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# A TAUBERIAN THEOREM FOR THE WEIGHTED MEAN METHOD OF SUMMABILITY

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*In this paper we obtain a Tauberian condition in terms of the weighted classical control modulo for the weighted mean method of summability. Some additional results are also given.*

**Keywords:** summability by the weighted mean, Tauberian conditions and theorems, weighted classical and general control modulo.
**MSC2010:** 40A 10, 40E 05, 40G 05.

## 1. Introduction

Let $p = (p_n)$ be a sequence of nonnegative real numbers with

$$p_0 > 0 \quad \text{and} \quad P_n := \sum_{k=0}^{n} p_k \to \infty \quad (n \to \infty). \tag{1}$$

The $n$-th weighted mean of $u = (u_n)$ are defined by

$$\sigma_{n,p}^{(1)}(u) := \frac{1}{P_n} \sum_{k=0}^{n} p_k u_k.$$

A sequence $(u_n)$ is said to be summable by the weighted mean method $(\overline{N}, p)$ to $\ell$, written as $u_n \to \ell \, (\overline{N}, p)$, if

$$\lim_{n\to\infty} \sigma_{n,p}^{(1)}(u) = \ell. \tag{2}$$

Let $X$ and $Y$ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix. If for each $x \in X$ the series $A_n(x) = \sum_{k=0}^{\infty} a_{nk}x_k$ converges for each $n$ and the sequence $Ax = (A_n x) \in Y$ we say that the matrix $A$ maps $X$ into $Y$. By $(X, Y)$ we denote the set of all matrices which map $X$ into $Y$. Let $c$ be the set of all convergent sequences. A matrix $A$ is called regular if $A \in (c, c)$ and $\lim_{n\to\infty} A_n x = \lim_{k\to\infty} x_k$ for all $x \in c$.

The matrix representation of weighted mean method $(\overline{N}, p)$ is denoted by $W = (w_{nk})$, where $w_{nk}$ is defined by $w_{nk} = \frac{p_k}{P_n}$ if $k \leq n$ and $w_{nk} = 0$ otherwise.

It is known that $(\overline{N}, p)$ summability method is regular, i. e, $W \in (c, c)_{reg}$ if and only if (1) holds.

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The Silverman-Toeplitz theorem states that $A = (a_{nk})$ is regular if and only if

(R1) $||A|| = \sup_n \sum_k |a_{nk}| < \infty$,
(R2) $\lim_{n\to\infty} a_{nk} = 0$ for each $k$,
(R3) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 1$.
If the limit

$$\lim_{n\to\infty} u_n = \ell \tag{3}$$

exists, then (2) is satisfied. However, the converse is not always true. Notice that (2) implies (3) under certain condition(s), which is called a Tauberian condition. Any theorem which states that convergence of sequences follows from $(\overline{N}, p)$ summability method and some Tauberian condition is said to be a Tauberian theorem for $(\overline{N}, p)$ summability method. If $p_n = 1$ for all nonnegative $n$, then $(\overline{N}, p)$ summability method reduces to Cesàro summability method. The backward difference of $(u_n)$ is defined by $\Delta u_n = u_n - u_{n-1}$ for all nonnegative $n$, where $u_{-1} = 0$. The difference between $u_n$ and its $n$-th weighted mean $\sigma_{n,p}^{(1)}(u)$, which is called the weighted Kronecker identity [2] is given by the identity

$$u_n - \sigma_{n,p}^{(1)}(u) = V_{n,p}^{(0)}(\Delta u), \tag{4}$$

where

$$V_{n,p}^{(0)}(\Delta u) := \frac{1}{P_n} \sum_{k=1}^{n} P_{k-1} \Delta u_k.$$

The weighted classical control modulo of $(u_n)$ is denoted by $\omega_{n,p}^{(0)}(u) = \frac{P_{n-1}}{p_n} \Delta u_n$ and the weighted general control modulo of integer order $m \geq 1$ of $(u_n)$ is defined in [2] by

$$\omega_{n,p}^{(m)}(u) = \omega_{n,p}^{(m-1)}(u) - \frac{1}{P_n} \sum_{k=0}^{n} p_k \omega_{k,p}^{(m-1)}(u).$$

