_1 = C_3 = 0, \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} C_2 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{38}$$

Here

$$\begin{cases} a_{11} = \dfrac{1}{\beta}\left[\dfrac{\Omega_2(1)}{\beta^2} + \displaystyle\sum_{j=1}^{N}\dfrac{\Pi_j}{\mu_j^*}\Omega_1\left(1-\xi_j\right)\right], \ a_{12} = \dfrac{1}{\beta}\left[\Omega_1(1) + \displaystyle\sum_{j=1}^{N}\dfrac{\Gamma_j}{\mu_j^*}\Omega_1\left(1-\xi_j\right)\right], \\ a_{21} = -I^*\dfrac{q_0^* + \mathrm{i}\omega q_1^*}{1+\mathrm{i}\omega p_1^*}\beta\left[\dfrac{\Omega_1(1)}{\beta^2} + \displaystyle\sum_{j=1}^{N}\dfrac{\Pi_j}{\mu_j^*}\Omega_2\left(1-\xi_j\right)\right], \\ a_{22} = -I^*\dfrac{q_0^* + \mathrm{i}\omega q_1^*}{1+\mathrm{i}\omega p_1^*}\beta\left[\Omega_2(1) + \displaystyle\sum_{j=1}^{N}\dfrac{\Gamma_j}{\mu_j^*}\Omega_2\left(1-\xi_j\right)\right]. \end{cases} \tag{39}$$

4. Numerical results and discussion

4.1. Validation

To verify the correctness and applicability of the present exact analytical method (EAM), the numerical example for comparisons have been provided. Let $E_1 \to \infty$ and $d_1 \to 0$, the present model is degenerated into the Kelvin-Voigt intact model. Lee and Oh [3] analyzed vibration of the simple-supported Kelvin-Voigt intact beam based on the spectral finite element method. The geometric and physical parameters are $L = 1\text{ m}$, $b = 0.2\text{ m}$, $h = 0.0015\text{ m}$, $\rho = 7800\text{ kg/m}^3$, $E_2 = 2\times10^{11}\text{ N/m}^2$, $E_1/E_2 = 9999$ and $\eta_2 = 6.8\times10^{-4}E_2$. The first five eigenfrequencies are shown in table 1. It is noticed that the results of the present method are in excellent agreement with those of reference [3].

Table 1

First five eigenfrequencies of the simply-supported Kelvin-Voigt beam

	EAM	Ref.[3]
1st	3.4439+0.0253i	3.444+0.025i
2nd	13.7702+0.4054i	13.771+0.405i
3rd	30.9283+2.0523i	30.930+2.052i
4th	54.7215+6.4862i	54.724+6.486i
5th	84.6325+15.8356i	84.636+15.836i

4.2. Vibration characteristic of a viscoelastic cracked beam

For a standard linear solid beam under the simple-supported boundary conditions, we suppose that the geometric parameters are $L = 1\text{ m}$, $\rho = 500\text{ kg/m}^3$ and $L/h = 20$. According to the fitting results of the Douglas fir beams by Yahyaei-Moayyed and Taheri [15], the material parameters are $E_1 = 14\text{ GPa}$, $E_2 = 39.68\text{ GPa}$ and $\eta_2 = 6.9\times10^3\text{ GPa}\cdot\text{h}$. Additionally, in order to analyze the effect of viscous coefficient on the vibration properties of the viscoelastic beam, the viscous coefficient is taken as $\eta_2 \in 6.9\times[10^4, 10^{12}]$ according to the references [4,7,14].

At first, the effect of viscous coefficient on the vibration properties of the simply-supported viscoelastic intact beam is considered. Based on the standard linear solid model (SLS) and Kelvin-Voigt model (KV), the first three eigenfrequencies are obtained by the present EAM in tables 2 and 3, respectively. Let $E_1 \to \infty$, the present solutions are degenerated into the results of the KV intact beam. For the sake of simplicity, the real part (natural frequency) and imaginary part (decrement coefficient) of the k-th eigenfrequency ω_k are defined by $\mathrm{Re}(\omega_k)$ and $\mathrm{Im}(\omega_k)$, respectively. With the viscous coefficient η_2 increasing, it is seen that the first three decrement coefficients $\mathrm{Im}(\omega_k)$ $(k = 1,2,3)$ first increase, and then decrease. In addition, when $\eta_2 \in 6.9\times[10^4, 10^7]$, $\mathrm{Im}(\omega_k)$ increases with the order of mode function increasing. While $\eta_2 \in 6.9\times[10^8, 10^{12}]$, the decrement coefficient seems to be a constant. A similar conclusion had been presented by Peng [16] based on the results of the Euler-Bernoulli elastic beam resting on the viscoelastic foundation.