For each integer $m \geq 0$, we define $\sigma_{n,p}^{(m)}(u)$ by

$$\sigma_{n,p}^{(m)}(u) = \begin{cases} \frac{1}{P_n} \sum_{k=0}^{n} p_k \sigma_{k,p}^{(m-1)}(u) & , m \geq 1 \\ u_n & , m = 0 \end{cases}$$

A sequence $(u_n)$ is said to be slowly oscillating [5] if

$$\lim_{1 \leq \frac{m}{n} \to 1, n\to\infty} (x_m - x_n) = 0.$$

In terms of $\epsilon > 0$ and $\delta$, this definition is equivalent to the following: for any given $\epsilon > 0$, there exist $\delta = \delta(\epsilon) > 0$ and the positive integer $N = N(\epsilon)$ such that $|x_m - x_n| < \epsilon$ if $n \geq N(\epsilon)$ and $n \leq m \leq (1 + \delta)n$.

Our aim in this paper is to obtain a Tauberian condition in terms of the weighted classical control modulo for $(\overline{N}, p)$ summability method. Some additional results are also given.

## 2. **The Result**

We prove the following Tauberian theorem for $(\overline{N}, p)$ summability method.

**Theorem 2.1.** *Let $(p_n)$ be a sequence of nonnegative numbers such that $p_0 > 0$,*

$$\frac{P_{n-1}}{p_n} = O(n), \quad n \to \infty, \tag{5}$$

$$\liminf_{n\to\infty} \frac{P_{[\lambda n]}}{P_n} > 1 \quad \textit{for every} \quad \lambda > 1, \tag{6}$$

*where $[\lambda n]$ denotes the integral part of the product $\lambda n$, and let $u_n \to \ell\,(\overline{N}, p)$. Then $(u_n)$ converges to $\ell$ if for some $t > 1$*

$$(\lambda - 1)^{t-1} \limsup_{n\to\infty} \sum_{j=n+1}^{[\lambda n]} j^{t-1} |\omega_{j,p}^{(0)}(u)|^t = o(1), \quad \lambda \to 1^+. \tag{7}$$

Note that the condition (6) imposed on the sequence $(p_n)$ was used in [4].

**Remark 2.1.** We note that if

$$\sum_{j=1}^{n} j^{t-1} |\omega_{j,p}^{(0)}(u)|^t = \log v_n, \quad t > 1 \tag{8}$$

for some O-Regularly varying sequence $(v_n)$, then (8) is equivalent to (see [6])

$$\frac{1}{n} \sum_{j=1}^{n} j^{t} |\omega_{j,p}^{(0)}(u)|^t = O(1), \quad n \to \infty \quad t > 1.$$

We remind the reader that a positive sequence $(u_n)$ is $O$-Regularly varying [1] if

$$\limsup_{n\to\infty} \frac{u_{[\lambda n]}}{u_n} < \infty, \quad \text{for}\, \lambda > 1.$$

## 3. **Lemmas**

We need the following Lemmas for the proof of Theorem 2.1.

**Lemma 3.1.** *([2]) Let $u = (u_n)$ be a sequence of real numbers.*
*For $\lambda > 1$ and sufficiently large $n$,*

$$u_n - \sigma_{n,p}^{(1)}(u) = \frac{P_{[\lambda n]}}{P_{[\lambda n]} - P_n} \left( \sigma_{[\lambda n],p}^{(1)}(u) - \sigma_{n,p}^{(1)}(u) \right) - \frac{1}{P_{[\lambda n]} - P_n} \sum_{k=n+1}^{[\lambda n]} p_k \sum_{j=n+1}^{k} \Delta u_j,$$