Besides, for SLS intact beam, the natural frequency $\mathrm{Re}(\omega_k)$ increases with the viscous coefficient η_2 increasing, and then it remains a constant when $\eta_2 \ge 6.9\times10^9$. While for KV intact beam, $\mathrm{Re}(\omega_k)$ decreases with η_2 increasing, and it reduces to zero when $\eta_2 = 6.9\times10^7$. The above conclusion is consistent with the results of the KV Timoshenko beam presented by Chen [14] to some degree. While $\eta_2 \geq 6.9\times10^9$, the natural frequencies of SLS beam and KV beam remain some certain constants.

Table 2

The first three eigenfrequencies of the simply-supported viscoelastic beam based on SLS model with different viscous coefficient η_2

η_2	Re(ω_1)	Im(ω_1)	Re(ω_2)	Im(ω_2)	Re(ω_3)	Im(ω_3)
6.9×10^4	648.09	0.09511	2592.38	1.52169	5832.895	7.70332
6.9×10^5	648.10	0.95102	2592.84	15.20608	5838.093	76.75614
6.9×10^6	648.81	9.46795	2636.86	141.0728	6234.933	508.7931
6.9×10^7	699.26	57.14699	2994.61	97.93233	6774.891	100.866
6.9×10^8	752.96	10.10045	3015.00	10.15595	6784.138	10.15892
6.9×10^9	753.80	1.01591	3015.21	1.01596	6784.23	1.01596
6.9×10^{10}	753.80	0.10160	3015.21	0.10160	6784.23	0.10160
6.9×10^{11}	753.80	0.01016	3015.21	0.01016	6784.23	0.01016
6.9×10^{12}	753.80	0.00102	3015.21	0.00102	6784.23	0.00102

Table 3

The first three eigenfrequencies of the simply-supported viscoelastic beam based on KV model with different viscous coefficient η_2

η_2	Re(ω_1)	Im(ω_1)	Re(ω_2)	Im(ω_2)	Re(ω_3)	Im(ω_3)
6.9×10^4	1268.86	1.398	5075.87	223.367	11420.3	113.23
6.9×10^5	1268.92	13.979	5071.03	223.67	11364.7	1132.3
6.9×10^6	1261.27	139.795	455.72	2236.89	1472.7	11327.8
6.9×10^7	0	1986.02	0	44503.7	0	235572
6.9×10^8	0	29408.7	417995	287924	1105187	287925
6.9×10^9	123588	28792.5	506779	28792.5	1141728	28792.5
6.9×10^{10}	126866	2879.25	507588	2879.25	1142087	2879.2
6.9×10^{11}	126898	287.92	507596	287.92	1142091	287.9
6.9×10^{12}	126899	28.79	507596	28.79	1142091	28.79

Table 4

The first eigenfrequency of the simply-supported SLS beam with a single crack for different viscous coefficient η_2 and crack location ξ_1

	ξ_1=0.1		ξ_1=0.2		ξ_1=0.3		ξ_1=0.4		ξ_1=0.5	
η_2	Re(ω_1)	Im(ω_1)	Re(ω_1)	Im(ω_1)	Re(ω_1)	Im(ω_1)	Re(ω_1)	Im(ω_1)	Re(ω_1)	Im(ω_1)
6.9×10^4	642.02	0.0933	626.75	0.0889	609.46	0.0841	596.71	0.0806	592.12	0.0794
6.9×10^5	642.03	0.9333	626.75	0.8894	609.47	0.8410	596.72	0.8062	592.12	0.7938
6.9×10^6	642.72	9.2921	627.40	8.8571	610.06	8.3772	597.27	8.0317	592.67	7.9089
6.9×10^7	692.13	56.592	674.22	55.166	654.00	53.502	639.12	52.240	633.77	51.778
6.9×10^8	745.89	10.099	728.10	10.096	707.97	10.923	693.13	10.090	687.78	10.089
6.9×10^9	746.73	1.0159	728.96	1.0159	708.86	1.0159	694.03	1.0159	688.69	1.0159
6.9×10^{10}	746.74	0.1016	728.97	0.1016	708.87	0.1016	694.04	0.1016	688.69	0.1016
6.9×10^{11}	746.74	0.0102	728.97	0.0102	708.87	0.0102	694.04	0.0102	688.69	0.0102
6.9×10^{12}	746.74	0.0010	728.97	0.0010	708.87	0.0010	694.04	0.0010	688.69	0.0010