*where $[\lambda n]$ denotes the integer part of $\lambda n$.*

**Lemma 3.2.** *([2]) For a sequence $(u_n)$,*

$$\frac{P_{n-1}}{p_n} \Delta\sigma_{n,p}^{(1)}(u) = V_{n,p}^{(0)}(\Delta u).$$

For a sequence $u=(u_n)$, we define

$$\left(\frac{P_{n-1}}{p_n}\Delta\right)_m u_n=\left(\frac{P_{n-1}}{p_n}\Delta\right)_{m-1}\left(\frac{P_{n-1}}{p_n}\Delta u_n\right)=\frac{P_{n-1}}{p_n}\Delta\left(\left(\frac{P_{n-1}}{p_n}\Delta\right)_{m-1}u_n\right),$$

where

$$\left(\frac{P_{n-1}}{p_n}\Delta\right)_0 u_n=u_n$$

, and

$$\left(\frac{P_{n-1}}{p_n}\Delta\right)_1 u_n=\frac{P_{n-1}}{p_n}\Delta u_n.$$

**Lemma 3.3.** *([3]) For a sequence $(u_n)$ and any integer $m\geq 1$,*

$$\omega_{n,p}^{(m)}(u)=\left(\frac{P_{n-1}}{p_n}\Delta\right)_m V_{n,p}^{(m-1)}(\Delta u). \tag{9}$$

## 4. Proof of Theorem 2.1

By Lemma 3.1,

$$\left|u_n-\sigma_{n,p}^{(1)}(u)\right|\leq\frac{P_{[\lambda n]}}{P_{[\lambda n]}-P_n}\left|\sigma_{[\lambda n],p}^{(1)}(u)-\sigma_{n,p}^{(1)}(u)\right|+\sum_{j=n+1}^{[\lambda n]}|\Delta u_j|. \tag{10}$$

By (6), we have

$$\limsup_{n\to\infty}\frac{P_{[\lambda_n]}}{P_{[\lambda_n]}-P_n}=\left\{1-\frac{1}{\liminf\limits_{n\to\infty}\frac{P_{[\lambda n]}}{P_n}}\right\}^{-1}<\infty. \tag{11}$$

Hence, we have

$$\begin{aligned}\limsup_{n\to\infty}\frac{P_{[\lambda n]}}{P_{[\lambda n]}-P_n}&\left|\sigma_{[\lambda n],p}^{(1)}(u)-\sigma_{n,p}^{(1)}(u)\right|\\ &\leq\limsup_{n\to\infty}\frac{P_{[\lambda n]}}{P_{[\lambda n]}-P_n}\limsup_{n\to\infty}\left|\sigma_{[\lambda n],p}^{(1)}(u)-\ell\right|\\ &+\limsup_{n\to\infty}\frac{P_{[\lambda n]}}{P_{[\lambda n]}-P_n}\limsup_{n\to\infty}\left|\sigma_{n,p}^{(1)}(u)-\ell\right|.\end{aligned}$$

Since $(u_n)$ is $(\overline{N},p)$ summable to $\ell$, both the limits

$$\lim_{n\to\infty}\sigma_{[\lambda_n],p}^{(1)}(u)=\ell$$

and

$$\lim_{n\to\infty}\sigma_{n,p}^{(1)}(u)=\ell$$

exist. Therefore, we have, by (11),

$$\limsup_{n\to\infty}\frac{P_{[\lambda n]}}{P_{[\lambda n]}-P_n}\left|\sigma_{[\lambda n],p}^{(1)}(u)-\sigma_{n,p}^{(1)}(u)\right|=0. \tag{12}$$

For the second term on the right-hand side of (10) we obtain

$$\begin{aligned}
\sum_{j=n+1}^{[\lambda n]}|\Delta u_j| &\le \sum_{j=n+1}^{[\lambda n]}\frac{p_j}{P_{j-1}}\frac{P_{j-1}}{p_j}|\Delta u_j| \\
&= \sum_{j=n+1}^{[\lambda n]}\frac{p_j}{P_{j-1}}|\omega^{(0)}_{j,p}(u)| \\
&\le \left(\sum_{j=n+1}^{[\lambda n]}\left(\frac{p_j}{P_{j-1}}\right)^s\right)^{\frac{1}{s}}\left(\sum_{j=n+1}^{[\lambda n]}|\omega^{(0)}_{j,p}(u)|^t\right)^{\frac{1}{t}}, \text{where } \tfrac{1}{s}+\tfrac{1}{t}=1 \\
&\le ([\lambda n]-n)^{\frac{1}{s}}\left(\sum_{j=n+1}^{[\lambda n]}\frac{j^{t-1}|\omega^{(0)}_{j,p}(u)|^t}{j^{t-1}}\right)^{\frac{1}{t}} \\
&\le \left(\frac{[\lambda n]-n}{n}\right)^{\frac{1}{s}}\left(\sum_{j=n+1}^{[\lambda n]}j^{t-1}|\omega^{(0)}_{j,p}(u)|^t\right)^{\frac{1}{t}} \qquad (13)
\end{aligned}$$