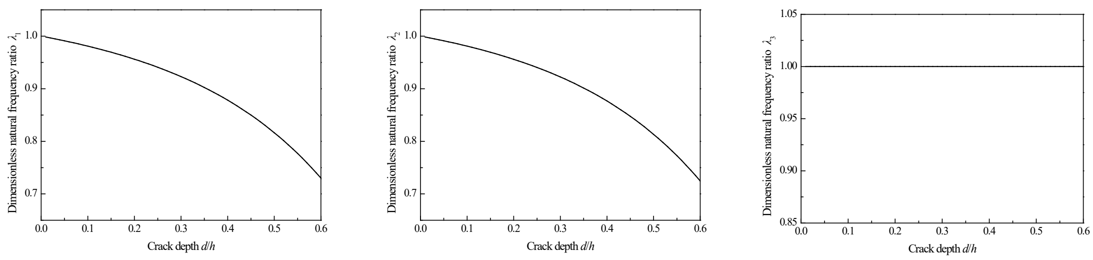
Table 5

The first eigenfrequency of the simply-supported SLS cracked beam for different viscous coefficient η_2 and crack number N

	N=0		N=1		N=2		N=4		N=8	
η_2	Re(ω_1)	Im(ω_1)	Re(ω_1)	Im(ω_1)	Re(ω_1)	Im(ω_1)	Re(ω_1)	Im(ω_1)	Re(ω_1)	Im(ω_1)
6.9×10^4	648.09	0.0951	592.12	0.0794	569.2	0.0734	530.3	0.0637	471.6	0.0504
6.9×10^5	648.10	0.9510	592.12	0.7938	569.2	0.7336	530.3	0.6367	471.6	0.5035
6.9×10^6	648.81	9.4680	592.67	7.9089	569.7	7.3105	530.7	6.3473	471.8	5.0231
6.9×10^7	699.26	57.1470	633.77	51.778	607.1	49.419	562.2	45.217	495.2	38.503
6.9×10^8	752.96	10.1005	687.78	10.089	661.1	10.083	615.7	10.071	547.3	10.048
6.9×10^9	753.80	1.0159	688.69	1.0159	662.0	1.0159	616.7	1.0159	548.4	1.0159
6.9×10^{10}	753.80	0.1016	688.69	0.1016	662.0	0.1016	616.7	0.1016	548.5	0.1016
6.9×10^{11}	753.80	0.0102	688.69	0.0102	662.0	0.0102	616.7	0.0102	548.5	0.0102
6.9×10^{12}	753.80	0.0010	688.69	0.0010	662.0	0.0010	616.7	0.0010	548.5	0.0010

Next, to consider the effect of cracks, a simple-supported viscoelastic beam with the symmetrically distributed cracks N is considered. Here the crack location is $\xi_j = j/(N+1)$ ($j=1,\cdots,N$), and crack depth is $d_j/h=0.4$. The effects of the viscous coefficient η_2 and crack number on the first eigenfrequency ω_1 for different viscoelastic beam models are analyzed, respectively. In tables 4 and 5, it is found that the decrement coefficient $\mathrm{Im}(\omega_1)$ and natural frequency $\mathrm{Re}(\omega_1)$ of the SLS beam decrease with the crack location ($\xi_1 \le 0.5$) and crack number increasing when $\eta_2 \in 6.9\times[10^4,10^7]$, which indicates that the crack has a significant influence on the vibration characteristics of the viscoelastic beam. While $\eta_2 \in 6.9\times[10^8,10^{12}]$, $\mathrm{Im}(\omega_1)$ remains a certain constant, that reveals the crack has less effect on the decrement coefficient for a higher value of η_2.