From (13), we have

$$\limsup_{n\to\infty}\sum_{j=n+1}^{[\lambda n]}|\Delta u_j| \le (\lambda-1)^{\frac{1}{s}}\limsup_{n\to\infty}\left(\sum_{j=n+1}^{[\lambda n]}j^{t-1}|\omega^{(0)}_{j,p}(u)|^t\right)^{\frac{1}{t}}. \qquad (14)$$

From (12) and (14), we have

$$\limsup_{n\to\infty}\left|u_n-\sigma^{(1)}_{n,p}(u)\right| \le (\lambda-1)^{\frac{1}{s}}\limsup_{n\to\infty}\left(\sum_{j=n+1}^{[\lambda n]}j^{t-1}|\omega^{(0)}_{j,p}(u)|^t\right)^{\frac{1}{t}} = 0. \qquad (15)$$

Letting $\lambda\to 1^+$ in (15) and taking (7) into account, we conclude that

$$\limsup_{n\to\infty}\left|u_n-\sigma^{(1)}_{n,p}(u)\right| = 0. \qquad (16)$$

This completes the proof of Theorem 2.1.

## 5. **Some additional results**

If we replace the $(\overline{N},p)$ summability of $(u_n)$ in Theorem 2.1 by the summability of $(\sigma^{(1)}_{n,p}(u))$ and $(V^{(0)}_{n,p}(\Delta u))$, we have the following theorems.

**Theorem 5.1.** *Let $(p_n)$ be a sequence of nonnegative numbers such that $p_0>0$, the conditions (5) and (6) are satisfied and let $\sigma^{(1)}_{n,p}(u)\to\ell\,(\overline{N},p)$. Then $u_n\to\ell\,(\overline{N},p)$ if for some $t>1$*

$$(\lambda-1)^{t-1}\limsup_{n\to\infty}\sum_{j=n+1}^{[\lambda n]}j^{t-1}|V^{(0)}_{j,p}(\Delta u)|^t = o(1), \quad \lambda\to 1^+. \qquad (17)$$

*Proof.* If we replace $u = (u_n)$ by $\sigma(u) = (\sigma^{(1)}_{n,p}(u))$ in $\omega^{(0)}_{n,p}(u) = \frac{P_{n-1}}{p_n}\Delta u_n$, we obtain that $\omega^{(0)}_{n,p}(\sigma(u)) = \frac{P_{n-1}}{p_n}\Delta\sigma^{(1)}_{n,p}(u)$. By Lemma 3.2,

$$\frac{P_{n-1}}{p_n}\Delta\sigma^{(1)}_{n,p}(u) = V^{(0)}_{n,p}(\Delta u).$$

All the conditions of Theorem 2.1 are satisfied and the condition (7) becomes (17). This completes the proof of Theorem 5.1. □

**Theorem 5.2.** *Let $(p_n)$ be a sequence of nonnegative numbers such that $p_0 > 0$, the conditions (5) and (6) are satisfied and and let $V^{(0)}_{n,p}(\Delta u) \to \ell\,(\overline{N}, p)$. Then $(u_n)$ is slowly oscillating if for some $t > 1$*

$$(\lambda-1)^{t-1}\limsup_{n\to\infty}\sum_{j=n+1}^{[\lambda n]} j^{t-1}|\omega^{(1)}_{j,p}(u)|^t = o(1), \quad \lambda\to 1^+. \tag{18}$$

*Proof.* If we replace $u = (u_n)$ by $V^{(0)}(\Delta u) = (V^{(0)}_{n,p}(\Delta u))$ in $\omega^{(0)}_{n,p}(u) = \frac{P_{n-1}}{p_n}\Delta u_n$, we obtain that $\omega^{(0)}_{n,p}(V^{(0)}(\Delta u)) = \frac{P_{n-1}}{p_n}\Delta V^{(0)}_{n,p}(\Delta u)$. By Lemma 3.3,

$$\frac{P_{n-1}}{p_n}\Delta V^{(0)}_{n,p}(u) = \omega^{(1)}_{n,p}(u).$$

All the conditions of Theorem 2.1 are satisfied and the condition (7) becomes (18). So, we have convergence of $(V^{(0)}_{n,p}(\Delta u))$ to $\ell$.

It follows from Lemma 3.2 that

$$\sigma^{(1)}_{n,p}(u) - \sigma^{(1)}_{m,p}(u) = \sum_{k=m+1}^{n}\frac{p_k}{P_{k-1}}V^{(0)}_{k,p}(\Delta u)$$

for $n > m$. By the condition (5) and the boundedness of $(V^{(0)}_{n,p}(\Delta u))$, we have

$$|\sigma^{(1)}_{n,p}(u) - \sigma^{(1)}_{m,p}(u)| \le C\sum_{k=m+1}^{n}\frac{1}{k} \le C(\frac{n}{m}-1)$$

for some constant $C > 0$. Taking the limit of both sides of the last inequality as $\frac{n}{m}\to 1$, and $m\to\infty$, we obtain that $(\sigma^{(1)}_{n,p}(u))$ is slowly oscillating. By Kronecker identity, $(u_n)$ is slowly oscillating. This completes the proof of Theorem 5.2. □

## 6. **Examples and an application for Theorem 2.1**

If we take $p_n = 1$ for all nonnegative $n$, then summability by the weighted mean method $(\overline{N}, p)$ reduces to the Cesàro summability method. We have the following examples of Theorem 2.1.

**Example 6.1.** *A Cesàro summable sequence $(u_n)$ to $\ell$ converges to $\ell$ in the ordinary sense if*

$$(\lambda-1)^{t-1}\limsup_{n\to\infty}\sum_{j=n+1}^{[\lambda n]} j^{t-1}|\omega^{(0)}_{j,1}(u)|^t = o(1), \quad \lambda\to 1^+. \tag{19}$$

If $\omega_{n,1}^{(0)}(u) = \dfrac{a_n}{n}$ for some bounded sequence $(a_n)$ in Example 6.1, then we have the following example.

**Example 6.2.** *A Cesàro summable sequence $(u_n)$ to $\ell$ converges to $\ell$ in the ordinary sense if*

$$(\lambda - 1)^{t-1} \limsup_{n\to\infty} \sum_{j=n+1}^{[\lambda n]} \frac{|a_j|^t}{j} = o(1), \quad \lambda \to 1^+, \tag{20}$$

*where $(a_n)$ is a bounded sequence.*

We have the following result as an application of Theorem 2.1.

**An application.** Let $(p_n)$ be a sequence of nonnegative numbers such that $p_0 > 0$, the conditions (5) and (6) are satisfied and let $u_n \to \ell\,(\overline{N}, p)$. Then $(u_n)$ converges to $\ell$ if

$$\sum_{j=1}^{n} j^{t-1} |\omega_{j,p}^{(0)}(u)|^t = \log v_n \tag{21}$$

for some O-Regularly varying sequence $(v_n)$ and for some $t > 1$.

*Proof.* Let the conditions (5) and (6) be satisfied and let $u_n \to \ell\,(\overline{N}, p)$. If

$$\sum_{j=1}^{n} j^{t-1} |\omega_{j,p}^{(0)}(u)|^t = \log v_n$$

for some O-Regularly varying sequence $(v_n)$ and for some $t > 1$, then it is easy to show that the condition (7) is satisfied. Indeed, the left side of the condition (7) becomes

$$(\lambda - 1)^{t-1} \limsup_{n\to\infty} (\log v_{[\lambda n]} - \log v_n),$$

which is o(1) as $\lambda \to 1^+$ by the definition of O-Regularly varying sequence. □

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