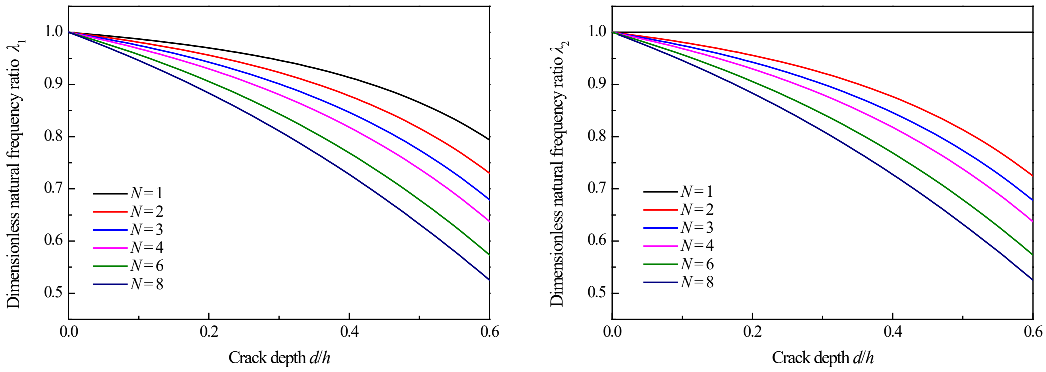
To sum up, for a higher value of η_2, the effects of crack depth and crack number on the decrement coefficient $\mathrm{Im}(\omega_k)$ of the viscoelastic beam are very limited. Therefore, the following analyses are mainly focused on the effects of crack depth and crack number on the natural frequency $\mathrm{Re}(\omega_k)$ of the viscoelastic beams.



(a) First frequency ratio (b) Second frequency ratio (c) Third frequency ratio

Fig. 1. The first three frequencies ratio of the simply-supported beam with two symmetric cracks

To consider the effect of crack, we suppose that ω_{0n} and ω_n are the n-th eigenfrequency of the viscoelastic intact and cracked beam, respectively, then $\lambda_n = \mathrm{Re}(\omega_n)/\mathrm{Re}(\omega_{0n})$ is the corresponding n-th natural frequency ratio. In the case of a viscoelastic beam with two symmetric cracks, the depths of cracks are equal to each other. Fig. 1 shows the first three natural frequency ratios of the cracked beam based on the present EAM. It is noticed that, when the cracks are located at the critical positions, i.e. $\xi_1 = 1/3$ and $\xi_2 = 2/3$, the 3rd natural frequency ratio is $\lambda_3 = 1$, which reveals that λ_3 is independent with the crack depth, in fig. 1(c).



(a) First frequency ratio (b) Second frequency ratio

Fig. 2. Variations of the first two frequencies ratio versus crack depth d/h of the simply-supported cracked beam with different crack number N

In the case of a viscoelastic beam with N symmetric cracks, the crack depths are equal to each other. The first two natural frequency ratios of the cracked beam are present in fig. 2. It can be seen that the first two natural frequency ratios decrease with the crack depth and crack number increasing generally. In addition, in fig. 2(b), when $N = 1$ that means the crack is located at the mid-span position, the 2nd natural frequency ratio is $\lambda_2 = 1$. The reason is possibly that the mid-span moment of the 2nd modal functions is null.

5. Conclusions

In this paper, the vibration characteristics of an Euler-Bernoulli viscoelastic cracked beams based on the standard linear solid model and Kelvin-Voigt model are investigated. Some conclusions arising from the numerical results can be summarized as follows: (1) For the simple- supported viscoelastic intact beam with SLS and KV models, the viscous coefficient has a significantly different effect on the first three decrement coefficients. (2) The crack has a complicated influence on the vibration characteristics of the viscoelastic beams. And for a higher value of viscous coefficient, the effects of crack depth and crack number on the decrement coefficient are very limited. (3) For the simple-supported cracked beam with SLS model, the first three natural frequencies decrease with the crack number and crack depth increasing.

R E F E R E N C E S

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